NETAJI SUBHAS OPEN UNIVERSITY

STUDY MATERIAL

MATHEMATICS

POST GRADUATE

PG (MT) 03

GROUPS : A & B

Ordinary Differential Equations and Special Functions

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Partial Differential Equations

PREFACE

In the curricular structure introduced by this University for students of Post-Graduate Degree Programme, the opportunity to pursue Post-Graduate course in any subject introduced by this University is equally available to all learners. Instead of being guided by any presumption about ability level, it would perhaps stand to reason if receptivity of a learner is judged in the course of the learning process. That would be entirely in keeping with the objectives of open education which does not believe in artificial differentiation.

Keeping this in view, study materials of the Post-Graduate level in different subjects are being prepared on the basis of a well laid-out syllabus. The course structure combines the best elements in the approved syllabi of Central and State Universities in respective subjects. It has been so designed as to be upgradable with the addition of new information as well as results of fresh thinking and analysis.

The accepted methodology of distance education has been followed in the preparation of these study materials. Co-operation in every form of experienced scholars is indispensable for a work of this kind. We, therefore, owe an enormous debt of gratitude to everyone whose tireless efforts went into the writing, editing and devising of proper lay-out of the materials. Practically speaking, their role amounts to an involvement in 'invisible teaching'. For, whoever makes use of these study materials would virtually derive the benefit of learning under their collective care without each being seen by the other.

The more a learner would seriously pursue these study materials, the easier it will be for him or her to reach out to larger horizons of a subject. Care has also been taken to make the language lucid and presentation attractive so that they may be rated as quality self-learning materials. If anything remains still obscure or difficult to follow, arrangements are there to come to terms with them through the counselling sessions regularly available at the network of study centres set up by the University.

Needless to add, a great deal of these efforts is still experimental–in fact, pioneering in certain areas. Naturally, there is every possibility of some lapse or deficiency here and there. However, these do admit of rectification and further improvement in due course. On the whole, therefore, these study materials are expected to evoke wider appreciation the more they receive serious attention of all concerned.

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PG (MT)—03
Ordinary Differential Equations and Special Functions, Partial Differential Equations

Group A

Ordinary Differential Equations and Special Functions

Group B
Partial Differential Equations

Unit : 1 ❑ Existence and Nature of Solutions

1.0 Introduction : In our earlier course on differential equations we have learai to recognise certain types of ordinary differential equations (ODE) and solve them when they belong to some standard forms. Here our main object will be to learn under what conditions these equations do possess solutions though in many cases we are unable to express them in particular forms. In this context we have discussed the conditions of existence of unique, continuous solutions of (i) first-order ODE (ii) first-order system of ODE in n unknown functions and (iii) n-th order ODE. Before discussing the existence theorems we have touched upon some equations like first-order exact equations, integrating factors of non-exact equations, the linear equations of first order, the principle of duality and simultaneous systems in three variables. The Cauchy-Picard existence theorem for first-order ODE is the most important one as it gives us the conditions under which the equation does possess a unique, continuous solution. The method may be extended to find the existence of unique set of continuous solutions of the system of first-order ODE in n unknown functions and also the existence of solution of nth order ODE. Lastly we have discussed singular solutions of first-order equations.

1.1 Definitions : A relationship between the differentials dx and dy of the two variables x and y is called a **differential equation.** Such a relationship involves the variables x and y together with some constants c, b, c, etc. The differential equation expressing a relation between an independent variable, a dependent variable and one or more differential coefficients of the dependent variable with respect to the independent variable, is called an **(ODE) Ordinary Differential Equation.** The highest order differential coefficients involved in the differential equation is its order and the power to which the highest order differential coefficient is raised is the **degree** of the equation. The equation is said to be **linear** if the dependent variables and its derivatives occur in the first degree only and not as higher powers or products. Otherwise, the differential equation is said to be **non-linear.** The coefficients of a linear equation may be either constants or functions of the dependent variable. For example, the ordinary differential equation (ODE)

$$\frac{d^4y}{dx^2}+x^2\frac{d^3y}{dx^3}+x^3\frac{dy}{dx}=xe^x$$

is linear of order 4 and degree 1, while the equation

$$\frac{d^2y}{dx^2}+5\left(\frac{dy}{dx}\right)^3+6y=0$$

is non-linear of order 2 and degree 3.

Let us now consider an equation of the form

$$f(x,\ y,\ c_1,\ c_2,\ \dots,\ c_n)=0 \tag{1.1}$$

where x and y are variables and $c_1, c_2, \dots, c_n$ are constants. This equation determines y as a function of x. Differentiating (1.1) with respect to x in succession, we get n equations

$$\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}y'=0$$

$$\frac{\partial^2 f}{\partial x^2}+2\frac{\partial^2 f}{\partial x\partial y}y'+\frac{\partial^2 f}{\partial y^2}y'^2+\frac{\partial f}{\partial y}y''=0 \tag{1.2}$$

$$\dots$$

$$\frac{\partial^n f}{\partial x^n}+\cdots+\frac{\partial f}{\partial y}y^{(n)}=0$$

where $y'=\dfrac{dy}{dx},\ y''=\dfrac{d^2y}{dx^2},\ \dots,\ y^{(n)}=\dfrac{d^ny}{dx^n}$.

Eliminating the n constants from the $(n+1)$ equations (1.1) and (1.2) we get the differential equation

$$F(x,\ y,\ y',\ y'',\ \dots,\ y^{(n)})=0 \tag{1.3}$$

This differential equation is satisfied by every function $y=\phi(x)$ defined by the relation (1.1). This relation is called **the primitive** of the differential equation (1.3) and every function $y=\phi(x)$ satisfying the differential equation is termed as a **solution.** A solution involving the number of distinct arbitrary constants equal to the order of the equation is known as the **general solution** Thus the general solution is the primitive of the differential equation.

The primitive of the first-order differential equation

$$\frac{dy}{dx} = f(x,y) \tag{1.4}$$

is a relation between the two variables x and y and a parameter c. The differential equation is then said to represent a **one-parameter family of plane curves** and each curve of the family is called an **integral-curve** or the **solution-curve** of the differentia) equation, In the equation (1.4) we assume that the function $f(x, y)$ is single-valued and continuous in a domain D in the (x, y)-plane. Through every point in the domain D, there passes one and only one integral-curve. However, there may be points outside D at which $f(x, y)$ is not continuous or single-valued; such points are known as **singular points,** where the behaviour of the integral-curves may be exceptional. In general, an ordinary differential equation of order n forms an n-parameter family of integral-curves and through each non-singuiar point there passes in genera! an $(n-1)$-fold infinity of integral curves.

1.2 Simultaneous System of Equations : Suppose that

$$\phi(x, y, z, c_1, c_2) = 0 \text{ and } \psi(x, y, z, c_1, c_2) = 0$$

are two equations, each involving three variables x, y, z and two arbitrary constants c_1, c_2. Diffemtiating these two equations with respect to x and then eliminating the constants c_1 and c_2 between these four equations, we obtain a pair of simultaneous ordinary differential equations of the first-order of the form

$$\Phi(x, y, y', z, z') = 0, \qquad \Psi(x, y, y', z, z') = 0$$

By introducing a sufficient number of new variables, we may obtain either a single equation of any order or any system of simultaneous equations of first-order. Let us consider an equation of the form

$$\frac{d^n y}{dx^n} = F\left(x, y, \frac{dy}{dx}, \cdots, \frac{d^{n-1}y}{dx^{n-1}}\right) \tag{1.5}$$

Introducing the new variables $y_1, y_2, y_3, \ldots, y_n$ by

$$y_1 = y,\ y_2 = \frac{dy_1}{dx},\ y_3 = \frac{d^2 y}{dx^2} = \frac{dy^2}{dx}, \ldots, y_n = \frac{dy_{n-1}}{dx},$$

we get from (1.5)

$$\frac{dy_n}{dx} = F(x, y_1, y_2, \ldots, y_n) \tag{1.6}$$

which form a system of n simultaneous equations of the first-order. It may be noted that if the original equation is linear, the equations of the equivalent system are also linear.

1.3 Exact Equations of the First-Order and of First Degree : An ordinary differential of the first-order and of first degree can always be expressible in the form of a total differential equation

$$P(x, y)dx + Q(x, y)dy = 0 \tag{1.7}$$

which do not involve $p(= dy / dx)$. The expression $P(x, y)dx + Q(x, y)dy$ is said to be exact differential if it can be expressed in the form du, where $u \sim u(x, y)$ and then the differential equation (1.7) is said to be exact. **The necessary and sufficient condition of integrability** of the exact equation (1.7) is

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \tag{1.8}$$

For example, the equation

$$(a^2 - 2xy - y^2)dx - (x + y)^2 dy = 0$$

where a is constant, is exact.

In particular, if P is a function of x alone and Q is a function of y alone, the equation $P(x)dx + Q(y)dy$ is said to **have separated variables.** As an example, consider the equation

$$x\sqrt{y}dx + (1 + y)\sqrt{1 + x}dy = 0$$

which can be written as

$$\frac{x}{\sqrt{1 + x}}dx + \frac{1 + y}{\sqrt{y}}dy = 0.$$

This equation is exact and has separated variables.

If P and Q are homogeneous functions of x and y of the same degree, then the equation (1.7) is reducible by the substitution y-vx to one whose variables are separable and then the equation (1.7) is called **homogeneous equation.** For example, the equation

$$x\cos\frac{y}{x}(y\,dx + x\,dy) = y\sin\frac{y}{x}(xdy - ydx)$$

can be reduced by the substitution $y = vx$ into the form

$$\frac{\cos v - v\sin v}{v\cos v}\,dv + 2\frac{dx}{x} = 0$$

which is separable.

1.4 Linear Equations of the First-order : The most general **linear equation of the first-order** is of the type

$$\frac{dy}{dx} + 2(x)y = Q(x) \tag{1.9}$$

To solve this equation, we consider first the homogeneous linear equation

$$\frac{dy}{dx} + P(x)y = 0$$

whose solution is $y = ce^{-\int Pdx}$ Substituting in (1.9), the expression $\frac{dv}{dx}e^{-\int Pdx} = Q$ leads to

whence $v = C + \int Qe^{\int Pdx}$, i.e., $y = Ce^{-\int Pdx} + e^{-\int Pdx}\int Qe^{\int Pdx}dx$.

The equation of the form

$$\frac{dy}{dx} + P(x)y = Q(x)y^n, \tag{1.10}$$

known as **Bernoulli equation,** can be reduced to the linear form (1.9) by changing the variable y by $z = y^{1-n}$

1.5 The Integrating Factor : If the differential equation $Pdx + Qdy = 0$ is not exact, then this equation can be made exact by introducing a function $\mu(x, y)$ such that the expression $\mu(Pdx + Qdy)$ is a total differential du (say). If μ can be found, the problem reduces to a mere quadrature. It can be proved that if the differential equation admits one and only one solution, there exists an infinite number of integrating factors.

1.6 The Principle of Duality : By the use of a certain transformation, due to Legendre, a dual relationship can be set up between two first-order differential equations. We introduce two new variables defined by $X = p,\ Y = xp - y,\ \left(p = \frac{dy}{dx} \neq 0\right)$ and let $P = \frac{dY}{dX}$. Then

$$dX = dp,\ dY = xdp + pdx - dy = xdp$$

and, therefore, $P = x$. Also $y = xp - Y = XP - Y$. Hence the transformations $X = p,\ Y = xp - y$ and $x = P,\ y = XP - y$ are equivalent. Thus the equations

$$F(x, y, p) = 0 \text{ and } F(P, XP - Y, X) = 0$$

can be transformed into one another and, therefore, there exists a dual relationship between them. If one of the equations is integrabie, the other can be integrated by purely algebraical process.

As for example, the equation $(y - px)x = y$ can be transformed into the homogeneous equation $P = \frac{Y}{X + Y}$ having solution $\log Y - \frac{X}{Y} =$ constant. Differentiating with respect to X, $\frac{P}{Y} - \frac{Y - XP}{Y^2} = 0$ whence $Y = \frac{Y - XP}{P} = -\frac{y}{x}$ and consequently $\frac{X}{Y} = \frac{1}{P} - 1 = \frac{1}{x} - 1$. Hence the solution of the given equation is $\log\left(-\frac{y}{x}\right) - \frac{1}{x} =$ constant, i.e., $y = cxe^{1/x}$.

1.7 Simultaneous Systems in Three Variables : Let us consider the system

$$\frac{dx}{\xi} = \frac{dy}{\eta} = \frac{dz}{\zeta} \tag{1.11}$$

where $\xi = \xi(x,y,z)$, $\eta = \eta(x,y,z)$ and $\zeta = \zeta(x,y,z)$. In particular if ξ and η are independent of z, then the equation $\frac{dx}{\xi} = \frac{dy}{\eta}$ involves only x and y; it is supposed that this equation can be integrated to give the solution $\Phi(x, y, \alpha) = 0$, α being the constant of integration. Let this equation be solved for y and $y = \phi(x, \alpha)$. Let ξ_1 and ζ_1 be what ξ and ζ become when y is replaced by $\Phi(x, \alpha)$. Then the equation $\frac{dx}{\xi_1} = \frac{dy}{\zeta_1}$ do not involve y and its solution is of the form $\theta(x, z, \alpha, \beta) = 0$, β being constant of integration. Let α be eliminated between the two solutions $\Phi(x, y, \alpha) = 0$ and $\theta(x, z, \alpha, \beta) = 0$ to give $\Psi(x, y, z, \beta) = 0$. The solutions then take the form

$$\Phi(x, y, \alpha) = 0 \text{ and } \Psi(x, y, z, \beta) = 0 \qquad (1.12)$$

Now consider a simultaneous linear system with constant coefficients of the form (1.11) where

$$\xi = a_1x + b_1y + c_1z + d_1$$

$$\eta = a_2x + b_2y + c_2z + d_2$$

$$\zeta = a_3x + b_3y + c_3z + d_3$$

where a_i, b_i, c_i, d_i $(i = 1, 2, 3)$ are constants. We introduce a new variable t such that

$$\frac{dx}{\xi} = \frac{dy}{\eta} = \frac{dz}{\zeta} = \frac{dt}{t}.$$

Then, we have

$$\frac{dt}{t} = \frac{l dx + m dy + n dz}{l\xi + m\eta + n\zeta}$$

whatever be the constants l, m, n. We choose l, m, n such that

$$la_1 + ma_2 + na_3 = l\rho$$

$$lb_1 + mb_2 + nb_3 = m\rho$$

$$lc_1 + mc_2 + nc_3 = n\rho$$

so that

$$\frac{dt}{t}=\frac{d(lx+my+nz)}{\rho(lx+my+nz+r)}$$

with $\rho r = ld_1 + md_2 + nd_3$. This choice of l, m, n is possible provided ρ is a root of the equation

$$\begin{vmatrix} a_1-\rho & a_2 & a_3 \\ b_1 & b_1-\rho & b_3 \\ c_1 & c_2 & b_3-\rho \end{vmatrix}=0$$

If $1/\lambda_i (i = 1, 2, 3)$ be the distinct roots of this equation and the corresponding values of l, m, n, r be l_i, m_i, n_i, r_i $(i = 1, 2, 3)$, then

$$\frac{dt}{t}=\frac{\lambda_i x(l_i x+m_i y+n_i z)}{l_i x+m_i y+n_i z+r_i}$$

whence $t = C_i\left(l_i x + m_i y + n_i z + r_i\right)^{\lambda_i}$. The solution is, therefore,

$$C_1\left(l_1 x+m_1 y+n_1 z+r_1\right)^{\lambda_1}=C_2\left(l_2 x+m_2 y+n_2 z+r_2\right)^{\lambda_2}$$

$$=C_3\left(l_3 x+m_3 y+n_3 z+r_3\right)^{\lambda_3}$$

containing three constants C_1, C_2, C_3 of which two are arbitrary.

1.8 The Existence Theorem : A differential equation may or may not have a solution, and if a solution exists, it may not be unique. We now proceed to find the conditions which guarantee the existence and uniqueness of a solution of a differential equation. The method of finding is known as **Picard's method or the method of successive approximations.**

Picard's theorem : The differential equation

$$\frac{dy}{dx}=f(x,y)$$

has a unique solution $y = y(x)$ satisfying $y_0 = y(x_0)$ over $[x_0 - h, x_0 + h]$ if

(i) $f(x, y)$ is continuous over a rectangular domain

$$D=\{|x-x_0|\le a, |y-y_0|\le b\} \quad (h\le a,\ b/M)$$

(ii) $|f(x,y)| \le M \in \mathbf{R} \; \forall (x,y) \in D$

(iii) $f(x, y)$ satisfies the Cauchy-Lipschitz condition

$|f(x, Y) - f(x, y)| < k|Y - y|$ for $k \in \mathbf{R}$ and (x, y), $(x, Y) \in D$

Proof : Let us construct a sequence of functions $y_n(x)$ defined over the common domain $[x_0 - a, x_0 + a]$ as follows :

$$y_1(x) = y_0 + \int_{x_0}^{x} f\{t, y_0\}dt$$

$$y_2(x) = y_0 + \int_{x_0}^{x} f\{t, y_1(t)\}dt$$

$$\cdots \quad \cdots$$

$$y_n(x) = y_0 + \int_{x_0}^{x} f\{t, y_{n-1}(t)\}dt$$

We shall show that

(i) as $n \to \infty$, the sequence of functions $y_n(x)$ tends to a function which is a continuous function of x;

(ii) the limit function satisfies the differential equation;

(iii) the solution thus defined assumes the value y_0, when $x = x_0$ and is the only continuous solution.

Let, x lie in the interval considered and $|y_n(x) - y_0| \le b$ so that $|y_{n-1}(x) - y_0| \le b$. Since $|f\{t, y_{n-1}(t)\}| \le M$, we have

$$|y_n(x) - y_0| \le \int_{x_0}^{x} |f\{t, y_{n-1}(t)\}| dt \le M(x - x_0) \le Mh \le b$$

But $|y_1(x) - y_0| \le b$ and, therefore, $|y_n(x) - y_0| \le b, \forall n$. It follows that when

$$x_0 \le x \le x_0 + h, \; f\{x, y_n(x)\} \le M$$

It will now be proved in a similar way that

$$|y_n(x) - y_{n-1}(x)| < \frac{Mk^{n-1}}{n!}(x - x_0)^n$$

For if we suppose it to be true when $x_0 \le x \le x_0 + h$ that

$$|y_{n-1}(x) - y_{n-2}(x)| < \frac{Mk^{n-2}}{(n-1)!}(x - x_0)^{n-1}$$

then

$$|y_n(x) - y_{n-1}(x)| \le \int_{x_0}^{x} |f\{t, y_{n-1}(t)\} - f\{t, y_{n-2}(t)\}| dt$$

$$< \int_{x_0}^{x} k|y_{n-1}(t) - y_{n-2}(t)| dt$$

[by Lipscbitz condition]

$$< \frac{Mk^{n-1}}{(n-1)!}\int_{x_0}^{x} |t - x_0|^{n-1} dt$$

$$= \frac{Mk^{n-1}}{n!}|x - x_0|^n$$

But this inequality is true tor $n = 1$ and, therefore, it is true for all n. In a similar way, the result can be proved for $x_0 - h \le x \le x_0$. Hence the result is true for $|x - x_0| \le h$.

Now the series $y_0 + \sum_{r=1}^{\infty} \{y_r(x) - y_{r-1}(x)\}$ is absolutely and uniformly convergent for $|x - x_0| \le h$ and each term is continuous in x; consequently the **limit function $y(x) = \lim_{n\to\infty} y_n(x)$ exists and is a continuous function of x in the interval** $[x_0 - h, x_0 + h]$.

Noting that

$$\lim_{n\to\infty} y_n(x) = y_0 + \lim_{n\to\infty} \int_{x_0}^{x} f\{t, y_n(t)\} dt$$

$$= y_0 + \int_{x_0}^{x} \lim_{n\to\infty} f\{t, y_n(t)\} dt$$

it follows that $y(x)$ is a solution of the integral equation

$$y(x) = y_0 + \int_{x_0}^{x} f\{t, y(t)\}dt.$$

That the inversion of the order of integration and the procedure of limit is permissible, can be proved as follows :

$$\left|\int_{x_0}^{x}[f\{t, y(t)\} - f\{t, y_{n-1}(t)\}]dt\right| < k\int_{x_0}^{x}|y(t) - y_{n-1}(t)|dt$$

$$< k\varepsilon_n|x - x_0|$$

$$< k\varepsilon_n h$$

where ε_n is independent of x and tends to zero as $n \to \infty$.

The function $f\{t, y(t)\}$ is continuous in the interval $x_0 - h \le t \le x_0 + h$ and consequently

$$\frac{dy(x)}{dx} = \frac{d}{dx}\int_{x_0}^{x} f\{t, y(t)\} = f\{x, y(x)\}$$

Hence **the limit function $y(x)$ satisfies the differential equation and reduces to y_0 when $x = x_0$.**

Next we show that the solution $y(x)$ is unique. If it is not, we suppose that $Y(x)$ be another solution different from $y(x)$, satisfying the initial condition $Y(x_0) = y_0$ and continuous in an interval $(x_0, x_0 + h')$, where $h' \le h$ and h' is such that the condition $|Y(x_0) - y_0| < b$ satisfied for this interval. Then since $Y(x)$ is a solution of the given equation, it satisfies the integral equation

$$Y(x) = y_0 + \int_{x_0}^{x} f\{t, Y(t)\}dt$$

and consequently,

$$Y(x) - y_n(x) = \int_{x_0}^{x}[f\{t, Y(t)\} - f\{t, y_{n-1}(t)\}]dt$$

For $n = 1$

$$Y(x) - y_1(x) = \int_{x_0}^{x}[f\{t, Y(t)\} - f\{t, y_0(t)\}]dt$$

so that

$$|Y(x) - y_1(x)| < \int_{x_0}^{x} |f(t, Y(t)\} - f\{t, y_0(t)\}| dt$$

$$< k\int_{x_0}^{x} |Y(t) - y_0(t)| dt \quad \text{(by Lipschitz condition)}$$

Hence

$$|Y(x) - y_1(x)| < kb(x - x_0)$$

For $n = 2$

$$|Y(x) - y_2(x)| < \left|\int_{x_0}^{x} [f\{t, Y(t)\} - f\{t, y_1(t)\}] dt\right|$$

$$< k\int_{x_0}^{x} |Y(t) - y_1(t) dt|$$

$$< k\int_{x_0}^{x} kb(t - x_0) dt$$

so that

$$|Y(x) - y_2(x)| < \frac{1}{2} k^2 b(x - x_0)^2$$

Hence, in general, we have

$$|Y(x) - y_n(x)| < \frac{k^n b(x - x_0)^n}{n!}$$

whence $Y(x) = \lim_{n \to \infty} y_n(x) = y(x)$ for all values of x in the interval $(x_0, x_0 + h')$ and, therefore, $Y(x)$ is identical with $y(x)$. Thus **there is one and only one continuous solution which satisfies the differential equation satisfying the given initial conditions.**

1.9 Existence Theorem for a System of Equations of the First-order

Let the system of equations be

$$\frac{dy_1}{dx} = f_1(x, y_1, y_2, \ldots, y_m)$$

$$\frac{dy_2}{dx} = f_2(x, y_1, y_2, \ldots, y_m)$$

$$\ldots \quad \ldots \quad \ldots$$

$$\frac{dy_m}{dx} = f_m(x, y_1, y_2, \ldots, y_m)$$

then, under conditions which will be stated, **there exists a unique set of continuous solutions of this system of equations which assume given values** $y_1^0, y_2^0, \ldots, y_m^0$ when $x = x_0$. outline of the proof will be given; the method follows exactly on the lines of the preceding section.

The functions $f_1, f_2, \ldots, f_m$ are supposed to be single-valued and continuous in their $(m+1)$ arguments when these arguments are restricted to lie in the domain D defined by

$$|x - x_0| \le a, \left|y_1 - y_1^0\right| \le b_1, \ldots, \left|y_m - y_m^0\right| \le b_m$$

Let the greatest of the upper bounds of $f_1, f_2, \ldots, f_m$ in this domain be M; if h is the least of a, $b_1 / M, \ldots, b_m / M$, let x be further restricted, if necessary, by the condition $|x - x_0| < h$.

The Lipschitz condition is

$$|f_r(x, Y_1, Y_2, \ldots, Y_m)| - f_r(x, y_1, y_2, \ldots, y_m)$$

$$| < K_1|Y_1 - y_1| + K_2|Y_2 - y_2| + \cdots + K_m|Y_m - y_m|$$

for $r = 1, 2, \ldots, m$.

We now define the functions $y_1^n(x), y_2^n(x), \ldots, y_m^n(x)$ by the relations

$$y_r^n(x) = y_r^n + \int_{x_0}^{x} f_r\left[t, y_1^{n-1}(t), y_2^{n-1}(t), \ldots, y_m^{n-1}(t)\right]dt$$

then it can be proved by induction that

$$\left|y_r^{(n)}(x) - y_r^{n-1}(x)\right| < \frac{M(K_1 + K_2 + \cdots + K_m)^{n-1}}{n!}|x - x_0|^n$$

and the existence, continuity and uniqueness of the set of solutions follow immediately.

Since the differential equation of order m

$$\frac{d^m y}{dx^m} = f\left(x, y, \frac{dy}{dx}, \cdots, \frac{d^{m-1}}{dx^{m-1}}\right)$$

is equivalent to the set of m equations of the first-order

$$\frac{dy}{dx} = y_1, \frac{dy_1}{dx} = y_2, \cdots, \frac{dy_{m-2}}{dx} = y_{m-1}, \frac{dy_{m-1}}{dx} = f(x, y, y_1, \cdots, y_{m-1})$$

it follows that if f is continuous and satisfies a Lipschitz condition in a domain D, **the equation admits of a unique continuous solution which, together with its first $m - 1$ derivatives, which are also continuous, will assume an arbitratry set of initial conditions for the initial value $x = x_0$.**

1.10 Singular Solution : Sometimes a solution of the differential equation can be found which involves no arbitrary constant and it is not a particular case of the general solution. Such a solution is known as **Singular solution.**

Let us consider the differential equation $f(x, y, p) = 0, \left(p = \frac{dy}{dx}\right)$, whose general solution is $\phi(x, y, c) = 0$, c being an arbitrary constant. The c-discriminant is obtained by eliminating c between

$$\phi(x, y, c) = 0 \text{ and } \frac{\partial\phi}{\partial c} = 0 \qquad (1.13)$$

while the p-discriminant is obtained by eliminating p between the equations

$$f(x, y, p) = 0 \text{ and } \frac{\partial f}{\partial p} = 0 \qquad (1.14)$$

Now when a continuous succession of line-elements build up an integral-curve which is singular and the corresponding solution is singular solution, these line-elements can occur only at points on the p-discriminant locus and a singular integral curve is a branch of the p-dicriminant locus.

The direction of the tangent at any point of the p-discriminant locus is obtained by differentiating the equation $f(x, y, p) = 0$ with respect to x. Thus

$$\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}\frac{dy}{dx}+\frac{\partial f}{\partial p}\frac{dp}{dx}=0$$

But at any point on the p-discriminant locus $\frac{\partial f}{\partial p}=0$ so that the direction of the tangent is given by $\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}\frac{dy}{dx}=0$. But the tangent to the p-discriminant locus coincides with the tangent to the integral-curve and, therefore a **necessary condition for the existence of a singular solution is that the three equations**

$$f(x,y,p)=0,\ \frac{\partial f}{\partial p}=0 \text{ and } \frac{\partial f}{\partial x}+p\frac{\partial f}{\partial y}=0 \tag{1.15}$$

should be satisfied simultaneously for a continuous set of values (x, y).

Conversely, we suppose that the three equations given by

$$f(x,y,\lambda)=0,\ \frac{\partial f}{\partial x}=0,\ \frac{\partial f}{\partial x}+\lambda\frac{\partial f}{\partial y}=0$$

λ being a parameter, represent a curve. Differentiating the first equation with respect x and using the second equation, we find that the direction p of the tangent at any point of the curve is given by

$$\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial y}=0$$

and, therefore, in view of the third equation, we have

$$(p-\lambda)\frac{\partial f}{\partial y}=0$$

If $f_y \neq 0$, we find that $\lambda = p$ and the curve is an integral-curve of the differential equation $f(x, y, p) = 0$.

Hence the conditions (1.15) together with the condition $\frac{\partial f}{\partial y} \neq 0$ are sufficient for the existence of a singular solution.

As an example, we find out the singular solution of the differential equation satisfied by the family of curves

$$c^2 + 2cy - x^2 + 1 = 0 \tag{1.16}$$

where c is a parameter. Differentiating both sides of (1.16) with respect to x, we get $2c\frac{dy}{dx} - \partial x = 0$ giving $c = x/p$ so that from (1.16) we have by putting the value of c

$$x^2 + 2xyp + (1 - x^2)p^2 = 0. \tag{1.17}$$

From (1.16) the c-discriminant relation is

$$4y^2 - 4(1 - x^2) = 0, \text{ i.e., } x^2 + y^2 - 1 = 0$$

From (1.17) the p-discriminant relation is

$$4x^2y^2 - 4x^2(1 - x^2) = 0, \text{ i.e., } x^2(x^2 + y^2 - 1) = 0$$

Hence the singular solution is $x^2 + y^2 - 1 = 0$, i.e., $x^2 + y^2 = 1$.

EXERCISES

1. Show that the equations (i) $4p^2 = 9x$ and (ii) $x^2 + y = p^2$ have no singular solutions.

2. Solve and find the singular solution of the following equations :

   (i) $p^4 = 4y(xp - 2y)^2$

   [Hints : Put $y = x^2$: Ans. $y = c^2(x - c^2)$; $x^4 = 16y$, $y = 0$]

(ii) $xyp^2 = xy - (x^2 - y^2 - b^2)p$

[Ans. $y^2 = x^2c - \frac{b^2c}{1+c}$; $(x^2 - y^2 - b^2)^2 + 4x^2y^2 = 0$]

(iii) $p^2y^2\cos^2\alpha - 2pxy\sin^2\alpha + y^2 - x^2\sin^2\alpha = 0$

[Ans. $x^2\tan^2\alpha - y^2 = 0$]

3. Transforming the differential equation $(px - y)(x - py) = 2p$ to Clairaut's form by the substitutions $x^2 = u$ and $y^2 = v$, solve it and find the singular solution, if any.

[Ans. $c^2x^2 - c(x^2 + y^2 - 2) + y^2 = 0$;

$(x - y + \sqrt{2})(x - y - \sqrt{2})(x + y + \sqrt{2})(x + y - \sqrt{2}) = 0$]

4. Verify whether the solutions of the first-order ODE exists or not :

$\frac{dy}{dx} = f(x,y)$, where $f(x,y) = y^{1/2}, \frac{y}{x}, \frac{x+y}{x}$,

$\frac{x+y}{x-y}, -\frac{x}{y}$ and $3y^{2/3}$

5. Find the first four successive approximations y_0, y_1, y_2, y_3 for the first-order ODE :

(i) $y' = x^2 + y^2$, $y(0) = 0$

(ii) $y' = 1 + xy$, $y(0) = 1$

(iii) $y' = y^2$, $y(0) = 1$

1.11 Summary : The main theorem discussed in this chapter is the **Cauchy-Picard Existence Theorem** for the solution of the initial value problem for the first order ODE $\frac{dy}{dx} = f(x,y)$ with the initial condition $y = y_0$ when $x = x_0$. A unique, continuous solution of the problem does exist if $f(x, y)$ is **continuous** in a domain D of the xy-plane and also if $f(x, y)$ satisfies the **Lipschitz Condition.** The existence of solutions for the first-order system in n dependent variables $y_1, y_2, ..., y_n$: $\frac{dy_i}{dx} = f_i\ (x, y_1, y_2, ..., y_n)$ $(i = 1, 2, ..., n)$ which assume the initial values $y_1^0, y_2^0, ..., y_n^0$ respectively, when $n = x_0$ can be deduced if the functions f_i are continuous and they satisfy Lipschitz conditions. Lastly, the existence of the solution $y = y(x)$ of nth order ODE : $y^{(n)} = f(x, y, y', ..., y^{(n-1)})$ with the initial conditions $y = y_0$, $y' = y_1^0, ..., y^{(n-1)} = y_{n-1}^0$ can be estabvlished in a similar way.

Unit : 2 □ General Theory of Linear Differential Equations

2.0 Introduction : The general ODE of order n which is **linear** in the dependent variable y and its derivatives is

$$L[y] \equiv p_0(x)y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_n(x)y = r(x)$$

where L is the linear differential operator. If $r(x) \equiv 0$, the equation

$$L[y] \equiv p_0(x)y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_n(x)y = 0$$

is called a **linear homogeneous** ODE.

The method of solution of linear ODE depends on our ability of finding a **fundamental set** of solutions of the corresponding homogeneous equations. A set of n linearly independent set of solutions of $L(y) = 0$ is called a fundamental set and there may be an infinite number of possible fundamental sets of solutions of the homogeneous equations. In general it is very difficult to find a fundamental set of solutions of the general linear ODE's. Only in the case of linear equations with constant coefficients and the Euler linear equations of the form

$$a_0x^n y^{(n)} + a_1x^{n-1}y^{(n-1)} + \cdots + a_n y = 0$$

$$(a_0, a_1, \ldots, a_n \text{ are all constants})$$

we can find the fundamental set very easily. As a matter of fact the Euler linear equation can be transformed into a linear ODE with constant coefficients by changing the independent variable x to z by the substitution $x = e^z$. Then we can find the fundamental set in terms of elementary functions ϕ such as polynomials, the exponential and trigonometrical functions.

The idea of integrating factor which plays a crucial role in finding solutions of linear equation of first order may be introduced in the theory of linear ODE of higher order. Let $L[u] = 0$ be the homogeneous ODE and we seek for a function $v(x)$ such that $v\,L[u]dx$ is an exact differential. Substituting for L and after some easy calculation we can find the Lagrange identity in the form

$$vL[u] - u\bar{L}[v] = \frac{d}{dx}\{P(u,v)\}$$

where $\bar{L}$ is another nth order linear differential operator (adjoint operator) and $P(u, v)$ is the **bilinear concomitant**. The equation $\bar{L}[v] = 0$ is called the **adjoint equation** corresponding with $L[u] = 0$. If v is a known solution of the adjoint equation, the equation $L[u] = 0$ reduces to a linear equation of order $(n - 1)$:

$$P(u, v) = C$$

where C is an arbitrary constant. Thus by knowing a solution v of the equation $\bar{L}[v] = 0$ we can reduce the order of the original equation by one. It may sometimes happen that it is easier to find a solution of the adjoint equation.

2.1 Liaear Differential Operator and its Properties : The most general linear differential equation is given by

$$p_0(x)\frac{d^n y}{dx^n} + p_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + p_{n-1}(x)\frac{dy}{dx} + p_n(x)y = r(x)$$

or $$L[y] \equiv \{p_0 D^n + p_1 D^{n-1} + \cdots + p_{n-1}D + p_n\}y = r(x), \left(D \equiv \frac{d}{dx}\right), \qquad (2.1)$$

in which the coefficients $p_0, p_1, ..., p_n$ and r are single-valued continuous functions of x throughout an interval $a \leq x \leq b$ and that $p_0 \neq 0$ at any point in the interval. The existence theorem of **1.8 in Chapter I** asserts the existence of a unique continuous solution $y(x)$ assumes a given value y_0 at $x = x_0$; $x_0 \in (a, b)$ and whose first $(n - 1)$ derivatives are continuous and assume the values $y_0', y_0'', \cdots, y_0^{(n-1)}$ respectively at x_0. The expression

$$L \equiv p_0 D^n + p_1 D^{n-1} + \cdots + p_{n-1}D + p_n$$

is called a **linear differential operator of order *n*.** The differential equation

$$L[u] = 0 \qquad (2.2)$$

is known as the **homogeneous equation** or the **reduced equation.**

We give below some theorems which will bring out the nature of the operator L.

Theorem 2.1 : If $u = u_1$ is a solution of the homogeneous equation (2.2), then $u = cu_1$ is also its solution, where c is an arbitrary constant.

Proof : Noting the $D^r(cu_1) = cD^r u$, we find

$$L[cu_1] = \sum_{r=0}^{n} p_r D^{n-r}(cu_1) = c\sum_{r=0}^{n} p_r D^{n-r} u_1 = cL[u_1] = 0$$

which proves the theorem.

Theorem 2.2 : If $u = u_1, u_2, \ldots, u_m$ are m solutions of (2.2), and $c_1, c_2, \ldots, c_m$ are arbitrary constants, then $u = c_1u_1 + c_2u_2 + \ldots + c_mu_m$ is also a solution of (2.2).

Proof : Noting that

$$D^r\{c_1u_1 + c_2u_2 + \cdots + c_mu_m\} = c_1D^r u_1 + c_2D^r u_2 + \cdots + c_mD^r u_m$$

the result immediately follows by Theorem 2.1.

If we can find n linearly-distinct solutions $u_1, u_2, \ldots, u_n$ of the equation (2.2), then $u = c_1u_1 + c_2u_2 + \ldots + c_nu_n$, in which the constants $c_1, c_2, \ldots, c_n$ are arbitrary is the complete primitive of (2.2). The constants $c_1, c_2, \ldots, c_n$ may be chosen in one and only one only, so that

$$u(x_0) = y_0, u'(x_0) = y_0', \ldots, u^{(n-1)}(x_0) = y_0^{(n-1)} \tag{2.3}$$

Theorem 2.3 : If $u(x)$ is the complete primitive of the homogeneous equation (2.2) and $y = y_0(x)$ be any solution of the non-homogeneous equation (2.1), then $y = u(x) + y_0(x)$ is the most general solution of (2.1).

Proof : The operator L is distributive, because the operator D^r is distributive. So, $L[u(x) + y_0(x)] = L[u(x)] + L[y_0(x)] = r(x)$ for $L[u(x)] = 0$ and $L[y_0(x)] = r(x)$. Since the solution $y = u(x) + y_0(x)$ involves n arbitrary constants, it is the general solution of (2.1).

Thus the general solution of (2.1) may be considered to consist of two parts :

(i) The complete primitive $u(x) = c_1u_1 + c_2u_2 + \dots + c_nu_n$ of the corresponding homogeneous equation and containing n arbitrary constants; this is known as the complementary function.

(ii) The particular solution $y_0(x)$, known as the particular integral, containing no arbitrary constant. For definiteness, it may be that solution of (2.1) which, together with its first $(n-1)$ derivatives, vanishes at a point x_0 in the interval (a, b).

2.2 Linear Dependence : The set of functions $f_1(x), f_2(x), \cdots, f_n(x)$ are said to be **linearly dependent** on an interval (a, b), if there exists constants $c_1, c_2, \dots, c_n$ not all zero, such that

$$c_1f_1(x) + c_2f_2(x) + \cdots + c_nf_n(x) = 0 \tag{2.4}$$

for all $x \in (a, b)$.

Functions which are not linearly dependent are called **linearly independent.** Thus the functions $f_1(x), f_2(x), \cdots, f_n(x)$ are linearly independent if the constants $c_1, c_2, \dots, c_n$ satisfying the relation (2.4) imply that

$$c_1 = c_2 = \cdots = c_n = 0$$

for all $x \in (a, b)$.

Assuming that $c_r \neq 0$, the linear dependence of the functions $f_1, f_2, \cdots, f_n$ shows from (2.4) that

$$f_r(x) = \beta_1f_1(x) + \beta_2f_2(x) + \cdots + \beta_{r-1}f_{r-1}(x) + \beta_{r+1}f_{r+1} + \cdots + \beta_nf_n \tag{2.5}$$

in which $\beta_i = -\frac{c_i}{c_r} (i = 1, 2, \cdots, n;\ i \neq r)$. Thus the function $f_r(x)$ can be expressed as a linear combination of the remaining $(n-1)$ functions of the given set.

2.3 The Wronskian : We have seen that if $u_1, u_2, \dots, u_n$ are n solutions of the homogeneous equation $L[u] = 0$ of degree n, then its most general solution is $u = c_1u_1 + c_2u_2 + \cdots + c_nu_n$. But this is possible provided the solutions $u_1, u_2 \dots, u_n$ are linearly independent. We now proceed to find the conditions for which the n

functions $u_1, u_2 ..., u_n$, supposed to be differentiable $n - 1$ times in (a, b) are linearly independent.

Let the functions $u_1, u_2 ..., u_n$ are not linearly independent. Then we can determine the constants $c_1, c_2 ..., c_n$ such that

$$c_1 u_1 + c_2 u_2 + \cdots + c_n u_n = 0$$

identically in the interval (a, b). Differentiating this relation up to $n - 1$ times in that interval, we get

$$c_1 u_1' + c_2 u_2' + \cdots + c_n u_n' = 0$$

$$c_1 u_1'' + c_2 u_2'' + \cdots + c_n u_n'' = 0$$

$$\cdots \qquad \cdots \qquad \cdots$$

$$c_1 u_1^{(n-1)} + c_2 u_2^{(n-1)} + \cdots + c_n u_n^{(n-1)} = 0$$

Thus there are n equations to determine the constants $c_1, c_2 ..., c_n$. These equations are consistent provided

$$W(u_1, u_2, \cdots, u_n) \equiv \begin{vmatrix} u_1 & u_2 & \cdots & u_n \\ u_1' & u_2' & \cdots & u_n' \\ \vdots & \vdots & & \vdots \\ u_1^{(n-1)} & u_2^{(n-1)} & \cdots & u_n^{(n-1)} \end{vmatrix} = 0 \qquad (2.6)$$

The determinant W is known as the **Wronskian** of the functions $u_1, u_2 ..., u_n$. **Thus a necessary condition that the functions $u_1, u_2 ..., u_n$ are linearly dependent is that $W \equiv 0$. Hence the non-vanishing of the Wronskian of the functions $u_1, u_2 ..., u_n$ is sufficient for their linear independence.**

Now we suppose that the solutions $u_1(x), u_2(x), ..., u_k(x)$ $(k < n)$ and their $k - 1$ derivatives are finite in the interval $a \le x \le b$ and their Wronskian vanishes identically in (a, b). **Then we can write**

$$u_k(x) = c_1 u_1(x) + c_2 u_2(x) + \cdots + c_{k-1} u_{k-1}(x) \qquad (2.7)$$

where $c_1, c_2, ..., c_{k-1}$ are constants, provided that the Wronskian of $u_1(x)$, $u_2(x), ..., u_{k-1}(x)$ does not vanish identically, i.e., the solutions are linearly dependent.

To prove this, we suppose that $U_1, U_2, \ldots, U_k$ denote the minors of the elements in the last line of the Wronskian

$$\begin{vmatrix} u_1 & u_2 & \cdots & u_k \\ u_1' & u_2' & \cdots & u_k' \\ \vdots & \vdots & & \vdots \\ u_1^{(k-1)} & u_2^{(k-1)} & \cdots & u_n^{(k-1)} \end{vmatrix}$$

then we shall have k identities of the form

$$U_1 u_1^{(r)} + U_2 u_2^{(r)} + \cdots + U_k u_k^{(r)} = 0 (r = 0, 1, \cdots, k-1)$$

Differentiating each of first $(k-1)$ identities and then subtracting the identity from the result, we obtain

$$U_1' u_1^{(r)} + U_2' u_2^{(r)} + \cdots + U_k' u_k^{(r)} = 0 (r = 0, 1, \cdots, k-2)$$

Multiplying the rth of these $(k-1)$ identities by the cofactor of $u_1^{(r-1)}$ in the determinant U_k and adding the products, one obtains

$$U_1' U_k - U_k' U_1 = 0$$

and since U_k is not identically zero in (a, b) we have $U_1 = -c_1 U_k$ Similarly, we can show that $U_2 = -c_2 U_k, \ldots, U_{k-1} = -c_k U_k$. From the identity $U_1 u_1 + U_2 u_2 + \ldots + U_k u_k = 0$, it, therefore, follows that

$$U_k \{-c_1 u_1 - c_2 u_2 - \cdots - c_k u_{k-1} + u_k\} = 0$$

from which we have $u_k = c_1 u_1 + c_2 u_2 + \cdots + c_{k-1} u_{k-1}$, i.e., the result (2.7) is proved.

Thus we conclude that

I. If the Wronskian of the solutions $u_1, u_2, \ldots, u_n$ vanishes at any point of (a, b), these n solutions are linearly dependent.

II. If the wronskian of the k solutions $u_1, u_2, \ldots, u_n$ $(k < n)$ vanishes identically in (a, b), then these k solutions are linearly dependent.

Now it follows from (2.6) by differentiation with respect to x that

$$\frac{dW}{dx} = \begin{vmatrix} u_1 & u_2 & \cdots & u_n \\ u_1' & u_2' & \cdots & u_n' \\ \vdots & \vdots & & \vdots \\ u_1^{(n-2)} & u_2^{(n-2)} & \cdots & u_n^{(n-2)} \\ u_1^{(n)} & u_2^{(n)} & \cdots & u_n^{(n)} \end{vmatrix}$$

for all other determinants arising from differentiation have two rows alike and therefore, vanish. Since u_r is a solution of the homogeneous equation $L[u] = 0$, it folows that

$$p_0 u_r^{(n)} = -p_1 u_r^{(n-1)} - \cdots - p_{n-1} u_r' - p_n u_r$$

and we get after some elementary calculation that

$$\frac{dW}{dx} = -\frac{p_1}{p_0} W$$

or
$$W = W_0 \exp\left\{-\int_{x_0}^{x} \frac{p_1}{p_0} dx\right\} \tag{2.8}$$

where W_0 is the value of W at $x = x_0$. The relation (2.8) is known as **Abel identity.**

If $p_0(x) \neq 0$ in (a, b), then if W vanishes at x_0, $W \equiv 0$. If $W_0 \neq 0$, then W cannot be zero except at a singular point, i.e., $\frac{p_1}{p_0}$ becomes infinite at the point.

2.4 Fundamental Sets of Solutions : A linearly independent set of n solutions $u_1, u_2, \ldots, u_n$ of the homogeneous equation $L[u] = 0$ is said to form a **fundamental set** or **fundamental system.** The **condition** that these n solutions should be a fundamental set is that the **Wronskian of these solutions is not zero.**

The general solution $u = c_1u_1 + c_2u_2 + \ldots + c_nu_n$ of the equation $L[u] = 0$ cannot vanish identically unless the constant $c_1, c_2, \ldots, c_n$ are all zero. There may be an infinite number of possible fundamental sets of solutions of which one particular set is of importance due to its simplicity.

We choose the function $u_1(x)$ such that

$$u_1(x_0) = 1 \text{ and } u_1'(x_0) = u_1''(x_0) = \cdots = u_1^{(n-1)}(x_0) = 0$$

and define the functions $u_r(x)$ ($r = 2, 3, ..., n$) as that particular solution satisfying the initial conditions

$$u(x_0) = u'(x_0) = \cdots = u^{(r-2)}(x_0) = 0,\ u^{(r-1)}(x_0) = 1,$$

$$u^{(r)}(x_0) = u^{(r+1)}(x_0) = \cdots = u^{(n-1)}(x_0) = 0,$$

then the set $u_1, u_2, ..., u_n$ form a fundamental set and its Wronskian for $x = x_0$ is unity.

2.5 Depression of the Order of an Equation : If r independent solutions of the homogeneous equation $L[u] = 0$ of order n are known, then the order of the equation may be reduced to $n - r$. To prove this, we suppose that the r solutions $u_1, u_2, ..., u_r$ are known. We put $u = u_1 z$ where z is the new independent variable and the equation $L[u] = 0$ is reduced to a new equation of the same type in z. This new equation in z must have $z = 1$ for a solution which requires that the coefficient of z shall be zero, since the coefficient of z is precisely $L[u_1]$. The equation in z is therefore of the form

$$u_1 \frac{d^n z}{dx^n} + b_1 \frac{d^{n-1} z}{dx^{n-1}} + \cdots + b_{n-1} \frac{dz}{dx} = 0 \tag{2.9}$$

where $b_1, b_2, ..., b_{n-1}$ are functions of z. Putting $v = dz / dx$, this equation reduces to

$$u_1 \frac{d^{n-1} v}{dx^{n-1}} + b_1 \frac{d^{n-2} v}{dx^{n-2}} + \cdots + b_{n-1} v = 0 \tag{2.10}$$

which is a linear homogeneous equation of order $n - 1$. Since $u_1, u_2, ..., u_r$ are r solutions of $L[u] = 0$, the equation (2.9) has $r - 1$ solutions of the form $u_2 / u_1, ..., u_r / u_1$ and therefore, the solutions of (2.10) are

$$\frac{d}{dx}\left(\frac{u_2}{u_1}\right), \frac{d}{dx}\left(\frac{u_3}{u_1}\right), \ldots, \frac{d}{dx}\left(\frac{u_r}{u_1}\right)$$

which are linearly independent; otherwise a relation exists in the form

$$c_2 \frac{d}{dx}\left(\frac{u_2}{u_1}\right) + c_3 \frac{d}{dx}\left(\frac{u_3}{u_1}\right) + \cdots + c_r \frac{d}{dx}\left(\frac{u_r}{u_1}\right) = 0$$

where $c_2, c_3, \ldots, c_r$ are constants not all zero so that by integration, a relation $c_2u_2 + \ldots + c_ru_r + c_1u_1 = 0$ exists where c_1 is a new constant. If $r > 1$, we may obtain similarly a new linear equation of order $n - 2$, and so on.

Hence, if r independent particular integrals of a linear homogeneous equation are known, then its integration reduces to that of a linear homogeneous equation of order $n - r$. When $r = n - 1$, the last equation will be integrable by a quadrature.

As an example, consider the second-order linear equation

$$\frac{d^2u}{dx^2} + p(x)\frac{du}{dx} + 2(x)u = 0 \tag{2.11}$$

and let u_1 be a particular integral of this equation. If we put $u = u_1z$, then this equation reduces to

$$u_1 \frac{d^2z}{dx^2} + \left(2\frac{du_1}{dx} + pu_1\right)\frac{dz}{dx} = 0.$$

Setting $\frac{dz}{dx} = u$, this equation can be written as

$$\frac{dv}{v} + \left(2\frac{du_1}{dx} + pu_1\right)\frac{dx}{u_1} = 0$$

which, on integration, leads to

$$\log v + \int_{x_0}^{x} p\,dx + \log u_1^2 = \log c$$

$$\text{i.e., } v = \frac{c}{u_1^2} e^{-\int_{x_0}^{x} p\,dx}$$

A second quadrature gives z and consequently u. Thus the equation (2.11) has the integral u_2 given by

$$u_2 = u_1 \int_{x_0}^{x} \frac{dx}{u_1^2} e^{-\int_{x_0}^{x} p\,dx}$$

which is independent of u_1.

2.6 Solution of the Non-homogeneous Equation : We now proceed to find the solution of the general non-homogeneous equation

$$L[y] = r(x) \tag{2.12}$$

in which we suppose that a fundamental set of solutions $u_1(x), u_2(x), ..., u_n(x)$ of the reduced equation $L[u] = 0$ are known. The general solution of the reduced equation is

$$u = c_1u_1 + c_2u_2 + ... + c_nu_n$$

where $c_1, c_2, ..., c_n$ are constants.

To determine the general solution of the equation (2.12), we use the **method of variation of parameter**. Let

$$u = V_1u_1 + V_2u_2 + ... + V_nu_n$$

in which $V_1, V_2, ..., V_n$ are undetermined functions of x, satisfy the equation (2.12). Noting that the differential equation itself is equivalent to a single relation between the functions V and $r(x)$, $n-1$ other relations may be set up if these relations are consistent with one another. We choose the set of $n - 1$ relations as

$$\begin{aligned} &V_1'u_1 + V_2'u_2 + \cdots + V_n'u_n = 0, \\ &V_1'u_1' + V_2'u_2' + \cdots + V_n'u_n' = 0, \\ &\quad\cdots \qquad \cdots \qquad \cdots \\ &V_1'u_1^{(n-2)} + V_2'u_2^{(n-2)} + \cdots + V_n'u_n^{(n-2)} = 0. \end{aligned} \tag{2.13}$$

As a consequence of these relations, we have

$$y' = V_1u_1' + V_2u_2' + \cdots + V_nu_n'$$

$$y'' = V_1u_1'' + V_2u_2'' + \cdots + V_nu_n''$$

$$\cdots \quad \cdots \quad \cdots$$

$$y^{(n-1)} = V_1u_1^{(n-1)} + V_2u_2^{(n-1)} + \cdots + V_nu_n^{(n-1)} \quad \text{and}$$

$$y^{(n)} = V_1u_1^{(n)} + V_2u_2^{(n)} + \cdots + V_nu_n^{(n)} + V_1'u_1^{(n-1)} + V_2'u_2^{(n-1)} + \cdots + V_n'u_n^{(n-1)}$$

Hence the expression $y = V_1u_1 + V_2u_2 + \cdots V_nu_n$ satisfies the differential equation (2.12) in which the coefficient of $y^{(n)}$ is supposed to be unity, provided that

$$V_1'u_1^{(n-1)} + V_2'u_2^{(n-1)} + \cdots + V_n'u_n^{(n-1)} = r(x) \tag{2.14}$$

Since the solutions $u_1, u_2, \ldots, u_n$ form a fundamental set, the n equations in (2.13) and (2.14) are sufficient for the determination of $V_1', V_2', \cdots, V_n'$ uniquely in terms of $u_1, u_2, \ldots, u_n$. $V_1, V_2, \ldots, V_n$ can then be obtained by quadrature

In particular, for a second-order equation

$$V_1 = -\int \frac{u_2(x)r(x)}{W(u_1, u_2)}dx \text{ and } V_2 = -\int \frac{u_1(x)r(x)}{W(u_1, u_2)}dx$$

$W(u_1, u_2)$ being Wronskian of u_1 and u_2.

2.7 The Adjoint Equation : The idea of integrating factor arising in the theory of first-order linear equations can be extended to the theory of higher order linear equations. Let

$$L[u] = (p_0D^n + p_1D^{n-1} + \cdots + p_{n-1}D + p_n)u$$

and suppose that there exists a function $v(x)$ such that $vL[u]dx$ is a perfect differential. Then the result

$$U^{(r)}V = \frac{d}{dx}\{U^{(r-1)}V - U^{(r-2)}V' + \cdots + (-1)^{r-1}UV^{(r-1)}\} + (-1)^r UV^{(r)}$$

gives

$$vL[u] = \frac{d}{dx}\{u^{(n-1)}(p_0v) - u^{(n-2)}(p_0v)' + \cdots + (-1)^{n-1}u(p_0v)^{(n-1)}\}$$

$$+\frac{d}{dx}\{u^{(n-2)}(p_1v) - u^{(n-2)}(p_1v)' + \cdots + (-1)^{n-2}u(p_1v)^{(n-2)}\}$$

$$+\cdots+\frac{d}{dx}\{u'(p_{n-2}v) - u(p_{n-2}v)'\} + \frac{d}{dx}(up_{n-1}v) + u\overline{L}[v] \quad (2.14)$$

where

$$\overline{L}[v] = (-1)^n(p_0v)^{(n)} + (-1)^{n-1}(p_1v)^{(n-1)} + \cdots + (p_{n-1}v)' + p_nv \quad (2.15)$$

The differential expression **is said to be adjoint to $L[u]$** and the equation

$$\overline{L}[v] = 0 \quad (2.16)$$

is called an **adjoint equation** corresponding to $L[u] = 0$.

Now the relation (2.14) may be expressed in the form

$$vL[u] - u\overline{L}[v] = \frac{d}{dx}\{P(u,v)\} \quad (2.17)$$

The relation (2.17) is known as **Lagrange identity**. The expression $P(u, v)$ is linear and homogeneous in $u, u', \cdots, u^{(n-1)}$ and $v, v', \cdots, v^{(n-1)}$ and is known as **bilinear contocomitant.**

In order that v may be an integrating factor for $L[u]$, the necessary and sufficient condition is that v should satisfy the adjoint equation $\overline{L}[v] = 0$.

Now we show that **the relation between $L[u]$ and $\overline{L}[v]$ is reciprocal to each other,** or in other words, **if $\overline{L}[v]$ is adjoint to $L[u]$, then $L[u]$ is adjoint to $\overline{L}[v]$.** For, if not, let $L_1[u]$ ($\neq L[u]$) be adjoint to $\overline{L}[v]$. Then there exists a function $P_1(u, v)$ such that

$$vL_1[u] - u\overline{L}[v] = \frac{d}{dx}\{P_1(u,v)\}$$

But

$$vL[u] - u\overline{L}[v] = \frac{d}{dx}\{P(u,v)\}$$

Hence, $$v\{L_1[u] - L[u]\} = \frac{d}{dx}\{P_1(u,v) = P(u,v)\}$$

But $P_1(u,v) - P(u,v)$ is linear and homogeneous in $v, v', \cdots, v^{(n-1)}$ and, therefore, $v\{L_1[u] - L[u]\}$ does not involve $v^{(n)}$ so that the coefficient of $v^{(n-1)}$ in $P_1(u,v) - P(u,v)$ is zero. Repeting the argument, we prove that $P_1(u,v) - P(u,v) = 0$ and, therefore $L_1[u] = L[u]$.

An equation is said to be **self-adjoint** if it is identical with its adjoint.

EXAMPLES

1. Show that the solutions e^x, e^{-x} and e^{2x} of

$$\frac{d^3y}{dx^3} - 2\frac{d^2y}{dx^2} - \frac{dy}{dx} + 2y = 0$$

are linearly independent on every real line.

Solution : The Wronskian of the solutions is

$$W(e^x, e^{-x}, e^{2x}) = \begin{vmatrix} e^x & e^{-x} & e^{2x} \\ e^x & -e^{-x} & 2e^{2x} \\ e^x & e^{-x} & 4e^{2x} \end{vmatrix} = e^{2x}\begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 1 & 1 & 4 \end{vmatrix} = -6e^{2x} \neq 0$$

Hence, the given solutions are linearly independent for all real x.

2. Given that $y = x$ is a solution of

$$(x^2+1)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 0$$

Find a linearly independent solution by reducing the order.

Solution : Let $y = zx$. Then the given equation is reduced to

$$(x^2+1)\left(x\frac{d^2z}{dx^2} + 2\frac{dz}{dx}\right) - 2x\left(x\frac{dz}{dx}\right) + 2zx = 0$$

$$\text{Or,}\quad x(x^2+1)\frac{d^2z}{dx^2} + 2\frac{dz}{dx} = 0$$

Putting $\dfrac{dz}{dx} = v$, we have a first-order linear homogeneous equation

$$x(x^2+1)\frac{dy}{dx} + 2v = 0 \Rightarrow \frac{dv}{v} = \left(\frac{2x}{x^2+1} - \frac{2}{x}\right)dx$$

the integration of which leads to $v = \dfrac{c(x^2+1)}{x}$.

Choosing $c = 1$, we have $\dfrac{dz}{dx} = v = \dfrac{x^2+1}{x}$. Integrating again, we get $v = x - \dfrac{1}{x}$.

Hence $y = x^2 - 1$ is the desired linearly independent solution.

3. Find the general solution of

$$(x^2+1)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 6(x^2+1)^2$$

given that $y = x$ and $y = x^2 - 1$ are linearly independent solutions of the corresponding homogeneous equation.

Solution : The complementary function of the given equation is

$$u_c(x) = c_1 x + c_2(x^2 - 1)$$

c_1, c_2 being constants. To find the general solution, we let the particular integral

$$u(x) = V_1(x)x + V_2(x)(x^2 - 1)$$

so that

$$u'(x) = V_1 + 2V_2 x + xV_1' + (x^2 - 1)V_2'$$

Imposing the condition $xV_1' + (x^2 - 1)V_2' = 0$ (A)

we

$$u'(x) = V_1 + 2V_2 x, \quad \therefore \quad u''(x) = V_1' + 2V_2' x + 2V_2$$

Substituting these in the given equation we obtain

$$(x^2+1)[V_1' + 2V_2'x + 2V_2] - 2x[V_1 + 2xV_2]$$
$$+2[xV_1 + (x^2-1)V_2] = 6(x^2+1)^2$$

or $$V_1' + 2V_2'x = 6(x^2+1) \qquad \text{(B)}$$

Solving for V_1' and V_2' from (A) and (B) we have

$$V_1' = -6(x^2-1),\ V_2' = 6x$$

so that $V_1 = -2x^3 + 6x$, $V_2 = 3x^2$ where the integration constants are chosen to be zero. Hence, the general solution of the given equation is

$$y = u_c(x) + u(x)$$
$$= c_1x + c_2(x^2-1) + (-2x^3+6x)x + 3x^2(x^2-1)$$

i.e., $$y = c_1x + c_2(x^2-1) + x^4 + 3x^2$$

4. Show that if $u(x)$ and $v(x)$ are solutions of the self-adjoint differential equation $(pu')' + q(x)u = 0$, then $p(x)(uv' - u'v)$ is constant.

Solution : Let $L[u] = (pu')' + qu = 0$, i.e., $L[u] = pu'' + p'u' + qu = 0$. Its adjoint equation is

$$\bar{L}[v] = pv'' + (2p' - p')v' + (p'' - p'' + q)v = 0$$

i.e., $$\bar{L}[v] = pv'' + p'v' + qv = 0$$

But the equation $L(u)$ is adjoint and $v(x)$ is also its solution.

Hence the L.H.S. of the Lagrange identity (2.17) vanishes, and, therefore $p(u'v - uv') =$ constant.

EXERCISE

1. Show that x^2 and $\frac{1}{x^2}$ are linearly independent solutions of the equation

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - 4y = 0 \text{ on the interval } 0 < x < \infty.$$

2. Consider the different equation

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 4y = 0$$

(a) Show that each of the functions e^x, e^{4x} and $2e^x - 3e^{4x}$ is a solution of the equation on the interval $-\infty < x < \infty$.

(b) Show that the solutions e^x and e^{4x} are linearly independent on $-\infty < x < \infty$.

(c) Show that the solutions e^x and $2e^x - 3e^{4x}$ are linearly independent on $-\infty < x < \infty$.

(d) Are the solutions e^{4x} and $2e^x - 3e^{4x}$ still another pair of linearly independent solutions on $-\infty < x < \infty$? Justify your answer.

3. Given that $y = x$ is a solution of

$$(x^2 - 1)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 0$$

Find a linearly independent solution by reducing the order and write the general solution.

[Ans. : $y = c_1x + c_2(1 + x^2)$]

4. Given that $y = e^{2x}$ is a solution of

$$(2x + 1)\frac{d^2y}{dx^2} - 4(x + 1)\frac{dy}{dx} + 4y = 0$$

Find a linearly independent solution by reducing the order and write the general solution.

[Ans. : $y = c_1e^{2x} + c_2(x + 1)$]

5. Find the general solution of

$$x^2\frac{d^2y}{dx^2} - 6x\frac{dy}{dx} + 10y = 3x^4 + 6x^3$$

given that $y = x^2$ and $y = x^5$ are linearly independent solutions of the corresponding homogeneous equation.

[Ans. : $y = c_1x^2 + c_2x^5 - 3x^3 - \frac{3}{2}x^4$]

6. Find the general solution of

$$(x^2 + 2x)\frac{d^2y}{dx^2} - 2(x+1)\frac{dy}{dx} + 2y = (x+2)^2$$

given that $y = x + 1$ and $y = x^2$ are linearly independent solutions of the corresponding homogeneous equation.

[Ans. : $y = c_1(x+1) + c_2x^2 - x^2 - 2x + x^2 \ln|x|$]

7. Find the general solution of

$$\sin^2 x \frac{d^2y}{dx^2} - 2\sin x.\cos x \frac{dy}{dx} + (1 + \cos^2 x)y = \sin^3 x$$

given that $y = \sin x$ and $y = x \sin x$ are linearly independent solutions of the corresponding homogeneous equation.

[Ans. : $y = c_1 \sin x + c_2 x \sin x + \frac{1}{2}x^2 \sin x$]

2.8. Summary :

The solution $y = y(x)$ of an nth order linear ODE

$$L(y) \equiv P_0(x)y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_n(x)y = r(x)$$

which assumes the value $y = y_0$ at a point $x = x_0$ and whose first $(n - 1)$ derivatives are continuous and assume the prescribed values $y_0', y_0'', \cdots, y_0^{(n-1)}$ at $n = x_0$ consists of two parts : the **complementary function** which is the general solution of the homogeneous (reduced) equation $L(y) = 0$ containing n arbitrary constants and the **particular integral** which is any particular solution of the non-homogeneous (complete) equation $L(y) = r(x)$. The complementary function can be obtained if we can find out n linearly independent solutions $u_1(x), u_2(x), \cdots, u_n(x)$ of the homogeneous equation.

The Wronskian of the n solutions $W(u_1, u_2, ..., u_n)$ plays a crucial role in determining whether the set of solutions is linearly independent or not. The vanishing of the Wronskian of the n solutions implies the linear dependence of the solutions and its non-vanishing is sufficient for linear independence of $u_1, u_2, ..., u_n$. We have also another important result in connection with the n independent solutions of $L[u] = 0$

$$W = W_0 \exp\left\{-\int_x^{x_0} \frac{p_1}{p_0}\, dx\right\}$$

which is known as **Abel's identity**.

The **Adjoint equation** $\bar{L}[v] = 0$ corresponding with the equation $L[u] = 0$ is also introduced which plays some role in solving the equation.

Unit : 3 □ System of Linear Differential Equations

2.0 Introduction : The system of linear differential equations in n unknown functions $x_1(t), x_2(t), \cdots, x_n(t)$ of the form

$$\frac{dx}{dt} = A(t)x + F(t)$$

where $A(t)$ is a $n \times n$ matrix, x and $F(t)$ are column vectors, t is the independent variable can be solved just like nth order ODE if we can find out a **fundamental matrix** of the homogeneous system

$$\frac{dx}{dt} = A(t)x$$

The fundamental matrix is the $n \times n$ matrix whose elements are $\vec{\phi}_1(t), \vec{\phi}_2(t) \cdots \vec{\phi}_n(t)$ and these functions are n linearly independent solutions of the same. In general, the problem of construction of the fundamental matrix is a very difficult one when the matrix A is a function of t.

However, if A is a constant matrix, the fundamental matrix can be constructed more easily. In this case the solutions $\vec{\phi}_1(t), \vec{\phi}_2(t) \cdots \vec{\phi}_n(t)$ are of the form $\vec{\phi} = re^{\lambda t}$

3.1 Basic Theory of Linear Systems in Normal Form : The normal form of a linear system of n differential equations in the generalised case with n unknowns x_1, x_2, ..., x_n is given by

$$\begin{aligned}
\frac{dx_1}{dt} &= a_{11}(t)x_1 + a_{12}(t)x_2 + \cdots + a_{1n}(t)x_n + F_1(t), \\
\frac{dx_2}{dt} &= a_{21}(t)x_1 + a_{22}(t)x_2 + \cdots + a_{2n}(t)x_n + F_2(t), \qquad (3.1) \\
\cdots & \qquad\qquad \cdots \qquad\qquad \cdots \\
\frac{dx_n}{dt} &= a_{n1}(t)x_1 + a_{n2}(t)x_2 + \cdots + an_{2n}(t)x_n + F_n(t)
\end{aligned}$$

or, more compactly,

$$\frac{dx_i}{dt} = \sum_{j=1}^{n} a_{ij}(t)x_j + F_i(t), \ (i = 1, 2, \cdots, n) \qquad (3.2)$$

Now we proceed to find a relationship of the normal linear system (3.1) or (3.2) with a single nth order linear differential equation in one unknown.

Consider the normalized (i.e., the coefficient of the higher order derivative is one) nth order linear differential equation

$$\frac{d^n x}{dt^n} + a_1(t)\frac{d^{n-1}x}{dt^{n-1}} + \cdots + a_{n-1}(t)\frac{dx}{dt} + a_n(t)x = F(t). \tag{3.3}$$

Let

$$x_1 = x, x_2 = \frac{dx}{dt}, x_3 = \frac{d^2x}{dt^2}, \cdots, x_{n-1} = \frac{d^{n-2}x}{dt^{n-2}}, x_n = \frac{d^{n-1}x}{dt^{n-1}}. \tag{3.4}$$

Then we have

$$\frac{dx_1}{dt} = x_2, \frac{dx_2}{dt} = x_3, \cdots, \frac{dx_{n-1}}{dt} = x_n, \frac{dx_n}{dt} = -a_n(t)x_1 - a_{n-1}(t)x_2 \cdots$$

$$-a_1(t)x_n + F(t) \tag{3.5}$$

which is a special case of the normal linear system (3.1) or (3.2) of n equations in n unknowns. Hence, a single nth order linear differential equation in one unknown function is intimately related to n first order normal linear system of differential equations in n unknown functions.

We now assume that all the functions $a_{ij}(t)$ ($i = 1, 2, ..., n$; $j = 1, 2, ..., n$) and $F_i(t)$ ($i = 1, 2, ..., n$) in (4.2) are continuous on a real interval $a \leq t \leq b$. If all $F_i(t) = 0$ ($i = 1, 2, ..., n$) for all t, then the system (3.2) said to be homogeneous, Otherwise, it is called **non-homogeneous**.

The system (3.2) can be expressed in a vector form by using vectors and matrices as following :

Let the matrix A be defined by

$$A(t) = \begin{pmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t) \\ \vdots & \vdots & & \vdots \\ a_{n1}(t) & a_{n2}(t) & \cdots & a_{nn}(t) \end{pmatrix} \tag{3.6}$$

and the vectors F and x by

$$F(t) = \begin{pmatrix} F_1(t) \\ F_2(t) \\ \vdots \\ F_n(t) \end{pmatrix} \text{ and } x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \tag{3.7}$$

Then the system (3.1) or (3.2) can be expressed as the linear vector differential equation

$$\frac{dx}{dt} = A(t)x + F(t) \tag{3.8}$$

The equation (3.8) is referred to as the **vector differential equation corresponding to the system** (3.1) and **the system (3.1) is the scalar form of the vector differential equation (3.8).**

Example 3.1 : The system

$$\begin{aligned} \frac{dx_1}{dt} &= 7x_1 - x_2 + 6x_3, \\ \frac{dx_2}{dt} &= -10x_1 + 4x_2 - 12x_3, \\ \frac{dx_3}{dt} &= -2x_1 + x_2 - x_3 \end{aligned} \tag{3.9}$$

is a homogeneous linear system with $n = 3$ having constant coefficients, while the system

$$\frac{dx_1}{dt} = 7x_1 - x_2 + 6x_3 - 5t - 6$$

$$\frac{dx_2}{dt} = -10x_1 + 4x_2 - 12x_3 - 4t + 23 \tag{3.10}$$

$$\frac{dx_3}{dt} = -2x_1 + x_2 - x_3 + 2$$

is a non-homogeneous system with $n = 3$, the non-homogeneous terms being $-5t - 6, -4t + 23$ and 2 respectively.

The vector differential equation corresponding to the non-homogeneous system is

$$\frac{dx}{dt} = A(t)x + F(t) \tag{3.11}$$

where,

$$A(t) = \begin{pmatrix} 7 & -1 & 6 \\ -10 & 4 & -12 \\ -2 & 1 & -3 \end{pmatrix}, \ x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \text{ and } F(t) = \begin{pmatrix} -5t - 6 \\ -4t + 23 \\ 2 \end{pmatrix} \tag{3.12}$$

Definition 3.1 : The solution of the vector differential equation (3.8) is given by a $n \times 1$ column vector function of the form

$$\vec{\phi} = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_n \end{pmatrix} \tag{3.13}$$

in which each of the components $\phi_1, \phi_2, ..., \phi_n$ has a continuous derivative on the real interval $a \le t \le b$ such that

$$\frac{d\vec{\phi}(t)}{dt} = A(t)\vec{\phi}(t) + F(t), \ \forall t \text{ in } a \le t \le b \tag{3.14}$$

It also follows that the components $\phi_1, \phi_2, ..., \phi_n$ of $\vec{\phi}$ are such that

$$x_1 = \phi_1(t), \ x_2 = \phi_2(t), \cdots, x_n = \phi_n(t)$$

simultaneously satisfy all the n equations (3.1) for $a \le t \le b$.

Example 3.2 : As an example, consider the homogeneous system (3.9) in vector form, i.e., $\frac{dx}{dt} = \begin{pmatrix} 7 & -1 & 6 \\ -10 & 4 & -12 \\ -2 & 1 & -1 \end{pmatrix} x$, where $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$. It follows that the column vector function $\vec{\phi}$ defined by $\vec{\phi} = \begin{pmatrix} e^{3t} \\ -2e^{3t} \\ -e^{3t} \end{pmatrix}$ is a solution of this equation on $a \le t \le b$; for $x = \vec{\phi}(t)$ satisfies the equation identically, i.e.,

$$\begin{pmatrix} 3e^{3t} \\ -2e^{3t} \\ -e^{3t} \end{pmatrix} = \begin{pmatrix} 7 & -1 & 6 \\ -10 & 4 & -12 \\ -2 & 1 & -1 \end{pmatrix} \begin{pmatrix} e^{3t} \\ -2e^{3t} \\ -e^{3t} \end{pmatrix}$$

Hence $x_1 = e^{3t}$, $x_2 = -2e^{3t}$, $x_3 = -e^{3t}$ simultaneously satisfy all the three equations of the given system for $a \le t \le b$ and, therefore, these are a solution of the system.

We now state the existence and uniqueness theorem for the vector differential equation (3.8).

Theorem 3.1 : Consider the vector differential equation

$$\frac{dx}{dt} = A(t)x + F(t) \tag{3.8}$$

in which the components of the $a_{ij}(t)$ ($i = 1, 2, ..., n$; $j = 1, 2, ..., n$) of the matrix $A(t)$ and the components $F_i(t)$ ($i = 1, 2, ..., n$) of the vector $F(t)$ are continuous on $a \le t \le b$. Let t_0 be any point of the interval $a \le t \le b$ and

$$C = \begin{pmatrix} C_1 \\ C_2 \\ \vdots \\ C_n \end{pmatrix}$$

be a $n \times 1$ column vector of n numbers $C_1, C_2, ..., C_n$. Then there exists a unique solution

$$\vec{\phi} = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_n \end{pmatrix}$$

of the vector differential equation (3.8) such that

$$\vec{\phi}(t_0) = C, \ \forall t \text{ in } a \le t \le b \qquad (3.15)$$

and this solution is unique in the interval.

3.2. Homogeneous Linear Systems : Let all $F_i(t) = 0$ $(i = 1, 2, ..., n)$ for all t in the linearised system (3.1) and we consider the homogeneous linear system

$$\frac{dx_1}{dt} = a_{11}(t)x_1 + a_{12}(t)x_2 + \cdots + a_{1n}(t)x_n,$$

$$\frac{dx_2}{dt} = a_{21}(t)x_1 + a_{22}(t)x_2 + \cdots + a_{2n}(t)x_n, \qquad (3.16)$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots$$

$$\frac{dx_n}{dt} = a_{n1}(t)x_1 + a_{n2}(t)x_2 + \cdots + a_{nn}(t)x_n$$

or the corresponding homogeneous vector equation

$$\frac{dx}{dt} = A(t)x \qquad (3.17)$$

Let t_0 be any point of $a \le t \le b$ and $\vec{\phi} = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_n \end{pmatrix}$ be a solution of (3.17) such that $\vec{\phi}(t_0) = 0$. Then $\vec{\phi}(t) = 0$ for all t on $a \le t \le b$. For, these conditions are of the same form (3.15) with $C = 0$, and by Theorem 3.1, there is a unique solution of the differential equation satisfying such a set of conditions. Thus $\vec{\phi}(t) = 0$ for all t on $a \le t \le b$ is the only solution of (3.17) such that $\vec{\phi}(t_0) = 0$.

Theorem 3.2 : If the vector functions $\vec{\phi}_1, \vec{\phi}_2, \cdots, \vec{\phi}_m$ are m solutions of (3.17), and $C_1, C_2, ..., C_m$ are m numbers, then a linear combination of these m solutions, i.e., the vector function $\vec{\phi} = \sum_{k=1}^{m} C_k \vec{\phi}_k$ is also a solution of (3.17).

Proof : Since each $\vec{\phi}_k$ is a solution of (3.17), i.e.,

$$\frac{d\vec{\phi}_k(t)}{dt} = A(t)\vec{\phi}_k(t) \quad \text{for } k = 1, 2, ..., m$$

we have

$$\frac{d}{dt}\left[\sum_{k=1}^{m} C_k \vec{\phi}_k(t)\right] = \sum_{k=1}^{m}\left[\frac{d}{dt} C_k \vec{\phi}_k(t)\right] = \sum_{k=1}^{m} C_k \left[\frac{d}{dt}\vec{\phi}_k(t)\right]$$

$$= \sum_{k=1}^{m} C_k A(t)\vec{\phi}_k(t) = \sum_{k=1}^{m} A(t)[C_k \vec{\phi}_k(t)]$$

$$= A(t)\sum_{k=1}^{m} C_k \vec{\phi}_k(t)$$

i.e., $\frac{d}{dt}\vec{\phi}(t) = A(t)\vec{\phi}(t)$, $\forall t$ on $a \le t \le b$. Hence the linear combination $\vec{\phi} = \sum_{k=1}^{m} C_k \vec{\phi}_k$ is a solution of (3.17).

Definition 3.2 : Let $\vec{\phi}_1, \vec{\phi}_2, \cdots, \vec{\phi}_n$ be n vector functions of t defined by

$$\vec{\phi}_1(t) = \begin{pmatrix} \phi_{11}(t) \\ \phi_{21}(t) \\ \vdots \\ \phi_{n1}(t) \end{pmatrix}, \vec{\phi}_2(t) = \begin{pmatrix} \phi_{12}(t) \\ \phi_{22}(t) \\ \vdots \\ \phi_{n2}(t) \end{pmatrix}, \cdots, \vec{\phi}_n(t) = \begin{pmatrix} \phi_{1n}(t) \\ \phi_{2n}(t) \\ \vdots \\ \phi_{nn}(t) \end{pmatrix} \qquad (3.18)$$

Then the $n \times n$ determinant

$$W(\vec{\phi}_1, \vec{\phi}_2, \cdots, \vec{\phi}_n) = \begin{vmatrix} \phi_{11} & \phi_{12} & \cdots & \phi_{1n} \\ \phi_{21} & \phi_{22} & \cdots & \phi_{2n} \\ \vdots & \vdots & & \vdots \\ \phi_{n1} & \phi_{n2} & \cdots & \phi_{nn} \end{vmatrix} \qquad (3.19)$$

is called the **Wronskian of the n vector functions** $\vec{\phi}_1, \vec{\phi}_2, \cdots, \vec{\phi}_n$ and its value at t is $W(\vec{\phi}_1, \vec{\phi}_2, \cdots, \vec{\phi}_n)(t)$.

Theorem 3.3 : If the n vector functions $\vec{\phi}_1, \vec{\phi}_2, \cdots, \vec{\phi}_n$ defined by (3.18) are linearly dependent on $a \le t \le b$, then their Wronskian vanishes for all t on $a \le t \le b$.

Proof : Since the vector functions $\vec{\phi}_1, \vec{\phi}_2, \cdots, \vec{\phi}_n$ are linearly dependent, there exist n numbers $C_1, C_2, ..., C_n$ not all zero, such that

$$C_1\vec{\phi}_1(t) + C_2\vec{\phi}_2(t) + \cdots + C_n\vec{\phi}_n(t) = 0, \ \forall t \in [a,b].$$

which in corresponding components reduces to

$$\sum_{j=1}^{n} C_j\phi_{ij}(t) = 0, \ \forall t \in [a,b] \ (i = 1,2,\ldots,n)$$

In particular, for a point $t_0 \in [a,b]$, we must have

$$\sum_{j=1}^{n} C_j\phi_{ij}(t_0) = 0 \quad (i = 1,2,\ldots,n)$$

Since the numbers $C_1, C_2, ..., C_n$ are not all zero, the determinant of this system is zero, i.e.,

$$\begin{vmatrix} \phi_{11}(t_0) & \phi_{12}(t_0) & \cdots & \phi_{1n}(t_0) \\ \phi_{21}(t_0) & \phi_{22}(t_0) & \cdots & \phi_{2n}(t_0) \\ \vdots & \vdots & & \vdots \\ \phi_{n1}(t_0) & \phi_{n2}(t_0) & \cdots & \phi_{nn}(t_0) \end{vmatrix}$$

$$\text{or, } W(\vec{\phi}_1, \vec{\phi}_2, \cdots, \vec{\phi}_n)(t_0) = 0$$

Since t_0 in arbitrary, we must have $W(\phi_1, \phi_2, ..., \phi_n) = 0$ for all $t \in [a, b]$

Example 3.3 : Consider the three vector functions

$$\vec{\phi}_1(t) = \begin{pmatrix} e^{2t} \\ 2e^{2t} \\ 5e^{2t} \end{pmatrix}, \vec{\phi}_2(t) = \begin{pmatrix} e^{2t} \\ 4e^{2t} \\ 11e^{2t} \end{pmatrix}, \vec{\phi}_3(t) = \begin{pmatrix} e^{2t} \\ e^{2t} \\ 2e^{2t} \end{pmatrix}$$

Noting that $3\vec{\phi}_1(t)+(-1)\vec{\phi}_2(t)+(-2)\vec{\phi}_3(t)=0$ for all $t \in [a, b]$, we find that the vector functions $\vec{\phi}_1, \vec{\phi}_2, \cdots, \vec{\phi}_3$ are linearly dependent and, therefore, by Theorem 3.3, their Wronskian is zero for all $t \in [a, b]$. Indeed, we have

$$W(\vec{\phi}_1,\vec{\phi}_2,\vec{\phi}_3)=\begin{vmatrix} e^{2t} & e^{2t} & e^{2t} \\ 2e^{2t} & 4e^{2t} & e^{2t} \\ 5e^{2t} & 11e^{2t} & 2e^{2t} \end{vmatrix}=0 \text{ for all } t$$

Theorem 3.4 : If the vector functions $\vec{\phi}_1, \vec{\phi}_2, \cdots, \vec{\phi}_n$ defined in (3.18) are the n solutions of the homogeneous linear vector differential equation.

$$\frac{dx}{dt}=A(t)x \tag{3.17}$$

and the Wronskian $W(\vec{\phi}_1, \vec{\phi}_2, \cdots, \vec{\phi}_n)(t_0)=0$ for some $t_0 \in [a, b]$, then $\vec{\phi}_1, \vec{\phi}_2, \cdots, \vec{\phi}_n$ are linearly dependent on $[a, b]$.

Proof : Consider the linear algebraic system

$$\sum_{j=1}^{n} C_j\phi_{ij}(t_0)=0 \ (i=1,2,\ldots,n) \tag{3.20}$$

in the unknowns $C_1, C_2, \ldots, C_n$. This system has a non-trivial solution, because the determinant of the coefficients $W(\vec{\phi}_1, \vec{\phi}_2, \cdots, \vec{\phi}_n)=0$ by hypothesis. Thus there exist numbers $C_1, C_2, \ldots, C_n$ not all zero such that

$$C_1\vec{\phi}_1(t_0)+C_2\vec{\phi}_2(t_0)+\cdots+C_n\vec{\phi}_n(t_0)=0 \tag{3.21}$$

Now we consider the vector function $\vec{\phi}$ defined by

$$\vec{\phi}(t)=C_1\vec{\phi}_1(t)+C_2\vec{\phi}_2(t)+\cdots+C_n\vec{\phi}_n(t),\ \forall t\in[a,b] \tag{3.22}$$

Since $\vec{\phi}_1, \vec{\phi}_2, \cdots, \vec{\phi}_n$ are solutions of (3.17), by Theorem (3.2), the linear combination $\vec{\phi}$ defined by (3.22) is also a solution of (3.17). But this function $\vec{\phi}$ is such that $\vec{\phi}(t_0)=0$ by (3.21). Hence from § 3.2, we find that $\vec{\phi}(t)=0$ for all $t\in[a,b]$, i.e., $C_1\vec{\phi}_1(t)+C_2\vec{\phi}_2(t)+\cdots+C_n\vec{\phi}_n(t)=0$ for all $t\in[a,b]$, in which

$C_1, C_2, \ldots, C_n$ are not all zero. Thus $\vec{\phi}_1, \vec{\phi}_2, \cdots, \vec{\phi}_n$ are linearly dependent on $a \le t \le b$.

Example 3.4 : It is easy to verify that the vector functions $\vec{\phi}_1, \vec{\phi}_2, \vec{\phi}_3$ defined by

$$\vec{\phi}_1(t) = \begin{pmatrix} e^{3t} \\ -2e^{3t} \\ -e^{3t} \end{pmatrix}, \vec{\phi}_2(t) = \begin{pmatrix} 2e^{3t} \\ -4e^{3t} \\ -2e^{3t} \end{pmatrix} \text{ and } \vec{\phi}_3(t) = \begin{pmatrix} -3e^{3t} \\ 6e^{3t} \\ 3e^{3t} \end{pmatrix}$$

are solutions of the linear homogeneous equation

$$\frac{dx}{dt} = \begin{pmatrix} 7 & -1 & 6 \\ -10 & 4 & -12 \\ -2 & 1 & -1 \end{pmatrix} x, \text{ where } x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

on any interval $a \le t \le b$. Also $W(\vec{\phi}_1, \vec{\phi}_2, \vec{\phi}_3) = 0$ for any $t \in [a,b]$ containing $t = 0$. Thus by Theorem 3.4, $\vec{\phi}_1, \vec{\phi}_2, \vec{\phi}_3$ are linearly dependent for any $t \in [a,b]$ containing $t = 0$. Indeed, we have

$$\vec{\phi}_1(t) + \vec{\phi}_2(t) + \vec{\phi}_3(t) = 0 \ \forall t \in [a,b].$$

Theorem 3.5 : Let the n solutions of the homogeneous linear vector differential equation

$$\frac{dx}{dt} = A(t)x \tag{3.17}$$

on the interval $a \le t \le b$ be given by the vector functions $\vec{\phi}_1, \vec{\phi}_2, \cdots, \vec{\phi}_n$ defined in (3.18). Then

either $W(\vec{\phi}_1, \vec{\phi}_2, \cdots, \vec{\phi}_n) = 0$ for all $t \in [a,b]$

or $W(\vec{\phi}_1, \vec{\phi}_2, \cdots, \vec{\phi}_n) = 0$ for no $t \in [a,b]$.

Proof : If $W(\vec{\phi}_1, \vec{\phi}_2, \cdots, \vec{\phi}_n) = 0$ for some $t \in [a,b]$, then the solutions $\vec{\phi}_1, \vec{\phi}_2, \cdots, \vec{\phi}_n$ are linearly dependent on $[a, b]$ by Theorem 3.4 and $W(\vec{\phi}_1, \vec{\phi}_2, \cdots, \vec{\phi}_n)(t) = 0$ for all $t \in [a,b]$ by Theorem 3.3. Hence $W(\vec{\phi}_1, \vec{\phi}_2, \cdots, \vec{\phi}_n) = 0$ either for all $t \in [a,b]$ or for no $t \in [a,b]$.

Theorem 3.6 : If the vector functions $\vec{\phi}_1, \vec{\phi}_2, \cdots, \vec{\phi}_n$ defined in (3.18) be the n solutions of the homogeneous linear vector differential equation

$$\frac{dx}{dt} = A(t)x \tag{3.17}$$

on the interval $a \le t \le b$, then these n solutions are linearly independent on $a \le t \le b$ if and only if $W(\vec{\phi}_1, \vec{\phi}_2, \cdots, \vec{\phi}_n)(t) \ne 0$ for all t in $a \le t \le b$.

Proof : By Theorems 3.2 and 3.3, the solutions $\vec{\phi}_1, \vec{\phi}_2, \cdots, \vec{\phi}_n$ are linearly dependent on $[a, b]$ iff $W(\vec{\phi}_1, \vec{\phi}_2, \cdots, \vec{\phi}_n)(t) = 0$ for all $t \in [a,b]$. Hence these n solutions are linearly independent on $[a, b]$ iff $W(\vec{\phi}_1, \vec{\phi}_2, \cdots, \vec{\phi}_n)(t_0) \ne 0$ for some $t_0 \in [a,b]$. Hence by Theorem 3.5, $W(\vec{\phi}_1, \vec{\phi}_2, \cdots, \vec{\phi}_n)(t_0) \ne 0$ for some $t_0 \in [a,b]$ iff for all $t \in [a,b]$, we have $W(\vec{\phi}_1, \vec{\phi}_2, \cdots, \vec{\phi}_n)(t) \ne 0$.

Example 3.5 : The vector functions $\vec{\phi}_1, \vec{\phi}_2$ and $\vec{\phi}_3$ defined by

$$\vec{\phi}_1(t) = \begin{pmatrix} e^{2t} \\ -e^{2t} \\ -e^{2t} \end{pmatrix}, \vec{\phi}_2(t) = \begin{pmatrix} e^{3t} \\ -2e^{3t} \\ -e^{3t} \end{pmatrix}, \vec{\phi}_3(t) = \begin{pmatrix} 3e^{5t} \\ -6e^{5t} \\ -2e^{5t} \end{pmatrix}$$

are solutions of the homogeneous linear vector differential equation

$$\frac{dx}{dt} = \begin{pmatrix} 7 & -1 & 6 \\ -10 & 4 & -12 \\ -2 & 1 & -1 \end{pmatrix} x, \text{ where } x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \tag{3.22}$$

on any interval $a \le t \le b$, but

$$W(\vec{\phi}_1, \vec{\phi}_2, \vec{\phi}_3)(t) = \begin{vmatrix} e^{2t} & e^{3t} & 3e^{5t} \\ -e^{2t} & -2e^{3t} & -6e^{5t} \\ -e^{2t} & -e^{3t} & -2e^{5t} \end{vmatrix} = -e^{10t} \ne 0$$

so that by Theorem 3.6, the solutions $\vec{\phi}_1, \vec{\phi}_2, \vec{\phi}_3$ are linearly independent on every real interval $[a, b]$.

Definition 3.3 : Consider the homogeneous linear vector equation

$$\frac{dx}{dt} = A(t)x \tag{3.17}$$

in which x is an $n \times 1$ column vector. A set of n linearly independent solutions of (3.17) is called a **fundamental set of solutions** of (3.17) and a matrix whose individual column is a fundamental set of solutions of (3.17) is called a **fundamental matrix** of (3.17).

Example 3.6 : Since the solutions $\vec{\phi}_1, \vec{\phi}_2, \vec{\phi}_3$ in Example 3.5 are linearly independent, these three solutions form a fundamental set of the given differential equation, the fundamental matrix being given by

$$\begin{pmatrix} e^{2t} & e^{3t} & 3e^{5t} \\ -e^{2t} & -2e^{3t} & -6e^{5t} \\ -e^{2t} & -e^{3t} & -2e^{5t} \end{pmatrix}$$

Theorem 3.7 : There exist fundamental sets of solutions of the linear vector differential equation

$$\frac{dx}{dt} = A(t)x \qquad (3.17)$$

Proof : We define the constant vectors $u_i (i = 1, 2, ..., n)$ such that it has the ith component 1 and all other components 0, i.e.,

$$u_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, u_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, u_n = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

Let the n solutions $\vec{\phi}_1, \vec{\phi}_2, \cdots, \vec{\phi}_n$ of (3.17) satisfy the conditions $\vec{\phi}_i(t_0) = u_i$ $(i = 1, 2, ..., n)$ where $t_0 \in [a,b]$ is an arbitrary but fixed point. By Theorem 3.1, these solutions exist and unique. Now we have

$$W(\vec{\phi}_1, \vec{\phi}_2, \cdots, \vec{\phi}_n)(t_0) = W(u_1, u_2, \cdots, u_n) = \begin{vmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{vmatrix} = 1 \neq 0$$

Hence by Theorem 3.5, $W(\vec{\phi}_1, \vec{\phi}_2, \cdots, \vec{\phi}_n)(t) \neq 0 \ \forall t \in [a,b]$ and therefore the solutions $\vec{\phi}_1, \vec{\phi}_2, \cdots, \vec{\phi}_n$ are linearly independent on $[a, b]$. Thus the solutions $\vec{\phi}_1, \vec{\phi}_2, \cdots, \vec{\phi}_n$ form a fundamental set of (3.17).

Theorem 3.8 : If the solutions $\vec{\phi}_1, \vec{\phi}_2, \cdots, \vec{\phi}_n$ of the linear homogeneous vector differential equation

$$\frac{dx}{dt} = A(t)x \tag{3.17}$$

be a fundamental set of solutions of (3.17) and $\vec{\phi}$ be an arbitrary solution of (3.17), then $\vec{\phi}$ can be expressed as a linear combination of $\vec{\phi}_1, \vec{\phi}_2, \cdots, \vec{\phi}_n$ on $[a, b]$

Proof : Let

$$\vec{\phi}_1(t) = \begin{pmatrix} \phi_{11}(t) \\ \phi_{21}(t) \\ \vdots \\ \phi_{n1}(t) \end{pmatrix}, \vec{\phi}_2(t) = \begin{pmatrix} \phi_{12}(t) \\ \phi_{22}(t) \\ \vdots \\ \phi_{n2}(t) \end{pmatrix}, \cdots, \vec{\phi}_n(t) = \begin{pmatrix} \phi_{1n}(t) \\ \phi_{2n}(t) \\ \vdots \\ \phi_{nn}(t) \end{pmatrix}$$

and $\vec{\phi}(t_0) = u_0$, where $t_0 \in [a,b]$ and $u_0 = \begin{pmatrix} u_{10} \\ u_{20} \\ \vdots \\ u_{n0} \end{pmatrix}$ is a constant vector.

Consider the linear algebraic system

$$\begin{aligned} C_1\phi_{11}(t_0) + C_2\phi_{12}(t_0) + \cdots + C_n\phi_{1n}(t_0) &= u_{10}, \\ C_1\phi_{21}(t_0) + C_2\phi_{22}(t_0) + \cdots + C_n\phi_{2n}(t_0) &= u_{20}, \\ \vdots \qquad\qquad \vdots \qquad\qquad & \vdots \\ C_1\phi_{n1}(t_0) + C_2\phi_{n2}(t_0) + \cdots + C_n\phi_{nn}(t_0) &= u_{n0}, \end{aligned} \tag{3.23}$$

of n equations in the n unknowns C1, C_2, ..., C_n. Since $\vec{\phi}_1, \vec{\phi}_2, \cdots, \vec{\phi}_n$ is a fundamental set of solutions of (3.17), they are linearly independent on $[a, b]$ and hence $W(\vec{\phi}_1, \vec{\phi}_2, \cdots, \vec{\phi}_n) \neq 0$. Now $W(\vec{\phi}_1, \vec{\phi}_2, \cdots, \vec{\phi}_n)(t_0)$ is the determinant of the coefficients of the system (3.22) and hence this determinant is not zero. Thus the system (3.23) has a unique solution for C1, C_2, ..., C_n, i.e., there exists a unique set of numbers C1, C_2, ..., C_n such that

$$C_1\vec{\phi}_1(t_0) + C_2\vec{\phi}_2(t_0) + \cdots + C_n\vec{\phi}_n(t_0) = u_0$$

and hence

$$\vec{\phi}(t_0) = u_0 = \sum_{k=1}^{n} C_k\vec{\phi}_k(t_0) \tag{3.24}$$

Now if we define a vector function $\vec{\psi}(t) = \sum_{k=1}^{n} C_k\vec{\phi}_k(t)$ then by Theorem 3.2, $\vec{\psi}(t)$ is also a solution of (3.17). Since $\vec{\psi}(t_0) = \sum_{k=1}^{n} C_k\vec{\phi}_k(t_0)$ we have by using (3.24), $\vec{\psi}(t_0) = \vec{\phi}(t_0)$. Hence by Theorem 3.1, we must have $\vec{\psi}(t) = \vec{\phi}(t)$ for all $t \in [a,b]$, i.e., $\vec{\phi}(t) = \sum_{k=1}^{n} C_k\vec{\phi}_k(t)$ for all $t \in [a,b]$. Thus $\vec{\phi}$ is expressed as a linear combination of $\vec{\phi}_1, \vec{\phi}_2, \cdots, \vec{\phi}_n$, where $C_1, C_2, \ldots, C_n$ is the unique solution of the system (3.23).

Example 3.7 : Consider the vector functions $\vec{\phi}_1, \vec{\phi}_2, \vec{\phi}_3$ in Example 3.5 and these functions form a fundamental set of differential equations (3.22) (by Example 3.6) and the Theorem 3.8 shows that if $\vec{\phi}$ is an arbitrary solution of (3.22), the $\vec{\phi}$ can be represented as a suitable linear combination of $\vec{\phi}_1, \vec{\phi}_2, \vec{\phi}_3$ i.e., $C_1\vec{\phi}_1 + C_2\vec{\phi}_2 + C_3\vec{\phi}_3$ is a general solution of (3.22), where C_1, C_2, C_3 are arbitrary numbers, i.e., a general solution is given by

$$C_1\begin{pmatrix} e^{2t} \\ -e^{2t} \\ -e^{2t} \end{pmatrix} + C_2\begin{pmatrix} e^{3t} \\ -2e^{3t} \\ -e^{3t} \end{pmatrix} + C_3\begin{pmatrix} 3e^{5t} \\ -6e^{5t} \\ -2e^{5t} \end{pmatrix}$$

and can be written as

$$\begin{aligned} x_1 &= c_1e^{2t} + c_2e^{3t} + 3c_3e^{5t} \\ x_2 &= -c_1e^{2t} - 2c_2e^{3t} - 6c_3e^{5t} \\ x_3 &= -c_1e^{2t} - c_2e^{3t} - 2c_3e^{5t} \end{aligned}$$

3.3 Non-homogeneous Linear System : We now consider the non-homogeneous linear vector differential equation

$$\frac{dx}{dt} = A(t)x + F(t) \tag{3.8}$$

where $A(t)$ is given by (3.6) and $F(t)$ and x are given by (3.7). We shall see that the solutions of the equation (3.25) are closely related to those of the corresponding homogeneous linear equation

$$\frac{dx}{dt} = A(t)x. \tag{3.17}$$

Theorem 3.9 : If $\vec{\phi}_0$ be any solution of the non-homogeneous linear vector differential equation

$$\frac{dx}{dt} = A(t)x + F(t) \tag{3.8}$$

and $\vec{\phi}_1, \vec{\phi}_2, \cdots, \vec{\phi}_n$ be a fundamental set of solutions of the corresponding homogeneous differential equation

$$\frac{dx}{dt} = A(t)x \tag{3.17}$$

then (i) the vector function

$$\vec{\psi} = \vec{\phi}_0 + \sum_{k=1}^{n} C_k \vec{\phi}_k \tag{3.25}$$

is also a solution of (3.8) for every choice of the constants $C_1, C_2, ..., C_n$ and (ii) an arbitrary solution $\vec{\phi}$ of the non-homogeneous differential equation (3.8) is of the form (3.25)

Proof : (i) Since $\vec{\phi}_0$ satisfies (3.8), we have

$$\frac{d\vec{\phi}_0(t)}{dt} = A(t)\vec{\phi}(t) + F(t)$$

and since $\sum_{k=1}^{n} C_k \vec{\phi}_k$ satisfies (3.17), we have

$$\frac{d}{dt}\left[\sum_{k=1}^{n} C_k\vec{\phi}_k(t)\right] = A(t)\left[\sum_{k=1}^{n} C_k\vec{\phi}_k(t)\right]$$

Now $\frac{d}{dt}\left[\vec{\phi}_0(t) + \sum_{k=1}^{n} C_k\vec{\phi}_k(t)\right] = A(t)\vec{\phi}_0(t) + F(t) + A(t)\left[\sum_{k=1}^{n} C_k\vec{\phi}_k(t)\right]$

$$= A(t)\left[\vec{\phi}_0(t) + \sum_{k=1}^{n} C_k\vec{\phi}_k(t)\right] + F(t)$$

i.e., $\frac{d\vec{\psi}(t)}{dt} = A(t)\vec{\psi}(t) + F(t)$

and so $\vec{\psi} = \vec{\phi}_0 + \sum_{k=1}^{n} C_k\vec{\phi}_k$ is a solution of (3.8) for every choice of the constants $C_1, C_2, \ldots, C_n$.

(ii) Let us consider an arbitrary solution $\vec{\phi}$ of (3.8). Since both $\vec{\phi}$ and $\vec{\phi}_0$ satisfy (3.8), we have simultaneously

$$\frac{d\vec{\phi}(t)}{dt} = A(t)\vec{\phi}(t) + F(t) \text{ and } \frac{d\vec{\phi}_0(t)}{dt} = A(t)\vec{\phi}_0(t) + F(t)$$

so that $\frac{d}{dt}[\vec{\phi}(t) - \vec{\phi}_0(t)] = A(t)[\vec{\phi}(t) - \vec{\phi}_0(t)]$

Thus $\vec{\phi} - \vec{\phi}_0$ satisfies the homogeneous differential equation (3.17). Hence for suitable choice of the constants $C_1, C_2, \ldots, C_n$, we have

$$\vec{\phi} - \vec{\phi}_0 = \sum_{k=1}^{n} C_k\vec{\phi}_k$$

Thus for suitable choice of $C_1, C_2, \ldots, C_n$ we have

$$\vec{\phi} = \vec{\phi}_0 + \sum_{k=1}^{n} C_k\vec{\phi}_k \tag{3.20}$$

Example 3.8 : Let us consider the non-homogeneous differential equation

$$\frac{dx}{dt} = \begin{pmatrix} 7 & -1 & 6 \\ -10 & 4 & -12 \\ -2 & 1 & -1 \end{pmatrix} x + \begin{pmatrix} -5t-6 \\ -4t+23 \\ 2 \end{pmatrix}, \quad \text{where } x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

We have seen in Example 3.5 that

$$\vec{\phi}_1(t) = \begin{pmatrix} e^{2t} \\ -e^{2t} \\ -e^{2t} \end{pmatrix}, \vec{\phi}_2(t) = \begin{pmatrix} e^{3t} \\ -2e^{3t} \\ -e^{3t} \end{pmatrix}, \vec{\phi}_3(t) = \begin{pmatrix} 3e^{5t} \\ -6e^{5t} \\ -2e^{5t} \end{pmatrix}$$

form a fundamental set of the homogeneous differential equation

$$\frac{dx}{dt} = \begin{pmatrix} 7 & -1 & 6 \\ -10 & 4 & -12 \\ -2 & 1 & -1 \end{pmatrix} x$$

Noting that the vector function $\vec{\phi}_0$ defined by

$$\vec{\phi}_0(t) = \begin{pmatrix} 2t \\ 3t-2 \\ -t+1 \end{pmatrix}$$

is a solution of the given non-homogeneous equation, its general solution is

$$x = C_1\vec{\phi}_1(t) + C_2\vec{\phi}_2(t) + C_3\vec{\phi}_3(t) + \vec{\phi}_0(t)$$

$$\text{i.e.,} \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = C_1\begin{pmatrix} e^{2t} \\ -e^{2t} \\ -e^{2t} \end{pmatrix} + C_2\begin{pmatrix} e^{3t} \\ -2e^{3t} \\ -e^{3t} \end{pmatrix} + C_3\begin{pmatrix} 3e^{5t} \\ -6e^{5t} \\ -2e^{5t} \end{pmatrix} + \begin{pmatrix} 2t \\ 3t-2 \\ -t+1 \end{pmatrix}$$

where C_1, C_2, C_3 are arbitrary constants. The above can be written as

$$\begin{aligned} x_1 &= C_1e^{2t} + C_2e^{3t} + 3C_3e^{5t} + 2t \\ x_2 &= -C_1e^{2t} - 2C_2e^{3t} - 6C_3e^{5t} + 3t - 2 \\ x_3 &= -C_1e^{2t} - C_2e^{3t} - 2C_3e^{5t} - t + 1 \end{aligned}$$

3.4 Homogeneous Linear Systems with Constant Coefficients : Let us consider the normal form of a homogeneous linear system of n first order differential equations (3.16) in which the coefficient $a_{ij}(i = 1, 2, ..., n;\ j = 1, 2, ..., n)$ are real constants. Introducing the $n \times n$ constant matrix of real numbers

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \tag{3.27}$$

and the vector

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

the system (3.16) can be expressed as the homogeneous vector differential equation

$$\frac{dx}{dt} = Ax \tag{3.28}$$

in which the matrix A is called the coefficient matrix of (3.28).

We now seek non–trivial solutions of the system (3.28) of the form

$$\begin{aligned} x_1 &= \alpha_1 e^{\lambda t} \\ x_2 &= \alpha_2 e^{\lambda t} \\ &\vdots \\ x_n &= \alpha_n e^{\lambda t} \end{aligned}$$

where $\alpha_1,\ \alpha_2,\ \cdots,\ \alpha_n$ and λ are numbers. Letting $\bar{\alpha} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix}$, we find that the vector form of the desired solution of (3.28) is

$$x = \bar{\alpha} e^{\lambda t} \tag{3.29}$$

substituting (3.29) in (3.28) we obtain

$$\lambda\vec{\alpha}e^{\lambda t} = A\vec{\alpha}e^{\lambda t}$$

so that

$$(A-\lambda I)\vec{\alpha} = 0 \tag{3.30}$$

where I is the $n \times n$ identity matrix. In terms of the components, this is the system of n homogeneous linear algebraic equations

$$\begin{aligned}
(a_{11}-\lambda)\alpha_1 + a_{12}\alpha_2 + \cdots + a_{1n}\alpha_n &= 0\\
a_{21}\alpha_1 + (a_{22}-\lambda)\alpha_2 + \cdots + a_{2n}\alpha_n &= 0\\
\vdots \qquad\qquad \vdots \qquad\qquad &\vdots\\
a_{n1}\alpha_1 + a_{n2}\alpha_2 + \cdots + (a_{nn}-\lambda)\alpha_n &= 0
\end{aligned} \tag{3.31}$$

in the unknowns $\alpha_1, \alpha_2, \cdots, \alpha_n$. This system has a non-trivial solution iff

$$\begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n}\\ a_{21} & a_{22}-\lambda & \cdots & a_{2n}\\ \vdots & \vdots & & \vdots\\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{vmatrix} = 0 \tag{3.32}$$

In vector notation, this gives

$$|A-\lambda I| = 0 \tag{3.33}$$

The equation (3.32) or (3.33) is known as **characteristic equation** of the coefficient matrix $A = (a_{ij})$ of the differential equation (3.28). This is an nth degree polynomial equation in λ and its roots $\lambda_1, \lambda_2, \ldots, \lambda_n$ are called the **characteristics values** of A. Substituting each characteristic values $\lambda_i (i = 1, 2, \ldots, n)$ into the system (3.31), we obtain the corresponding non-trivial solution

$$\alpha_1 = \alpha_{1i}, \alpha_2 = \alpha_{2i}, \cdots, \alpha_n = \alpha_{ni} \quad (i = 1, 2, \ldots, n)$$

of the system (3.31). The vector defined by

$$\vec{\alpha}^{(i)} = \begin{pmatrix} \alpha_{1i}\\ \alpha_{2i}\\ \vdots\\ \alpha_{ni} \end{pmatrix} \quad (i = 1, 2, \cdots, n) \tag{3.34}$$

is a characteristic vector corresponding to the characteristic value $\lambda_i (i = 1, 2, ..., n)$

Thus if the vector differential equation (3.28) has a solution of the form $x = \bar{\alpha}e^{\lambda t}$, then the number λ must be a characteristic value λ_i of the coefficient matrix A and the vector $\bar{\alpha}$ must be a characteristic value $\bar{\alpha}^{(i)}$ corresponding to the characteristic value λ_i.

We now consider the following cases for characteristic values :

Case I : Characteristic Values are all Distinct : Let the characteristic values $\lambda_1, \lambda_2, ..., \lambda_n$ of the matrix A are all distinct.and let $\alpha^{(1)}, \alpha^{(2)}, \cdots, \alpha^{(n)}$ be a set of n respective corresponding characteristic vectors of A. Then the n distinct vector functions $x_1, x_2, ..., x_n$ defined by

$$x_1(t) = \alpha^{(1)}e^{\lambda_1(t)}, x_2(t) = \alpha^{(2)}e^{\lambda_2(t)}, \cdots, x_n(t) = \alpha^{(n)}e^{\lambda_n(t)} \tag{3.35}$$

are solutions of the equation (3.28) on any real interval $[a, b]$. This can be readily verified as follows : Noting from (3.30) that for each i, $\lambda_i \alpha^{(i)} = A\alpha^{(i)}$ $(i = 1, 2, ..., n)$ and using (3.35) we have

$$\frac{dx_i(t)}{dt} = \lambda_i \alpha^{(i)}e^{\lambda_i t} = A\alpha^{(i)}e^{\lambda_i t} = Ax_i(t)$$

showing that $x_i(t)$ satisfies the vector differential equation (3.28).

Next consider the Wronskian $W(x_1, x_2, ..., x_n)$ of the n solutions $x_1, x_2, ..., x_n$ defined by (3.35). We find

$$W(x_1, x_2, \cdots, x_n)(t) = \begin{vmatrix} \alpha_{11}e^{\lambda_1 t} & \alpha_{11}e^{\lambda_2 t} & \cdots & \alpha_{1n}e^{\lambda_n t} \\ 2\alpha_{11}e^{\lambda_1 t} & \alpha_{22}e^{\lambda_2 t} & \cdots & \alpha_{2n}e^{\lambda_n t} \\ \vdots & \vdots & & \vdots \\ \alpha_{n1}e^{\lambda_1 t} & \alpha_{n2}e^{\lambda_2 t} & \cdots & \alpha_{nn}e^{\lambda_n t} \end{vmatrix}$$

$$= e^{(\lambda_1 + \lambda_2 + \cdots + \lambda_n)t} \begin{vmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \vdots & \vdots & & \vdots \\ \alpha_{n1} & \alpha_{n2} & \cdots & \alpha_{nn} \end{vmatrix}$$

$\neq 0$ for all t on $[a, b]$

because the n characteristic vectors $\alpha^{(1)}, \alpha^{(2)}, \cdots, \alpha^{(n)}$ are linearly independent and, therefore,

$$\begin{vmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \vdots & \vdots & & \vdots \\ \alpha_{n1} & \alpha_{n2} & \cdots & \alpha_{nn} \end{vmatrix} \neq 0$$

and obviously $e^{(\lambda_1+\lambda_2+\cdots+\lambda_n)t} \neq 0$ for all t on $[a, b]$. Hence the solutions $x_1, x_2, \ldots, x_n$ of the vector differential equation (3.28) are linearly independent on $[a, b]$. Thus a general solution of (3.28) is given by

$$C_1x_1 + C_2x_2 + \cdots + C_nx_n$$

where $C_1, C_2, \ldots, C_n$ are constants,

The above results can be summarised in a Theorem as follows :

Theorem 3.10 : Consider the vector differential equation

$$\frac{dx}{dt} = Ax \tag{3.28}$$

where A is an $n \times n$ constant matrix having n distinct characteristic values $\lambda_1, \lambda_2, \ldots, \lambda_n$; and let $\alpha^{(1)}, \alpha^{(2)}, \cdots, \alpha^{(n)}$ be the corresponding characteristic vectors of A. Then on every real interval $[a, b]$, the n functions defined by

$$\alpha^{(1)}e^{\lambda_1(t)}, \alpha^{(2)}e^{\lambda_2(t)}, \cdots, \alpha^{(n)}e^{\lambda_n(t)}$$

from a linearly independent (fundamental) set of solutions of (3.28) and

$$x = C_1\alpha^{(1)}e^{\lambda_1(t)} + C_2\alpha^{(2)}e^{\lambda_2(t)} + \cdots + C_n\alpha^{(n)}e^{\lambda_n(t)}$$

where $C_1, C_2, \ldots, C_n$ are arbitrary constants, is a general solution of (3.28) on $[a, b]$.

Example 3.9 : For the homogeneous linear system

$$\frac{dx}{dt} = \begin{pmatrix} 7 & -1 & 6 \\ -10 & 4 & -12 \\ -2 & 1 & -1 \end{pmatrix} x, \text{ where } x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

we assume a solution of the form $x = \alpha e^{\lambda t}$, that is $x_1 = \alpha_1 e^{\lambda t}, x_2 = \alpha_2 e^{\lambda t}, x_3 = \alpha_3 e^{\lambda t}$. Then the given equation gives on the assumption of $e^{\lambda t} \neq 0$.

$$\alpha_1\lambda = 7\alpha_1 - \alpha_2 + 6\alpha_3$$
$$\alpha_2\lambda = -10\alpha_1 + 4\alpha_2 - 12\alpha_3$$
$$\alpha_3\lambda = -2\alpha_1 + \alpha_2 - \alpha_3$$

i.e.,

$$\begin{aligned} (7-\lambda)\alpha_1 - \alpha_2 + 6\alpha_3 &= 0 \\ -10\alpha_1 + (4-\lambda)\alpha_2 - 12\alpha_3 &= 0 \\ -\alpha_1 + \alpha_2 - (1+\lambda)\alpha_3 &= 0 \end{aligned} \tag{3.36}$$

Which has a nontrivial solution iff

$$\begin{vmatrix} 7-\lambda & -1 & 6 \\ -10 & 4-\lambda & -12 \\ -2 & 1 & -1-\lambda \end{vmatrix} = 0$$

$$\text{or, } \lambda^3 - 10\lambda^2 + 31\lambda - 30 = 0$$

which is the characteristic equation of the coefficient matrix

$$A = \begin{pmatrix} 7 & -1 & 6 \\ -10 & 4 & -12 \\ -2 & 1 & -1 \end{pmatrix}$$

of the given system. The roots of the characteristic equation are

$$\lambda_1 = 2,\ \lambda_2 = 3,\ \lambda_3 = 5$$

A characteristic vector corresponding to $\lambda_1 = 2$ is a non-zero vector $\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}$

whose components are a non-trivial solution α_1, α_2, α_3 of the algebraic system (3.36) when $\lambda = 2$. Equivalently, it is a non-zero vector such that

$$\begin{pmatrix} 7 & -1 & 6 \\ -10 & 4 & -12 \\ -2 & 1 & -1 \end{pmatrix}\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = 2\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}$$

that is α_1, α_2, α_3 must be a trivial solution of the system

$$\begin{aligned} 5\alpha_1 - \alpha_2 + 6\alpha_3 &= 0 \\ -10\alpha_1 + 2\alpha_2 - 12\alpha_3 &= 0 \\ -2\alpha_1 + \alpha_2 - 3\alpha_3 &= 0 \end{aligned}$$

It may be observed that the second equation is merely a constant multiple of the first. Solving this system for α_2 and α_3 we gets $\alpha_2 = -\alpha_1$ and $\alpha_3 = -\alpha_1$. Setting $\alpha_1 = k$, $\alpha_2 = \alpha_3 = -k$, the characteristic vectors corresponding to the characteristic value $\lambda = 2$

$$\begin{pmatrix} k \\ -k \\ -k \end{pmatrix}$$

In particular, letting $k = 1$, we obtain the particular characteristic vector

$$\alpha^{(1)} = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$$

Similarly, the characteristic vectors corresponding to $x_2 = 3$ and $x_3 = 5$ are

$$\alpha^{(2)} = \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix} \text{ and } \alpha^{(3)} = \begin{pmatrix} 3 \\ -6 \\ -2 \end{pmatrix}$$

respectively. Thus a fundamental set of solutions of the given equation is

$$\alpha^{(1)}e^{\lambda_1 t}, \alpha^{(2)}e^{\lambda_2 t}, \alpha^{(3)}e^{\lambda_3 t}$$

$$\text{i.e., } \begin{pmatrix} e^{2t} \\ -e^{2t} \\ -e^{2t} \end{pmatrix}, \begin{pmatrix} e^{3t} \\ -2e^{3t} \\ -e^{3t} \end{pmatrix}, \begin{pmatrix} 3e^{5t} \\ -6e^{5t} \\ -2e^{5t} \end{pmatrix}$$

respectively. A general solution of the given system may, therefore be expressed as

$$x_1 = C_1 e^{2t} + C_2 e^{3t} + 3C_3 e^{5t}$$

$$x_2 = -C_1 e^{2t} - 2C_2 e^{3t} - 6C_3 e^{5t}$$

$$x_3 = -C_1 e^{2t} - C_2 e^{3t} - 2C_3 e^{5t}$$

We now return to the vector differential equation

$$\frac{dx}{dt} = Ax \tag{3.28}$$

where A is an $n \times n$ real constant matrix. In Theorem 3.10, the n characteristic values $\lambda_1, \lambda_2, ..., \lambda_n$ of A were assumed to be distinct. But we do not require that they may be real. The characteristic values may be complex and since A is a real matrix, any complex characteristic values must occur in conjugate pairs. Suppose $\lambda_1 = a + ib$, and $\lambda_2 = a - bi$ form such a pair. Then the corresponding solutions are

$$\alpha^{(1)} e^{(a+ib)t}, \alpha^{(2)} e^{(a-ib)t}$$

which are complex, $\alpha^{(1)}$, $\alpha^{(2)}$ being the characteristic vectors corresponding to λ_1 and λ_2. Thus, if one or more conjugate complex pair of characteristic values occur, the fundamental set defined by $\alpha^{(j)} e^{\lambda_j t} (j = 1, 2, \cdots, n)$ contains complex functions. However, this fundamental set may be replaced by another fundamental set consisting of real functions.

Example 3.10 : Consider the homogeneous linear system

$$\frac{dx}{dt} = Ax$$

where $A = \begin{pmatrix} 3 & 2 \\ -5 & 1 \end{pmatrix}$ and $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$. As a solution, we take $x = \alpha e^{\lambda t}$ so that the given equation gives (assuming $e^{\lambda t} \neq 0$)

$$\begin{aligned} (3-\lambda)\alpha_1 + 2\alpha_2 &= 0 \\ -5\alpha_1 + (1-\lambda)\alpha_2 &= 0 \end{aligned} \tag{3.37}$$

For non-trivial solution, we have the characteristic equation

$$\begin{vmatrix} 3-\lambda & 2 \\ -5 & 1-\lambda \end{vmatrix} = 0, \text{ i.e., } \lambda^2 - 5\lambda + 13 = 0$$

This characteristic equation have complex roots $2 \pm 3i$. A characteristic vector $\alpha^{(1)}$ corresponding to $\lambda_1 = 2 + 3i$ has components α_1, α_2 as a non-trivial solution of the equation (3.37), i.e., α_1, α_2 must be non-trivial solution of the system

$$(1-3i)\alpha_1 + 2\alpha_2 = 0$$
$$-5\alpha_1 + (-1-3i)\alpha_2 = 0$$

A simple non-trivial solution of this system is $\alpha_1 = 2$, $\alpha_2 = -1 + 3i$. Thus

$$\alpha^{(1)} = \begin{pmatrix} 2 \\ -1+3i \end{pmatrix}$$

similarly the other characteristic vector $\alpha^{(2)}$ corrosponding $\lambda_2 = 2 - 3i$ is

$$\alpha^{(2)} = \begin{pmatrix} 2 \\ -1-3i \end{pmatrix}$$

Thus a fundamental set of solutions of the given equation is

$$\alpha^{(1)}e^{\lambda_1 t}, \alpha^{(2)}e^{\lambda_2 t}$$

$$\text{i.e., } \begin{pmatrix} 2 \\ -1+3i \end{pmatrix} e^{(2+3i)t}, \begin{pmatrix} 2 \\ -1-3i \end{pmatrix} e^{(2-3i)t} \tag{3.38}$$

For the first, we may write the solutions as

$$x_1 = e^{2t}[(2\cos 3t) + i.(2\sin 3t)]$$
$$x_2 = e^{2t}[(-\cos 3t - 3\sin 3t) + i(3\cos 3t - \sin 3t)]$$

Since both real and imaginary parts of this solution are themselves solutions of the given system, we obtain the two real solutions

$$x_1 = 2e^{2t}\cos 3t, x_2 = -e^{2t}(\cos 3t + 3\sin 3t)$$
$$\text{and } x_1 = 2e^{2t}\sin 3t, x_2 = e^{2t}(3\cos 3t - \sin 3t)$$

Since these solutions are linearly independent, we may write the general solutions as

$$x_1 = 2e^{2t}[C_1\cos 3t + C_2\sin 3t]$$

and $x_2 = e^{2t}[C_1(-\cos 3t - 3\sin 3t) + C_2(3\cos 3t - \sin 3t)]$

where C_1 and C_2 are constants

We obtain the same result if the second set of (3.38) is considered.

Case II. Repeated Characteristic Values : We now briefly discuss the case when the matrix A of the differential equation (3.28) has a repeated characteristic value. To be specific, let the matrix A has a real characteristic value λ_1 of multiplicity $m(1 < m \leq n)$ and all other characteristic values $\lambda_{m+1}, \lambda_{m+2}, \cdots, \lambda_n$ (if there be any) are distinct. Now the repeated characteristic value λ_1 of multiplicity m has p linearly characteristic vectors where $1 \leq p < m$. We consider two subcases : (i) $p = m$ and $p < m$.

In the subcase (i) : $p = m$ and there are m linearly independent characteristic vectors $\lambda^{(1)}, \lambda^{(2)}, \cdots, \lambda^{(m)}$ corresponding to λ_1. Then the n functions defined by $\alpha^{(1)}e^{\lambda_1 t}, \alpha^{(2)}e^{\lambda_1 t}, \cdots, \alpha^{(m)}e^{\lambda_1 t}, \alpha^{(m+1)}e^{\lambda_{m+1} t}, \cdots, \alpha^{(n)}e^{\lambda_n t}$ form a linearly independent set of n solutions of the differential equation (3.28); and a general solution of (3.28) is a linear combination of these n solutions having n arbitrary constants.

Example 3.11 : Let us consider the equation

$$\frac{dx}{dt} = Ax$$

where

$$A = \begin{pmatrix} 3 & 1 & -1 \\ 1 & 3 & -1 \\ 3 & 3 & -1 \end{pmatrix} \text{ and } x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

We assume a solution of the form $x = \alpha e^{\lambda t}$. Substituting this in the given equation we get

$$\begin{aligned}(3-\lambda)\alpha_1 + \alpha_2 - \alpha_3 &= 0\\ \alpha_1 + (3-\lambda)\alpha_2 - \alpha_3 &= 0\\ 3\alpha_1 + 3\alpha_2 + (-1-\lambda)\alpha_3 &= 0\end{aligned} \tag{3.39}$$

so that the characteristic equation of the coefficient matrix A is

$$\begin{vmatrix} 3-\lambda & 1 & -1 \\ 1 & 3-\lambda & -1 \\ 3 & 3 & -1-\lambda \end{vmatrix} = 0$$

The roots of this equation are $\lambda_1 = 1$, $\lambda_2 = \lambda_3 = 2$.

We first consider the distinct value $\lambda_1 = 1$. A Charactreristic vector $\alpha^{(1)}$ corresponding to $\lambda_1 = 1$ has components α_1, α_2, α_3 as the solution of (3.39) given by the solution of

$$\begin{aligned}2\alpha_1 + \alpha_2 - \alpha_3 &= 0\\ \alpha_1 + 2\alpha_2 - \alpha_3 &= 0\\ 3\alpha_1 + 3\alpha_2 - \alpha_3 &= 0\end{aligned}$$

Note that $\alpha_1 = k$, $\alpha_2 = k$ and $\alpha_3 = 3k$ is a solution of this system for every real k. Taking $k = 1$, the characteristic vector is

$$\alpha^{(1)} = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$$

and the corresponding solution of the given system is $\alpha^{(1)} e^{\lambda_1 t}$, i.e., $\begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} e^t$.

We turn now to the repeated characteristic value $\lambda_2 = \lambda_3 = 2$. This characteristic value has multiplicity $m = 2 < 3 = n$. A characteristic vector corresponding to the characteristic value $\lambda_2 = \lambda_3 = 2$ is a non-zero vector $\alpha^{(2)}$ whose components α_1, α_2, α_3 are a non-trivial solution of the algebraic system (3.39), i.e., of

$$\begin{aligned} \alpha_1 + \alpha_2 - \alpha_3 &= 0 \\ \alpha_1 + \alpha_2 - \alpha_3 &= 0 \\ 3\alpha_1 + 3\alpha_2 - 3\alpha_3 &= 0 \end{aligned}$$

so that $\alpha_1 + \alpha_2 - \alpha_3 = 0$. To satisfy this, if $\alpha_1 = 1, \alpha_2 = -1, \alpha_3 = 0$, we obtain the vector $\alpha^{(2)} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ while if $\alpha_1 = 1, \alpha_2 = 0, \alpha_3 = 1$, we obtain the vector $\alpha^{(3)} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$. Thus corresponding to the two $-f$old characteristic value $\lambda = 2$, we get two linearly independent solutions of the form $\alpha e^{\lambda t}$ of the given system. These are $\alpha^{(2)} e^{2t}$ and $\alpha^{(3)} e^{2t}$, that is $\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} e^{2t}$ and $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} e^{2t}$ respectively.

Hence a fundamental set of solutions of the given system consists of the three vectors $\alpha^{(1)}$, $\alpha^{(2)}$ and $\alpha^{(3)}$, i.e.,

$$\begin{pmatrix} e^t \\ e^t \\ 3e^t \end{pmatrix}, \begin{pmatrix} e^{2t} \\ -e^t \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} e^{2t} \\ 0 \\ e^{2t} \end{pmatrix}$$

A general solution may, therefore, expressed as

$$\begin{aligned} x_1 &= C_1 e^t + (C_2 + C_3) e^{2t} \\ x_2 &= C_1 e^t - C_2 e^{2t} \\ x_3 &= 3C_1 e^t + C_3 e^{2t} \end{aligned}$$

where C_1, C_2 and C_3 are arbitrary constants.

For the subcase (ii) : $p < m$. In the case, there are less than m linearly independent characteristic vectors $\alpha^{(1)}$ corresponding to the characteristic value λ_1 of multiplicity m. Hence there are less than m linearly independent solutions of the differential equation (3.28) of the form $\alpha^{(1)} e^{\lambda_1 t}$ corresponding to λ_1 and there is no fundamental set of solutions of the form $\alpha^{(k)} e^{\lambda_k t}$, λ_k being a characteristic value of A and $\alpha^{(k)}$ is a characteristic vector corresponding to λ_k. We, therefore, seek linearly independent solutions of another form.

For this, we suppose that λ_1 is a characteristic value of multiplicity $m = 2$ and $p = 1(< m)$. We then seek linearly independent solutions of the form $\alpha e^{\lambda_1 t}$, $\alpha t e^{\lambda_1 t} + \beta e^{\lambda_1 t}$, where is a characteristic vector corresponding to λ_1, that is, α satisfies the equation

$$(A - \lambda_1 I)\alpha = 0$$

and β is a vector that satisfies the equation

$$(A - \lambda_1 I)\beta = \alpha$$

If λ_1 is a characteristic vector of multiplicity $m > 2$ and $p < m$, then the forms of the m linearly independent solutions corresponding to λ_1 depend upon whether $p = 1, 2, \ldots, m - 1$. However, we omit the case here.

EXERCISES

1. In each of exercises (i) and (ii) determine whether the matrix B is a fundamental matrix of the corresponding linear system $\frac{dx}{dt} = Ax$ where $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$:

(i) $B = \begin{pmatrix} e^{4t} & 0 & 2e^{4t} \\ 2e^{4t} & 3e^{t} & 4e^{4t} \\ e^{4t} & e^{t} & 2e^{4t} \end{pmatrix}$, $A = \begin{pmatrix} 1 & -3 & 9 \\ 0 & -5 & 18 \\ 0 & -3 & 10 \end{pmatrix}$ [Ans. No]

(ii) $B = \begin{pmatrix} e^{t} & e^{2t} & e^{2t} \\ e^{t} & -e^{2t} & 0 \\ 3e^{t} & 0 & e^{2t} \end{pmatrix}$, $A = \begin{pmatrix} 3 & 1 & -1 \\ 1 & 3 & -1 \\ 3 & 3 & -1 \end{pmatrix}$ [Ans. Yes]

2. For each of the non-homogeneous linear system, find (a) fundamental matrix of the corresponding homogeneous system and (b) find a solution of the given non-homogeneous system :

(i) $\frac{dx}{dt}=\begin{pmatrix}6 & -3\\ 2 & 1\end{pmatrix}x+\begin{pmatrix}e^{2t}\\ -e^{2t}\end{pmatrix}$ $\left[\text{Ans (a)}\begin{pmatrix}e^{3t} & 3e^{4t}\\ e^{3t} & 2e^{4t}\end{pmatrix}, \text{(b)}\begin{pmatrix}2e^{2t}\\ 3e^{2t}\end{pmatrix}\right]$

(ii) $\frac{dx}{dt}=\begin{pmatrix}3 & 1\\ 4 & 3\end{pmatrix}x+\begin{pmatrix}-2\sin t\\ 6\cos t\end{pmatrix}$

$$\left[\text{Ans (a)}\begin{pmatrix}e^{t} & e^{5t}\\ -2e^{t} & 2e^{5t}\end{pmatrix}, \text{(b)}\begin{pmatrix}\cos t\\ \sin t-3\cos t\end{pmatrix}\right]$$

3. Find the general solution of each of the homogeneous linear systems where in each exercise $x=\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}$

(i) $\frac{dx}{dt}=\begin{pmatrix}1 & -1 & -1\\ 1 & 3 & 1\\ -3 & 1 & -1\end{pmatrix}x$ $\left[\text{Ans : }\begin{array}{l}x_1=C_1e^{2t}+C_2e^{3t}+C_3e^{-2t}\\ x_2=-C_2e^{3t}-C_3e^{-2t}\\ x_3=-C_1e^{2t}-C_2e^{3t}+4C_3e^{-2t}\end{array}\right]$

(ii) $\frac{dx}{dt}=\begin{pmatrix}-5 & -12 & 6\\ 1 & 5 & -1\\ -7 & -10 & 8\end{pmatrix}x$ $\left[\text{Ans : }\begin{array}{l}x_1=C_1e^{t}+2C_3e^{4t}\\ x_2=C_2e^{3t}-C_3e^{4t}\\ x_3=C_1e^{t}+2C_2e^{3t}+C_3e^{4t}\end{array}\right]$

(iii) $\frac{dx}{dt} = \begin{pmatrix} 1 & -2 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix} x$

$$\left[\text{Ans : } \begin{array}{l} x_1 = -2C_1 e^{(2+\sqrt{5})t} + 2C_2 e^{(2-\sqrt{5})t} \\ x_2 = (\sqrt{5}+1)C_1 e^{(2+\sqrt{5})t} + (\sqrt{5}-1)C_2 e^{(2-\sqrt{5})t} \\ x_3 = C_3 e^{2t} \end{array} \right]$$

3.5 Summary : The linear system of n differential in n unknown functions can be written in the vector-matrix from

$$\frac{d\vec{x}}{dt} = A(t)\vec{x} + \vec{F}(t)$$

The solution of the system consists of two parts : The **general solution** of the homogeneous linear system

$$\frac{d\vec{x}}{dt} = A(t)\vec{x}$$

which can be obtained if we can find out n linearly independent (fundamental) set of vector functions $\phi_1, \phi_2, \ldots, \phi_n$ which are the solutions of the above equation and a **particular solution** of the non-homogeneous equation. However, if $A(t)$ is a constant matrix, the solutions can be obtained very easily.

Unit : 4 ☐ Second Order Linear Differential Equations

4.0 Introduction : If we consider a second order linear equation with variable coefficients we cannot in general, find the explicit form of the solutions but in many cases we can predict the properties of the solutions even without solving them. It can also be seen that the non-trivial solutions of general second-order equations can vanish at most once or can vanish periodically infinite number of times. Further it can be shown that the zeros of two linearly independent solutions separate one another that is to say that between two successive zeros of one solution lies a zero of the other (Sturm separation theorem). It is also possible to compare the number of zeros of solutions of two different equations in normal form $u'' + p(x)u = 0$ and $v'' + q(x)v = 0$, where $p(x) \geq q(x)$. (Sturm comparison theorem). Lastly Sturm-Liouville systems witn separated boundary conditions is considered and notion of eigen solutions of this system is introduced. The orthogonal property of eigen functions of such S-L systems is established and this property enables us to expand an orbitrary function which is sufficiently smooth in an infinite series of constant multiples of eigen functions of a Sturm-Liouville system. The constants may be determined formally by integration.

4.1 Bases of Solutions : Let us consider the second order linear differential equations of the form

$$p_0(x)\frac{d^2u}{dx^2} + p_1(x)\frac{du}{dx} + p_2(x)u = p_3(x) \tag{4.1}$$

where $p_i(x)(i = 0, 1, 2, 3)$ are assumed to be continuous and real-valued on an interval $I : a \leq x \leq b$ (finite or infinite) of the real axis. In the present chapter we shall devote ourselves to the second order linear differential equations and the behaviour of their solutions.

The equation in its normal form is

$$\frac{d^2u}{dx^2} + p(x)\frac{du}{dx} + q(x)u = r(x) \tag{4.2}$$

where $$p(x) = \frac{p_1(x)}{p_0(x)},\ q(x) = \frac{p_2(x)}{p_0(x)}, r(x) = \frac{p_3(x)}{p_0(x)}, \tag{4.3}$$

provided $p_0(x) \neq 0$. If $p_0(x_0) = 0$ at some point $x = x_0$, then the functions $p(x)$ and $q(x)$ are not defined at $x = x_0$ and we say that the differential equation (4.1) has a singular point at x_0.

For example, the Legendre differential equation

$$\frac{d}{dx}\left\{(1-x^2)\frac{du}{dx}\right\} + \lambda u = 0$$

has singular points at $x = \pm 1$.

Now we know that if $f_1(x)$ and $f_2(x)$ are any two linearly independent solutions of the reduced equation (homogeneous linear differential equation) of (4.2) i.e., of

$$\frac{d^2u}{dx^2} + p(x)\frac{du}{dx} + q(x)u = 0 \tag{4.4}$$

then

$$f(x) = c_1 f_1(x) + c_2 f_2(x) \tag{4.5}$$

is also a solution of (4.4). A pair of functions with this property is termed as a **basis** of solutions.

We proceed to show how to construct a basis of solutions of any second-order linear differential equation with constant coefficients given by

$$u'' + pu' + q = 0 \tag{4.6}$$

where p and q are constants and prime denote differentiation with respect to x.

Putting $u = e^{-px/2}v(x)$ in (4.6) we get

$$v'' + \left(q - \frac{p^2}{4}\right)v = 0, \quad v = e^{px/2}u. \tag{4.7}$$

Here, three cases arise according as the discriminant $\Delta = p^2 - 4q$ is positive zero or negative.

Case 1 : If $\Delta > 0$, then (6,7) reduces to $v'' = k^2 v$, where $k = (\sqrt{\Delta}/2)$ and has the functions $v = e^{kx}$ and $v = e^{-kx}$ as a basis of solutions so that

$$u = e^{(\sqrt{\Delta}-p)x/2} \text{ and } u = e^{(-\sqrt{\Delta}-p)x/2} \tag{4.8a}$$

are a basis of solutions of (4.6) when $p^2 - 4q > 0$.

Case 2 : If $\Delta = 0$, (6.7) gives $v'' = 0$ having 1 and x as basis of solutions. Hence the pair

$$u = e^{-px/2} \text{ and } u = xe^{-px/2} \tag{4.8b}$$

is a basis of solutions of (4.6) when $p^2 = 4q$.

Case 3 : If $\Delta < 0$, (6.7) reduces to $v'' + k^2 v = 0$ with $k = \sqrt{-\Delta}/2$ and has cos kx and sin kx as a basis of solutions, thus

$$u = e^{-px/2}\cos(\sqrt{-\Delta}x/2) \text{ and } u = e^{-px/2}\sin(\sqrt{-\Delta}x/2) \tag{4.8c}$$

form a basis of solutions of (4.6) when $p^2 - 4q < 0$.

4.2 Initial Value Problems : From physical considerations, there arise differential equations satisfying additional initial and boundary conditions. As an example we consider the following initial value problem :

Example 4.1 : To solve the equation $u'' + u = 3 \sin 2x$ which satisfy the initial conditions $u(0) = 0$, $u'(0) = 0$.

It is easy to see that the solution of the given equation is

$$u(x) = A\cos x + B\sin x - \sin 2x$$

Since $u(0) = 0$, $u'(0) = 0$, we have $A = 0$, $B = 2$. Thus the solution of the given equation satisfying the given initial conditions is

$$u(x) = 2 \sin x - \sin 2x$$

4.3 Qualitative Behaviour—Stability : We note that when $\Delta < 0$, al non-trivial solutions of (4.6) are oscillatory in the sense of changing sign infinitely often. On the other hand, when $\Delta \geq 0$, a non-trivial solution of (4.6) cai vanish only when $ae^{\alpha x} = -be^{\beta x}$, i.e., $e^{(\alpha-\beta)x} = -b/a$ so that (i) a and b have opposite signs and (ii) $x = \ln|b/a|/(\beta - \alpha)$. Hence a non-trivial solution can chang sign at most once; it is non-oscillatory. Thus we have the result :

Theorem 4.1 : If $\Delta \geq 0$, then a non-trivial solution of (4.6) can vanish at most once; and if $\Delta < 0$, it vanishes periodically with period $\pi/\sqrt{-\Delta}$.

Definition : The homogeneous linear differential equation (4.4) is called **strictly stable** when every solution tends to zero as $x \to \infty$; it is **stable** wiv every solution is bounded as $x \to \infty$. If not stable, the solution is **unstable.**

6.4 Uniqueness Theorem : Let us consider the differential equation (4.1) or its normal form (4.2) and assume that there are no singular points, i.e., $p_0(x) \neq 0$. We consider the initial value problem and show that there always exists a solution for any initial value (u_0, u_1), $u(x_0) = u_0$, $u'(x_0) = u_1$.

Put $L[u] = (p_0D^2 + p_1D + p_2)u$, ($D \equiv d/dx$) and suppose that u and v are any two solutions of the inhomogeneous equation $L[u] = p_3(x)$. Noting that the operator L is linear, we have

$$L[u - v] = L[u] - L[v] = p_3(x) - p_3(x) = 0$$

i.e., $u - v$ is a solution of the homogeneous equation (4.4).

It is easy to verify that if $u(x)$ is a solution of $L[u] = r(x)$ and $v(x)$ is a solution of $L[u] = s(x)$ and A, B are constants, then $w(x) = Au(x) + Bv(x)$ is a solution of the differential equation $L[u] = Ar(x) + Bs(x)$.

Now we prove a uniqueness theorem for second-order linear differential equations.

Theorem 4.2—(Uniqueness Theorem) : If $p(x)$ and $q(x)$ are continuous, then at most one solution of (4.2) can satisfy the given initial conditions $u(a) = b_0$ and $u'(a) = b_1$.

Proof : We have seen that if $v(x)$ and $w(x)$ are any two solutions of (4.2), then $U = v - w$ satisfies the equation

$$U'' + p(x)U' + q(x)U = 0 \tag{4.9}$$

where $U(x)$ satisfies the initial conditions $U(a) = U'(a) = 0$. Now consider the non-negative function $V(x) = U^2 + U'^2$ with $V(a) = 0$, by definition. Differentiating we have,

$$V'(x) = 2U'(U + U'') = 2U'[U - p(x)U' - q(x)U]$$

$$= -2p(x)U'^2 + 2[1 - q(x)]UU'.$$

Since $(U \pm U') \geq 0$, it follows that $|2UU'| \leq U^2 + U'^2$ and, therefore,

$$2[1 - q(x)]UU' \leq (1 + |q(x)|)(U^2 + U'^2)$$

$$\text{and } V'(x) \leq [1 + |q(x)|]U^2 + [1 + |q(x)| + |2p(x)|]U'^2$$

Thus, if $k = 1 + \max[|q(x)| + 2|p(x)|]$, the maximum being over any finite closed interval $[a, b]$, we obtain $V'(x) \le kV(x),\ k < +\infty$.

$$\text{i.e., } 0 \ge e^{-kx}[V'(x) - kV(x)] = \frac{d}{dx}[V(x)e^{-kx}]$$

so that $V(x)e^{-kx} \le V(a)e^{-ka}$. Since $V(a) = 0$, we have $V(x) = 0,\ \forall x \in [a,b]$, i.e., $U(x) \equiv 0$ and $v(x) = w(x)$ on the interval.

By superposition principle, we can prove easily the following theorem.

Theorem 4.3 : Let $f(x)$ and $g(x)$ be two solutions of the homogeneous second order linear differential equation

$$u'' + p(x)u' + q(x)u = 0 \tag{4.4}$$

and $(f(x_0), f'(x_0))$ and $(g(x_0), g'(x_0))$ be the initial conditions of $f(x)$ and $g(x)$ for some $x = x_0$. Then every solution of this differential equation is equal to some linear combination $h(x) = A\,f(x) + B\,g(x)$ of $f(x)$ and $g(x)$, A and B being constants.

4.5 Separation aad Comparison Theorems : We now show by Sturm separation theorem thai all non-trivial solutions of 4.4 have essentially the same number of oscillations or zeros. (A 'zero' of a function is defined to be a point at which the value of the function is zero; functions have two zeros in each complete oscillation).

Theorem 4.4 (Sturm Separation Theorem) : Let $f(x)$ and $g(x)$ are two linearly independent solutions of the differential equation (4.4). Then $f(x)$ must vanish at one point between two successive zeros of $g(x)$ and, similarly $g(x)$ must vanish at one point between two successive zeros of $f(x)$. In other words, the zeros of $f(x)$ and $g(x)$ occur alternatively.

Proof : Let $g(x)$ vanishes at $x = x_i$. Since $f(x)$ and $g(x)$ are linearly independent, the Wronskian

$$W(f, g; x_i) = \begin{vmatrix} f(x_i) & f'(x_i) \\ g(x_i) & g'(x_i) \end{vmatrix} \ne 0$$

i.e., $f(x_i)g'(x_i) \ne 0$ if $g(x_i) = 0$, showing that $f(x_i) \ne 0,\ g'(x_i) \ne 0$. If x_1 and x_2 are two successive zeros of $g(x)$, then $g'(x_1)$, $g'(x_2)$, $f(x_1)$ and $f(x_2)$ are all non-zero. Moreover, the non-zero numbers $g'(x_1)$ and $g'(x_2)$ cannot have the same sign, because if the function is decreasing at $x = x_1$, then it must be increasing at $x = x_2$ and vice-versa. Since $W(f, g; x_i)$ has constant sign, it follows that $f(x_1)$ and $f(x_2)$ must have opposite signs. Therefore $f(x)$ must vanish somewhere between x_1 and x_2.

We can use a slightly refinement of the above Theorem to prove more useful result, also due to Sturm.

Theorem 4.5 (Sturm Comparison Theorem) : Let $f(x)$ and $g(x)$ be two non-trivial solutions of the differential equations $u'' + p(x)u = 0$ and $v'' + q(x)v = 0$ respectively where $p(x) \geq q(x)$. Then $f(x)$ vanishes at least once between two zeros of $g(x)$, unless $p(x) \equiv q(x)$ and f is a constant multiple of g.

Proof : Let x_1 and x_2 be two successive zeros of $g(x)$, i.e., $g(x_1) = g(x_2) = 0$ and $f(x) \neq 0$ at any point in $x_1 < x < x_2$. Replacing $f(x)$ and/or g by their negative, if necessary, we can obtain solutions f and g positive on $x_1 < x < x_2$. This would make

$$W(f, g; x_1) = f(x_1)g'(x_1) \geq 0 \text{ and } W(f, g; x_2) = f(x_2)g'(x_2) \leq 0$$

On the other hand, by noting $f > 0$, $g > 0$ and $p \geq q$ on $x_1 < x < x_2$, we have

$$\frac{d}{dx}[W(f, g; x)] = fg'' - gf'' = (p - q)fg \geq 0 \text{ on } x_1 < x < x_2$$

showing that W is non-decreasing, giving a contradiction unless

$$p - q \equiv W(f, g; x) \equiv 0.$$

In this event, $f \equiv kg$ for some constant k.

Corollary 4.1 : No non-trivial solution of $u'' + p(x)u = 0$ can have more than one zero if $p(x) \leq 0$.

Proof : We prove the corollary by contradiction. By Sturm comparison theorem, it is evident that the solution $v \equiv 1$ of the differential equation $v'' = 0$ must vanish at least once between the two zeros of any non-trivial solution of the differential equation $u'' + p(x)u = 0$. The preceding results show that the oscillations of $u'' + p(x)u = 0$ are largely determined by the sign and magnitude of $p(x)$. If $p(x) \leq 0$, oscillations are impossible and no solution can change its sign more than once. On the other hand, when $p(x) \geq k^2 > 0$, then any solution of $u'' + p(x)u = 0$ must vanish between any two successive zeros of any given solution $A\cos k(x-x_1)$ of the differential equation $u''+k^2u=0$ and hence in any interval of length π/k.

4.6 Sturm-Liouville Systems : A second order homogeneous linear differential equation of the form

$$\frac{d}{dx}\left\{p(x)\frac{du}{dx}\right\}+\{\lambda\rho(x)-q(x)\}u=0 \tag{4.10}$$

where λ is a parameter, p, ρ and q are real-valued functions of x, p and ρ being positive, is called **Sturm-Liouville equation.** Writing $L=D\{[p(x)D\}-q(x)$, the equation (4.10) is abbreviated in the form

$$L[u]+\lambda\rho(x)u=0 \tag{4.11}$$

Such type of equation is self-adjoint for real λ. The functions q and ρ are assumed to be continuous and the function p is continuously differentiable (of class c^1) so that the functions are bounded in an interval $a \le x \le b$ (say). The jf'irm-Liouville equation is said to be **regular** in the interval $a \le x \le b$ if $p(x)$ and $\rho(x)$ are positive in the interval.

For each λ, a regular Sturm-Liouville system (or S-L system) for $a \le x \le b$ has a basis of two linearly independent solutions of class c^2. An **S-L system is an S-L equation** with boundary conditions to be satisfied by the solutions, for example $u(a)=u(b)=0$, or, **two separated boundary conditions**

$$\alpha_1u(a)+\alpha_2u'(a)=0,\ \ \beta_1u(b)+\beta_2u'(b)=0 \tag{4.12}$$

where α_1, α_2, β_1, β_2 are given real numbers. We shall exclude the two trivial conditions $\alpha_1=\alpha_2=0$ and $\beta_1=\beta_2=0$.

A non-trivial solution of an S-L system is called an **eigenfunction** and the corresponding X is called its **eigenvalue.** Also each eigenfunction is said to **belong** to its eigenvalue. The set of all eigenvalues of a regular S-L system is called the **spectrum** of the system.

Example 4.2 : For the interval $0 \le x \le \pi$, the system of the differential equation $u''+\lambda u=0$ with the boundary conditions $u(0)=0$, $u(\pi)=0$ has the eigenfunctions $u_n(x)$ = sin nx and the eigenvalues $\lambda_n=n^2$, (n = 1, 2, 3, ...)

Example 4.3 : The Bessel equation given by

$$\frac{d}{dx}\left[x\frac{dx}{dx}\right]+\left(k^2x-\frac{n^2}{x}\right)u=0, \quad a\leq x\leq b$$

is an S-L equation with $p = \rho = x$, $\lambda = k^2$ and $q = n^2/x$. For $0 < a < b$, a regular S-L system is obtained by imposing the boundary conditions $u(a) = u(b) = 0$ or any other separated boundary conditions. The system does not define a regular S-L system for $a = 0$, because the coefficient $p(x)$ vanishes at $x = 0$. We then obtain a **singular S-L system.**

Periodic boundary conditions : If the coefficients of S-L equations are periodic functions with period $b - a$, sometimes the periodic boundary conditions

$$u(a) = u(b),\ u'(a) = u'(b) \tag{4.13}$$

are used and we get another type of S-L system, called a **periodic S-L system.**

Example 4.4 : For the system $u'' + \lambda u = 0$ in $-\pi \leq x \leq \pi$ with periodic boundary conditions $u(-\pi) = u(\pi)$ and $u'(-\pi) = u'(\pi)$ we have the eigen functions 1, cos nx and sin nx, where n is any positive integer. The corresponding eigenvalues are the squares of integers; for $n > 0$, there are two linearly independent eigenfunctions with the same eigenvalue n^2.

4.7 Sturm-Liouville Series : We now show that for the regular S-L systems generally and for the S-L systems with periodic boundary conditions, orthogonalities hold.

Definition : Two integrable real-valued functions f and g with weight function $\rho > 0$ on an interval I are said to be orthogonal if and only if

$$\int_I \rho(x)f(x)g(x)dx = 0 \tag{4.14}$$

where the interval I may be finite and open or closed at either end; or it may be semi-infinite or infinite.

Theorem 4.6 : Eigenfunctions of a regular S-L system (4.10) or (4.11) having different eigenvalues are orthogonal with weight function ρ, i.e., if u and v are different eigenfunctions to distinct eigenvalues λ and μ, then

$$\int_a^b \rho(x)u(x)v(x)dx = 0$$

Proof : Let $L[u] = [p(x)u']' - q(x)u$. Now the functions u and v are eigenfunctions of (4.10) with eigenvalues λ and μ iff

$$L[u] + \lambda\rho(x)u = L[v] + \mu\rho(x)v = 0 \tag{4.15}$$

To prove the theorem, we first establish the following lemma :

Lemma 4.1 : If u and v satisfy an S-L equation (4.10) on a closed interval $a \le x \le b$, then for all values of the parameters λ and μ

$$(\lambda - \mu)\int_a^b \rho(x)u(x)v(x)dx = [p(x)\{u(x)v'(x) - v(x)u'(x)\}]_a^b \tag{4.16}$$

Proof of the lemma : From Lagrange identity

$$uL[v] - vL[u] = \frac{d}{dx}[p(x)\{u(x)v'(x) - v(x)u'(x)\}]$$

we have by integration with respect to x between the boundary points $x = a$ and $x = b$ and substituting $L[u] = -\lambda\rho u$ and $L[u] = -\mu\rho v$ from (4.15), the required result (4.16) is obtained.

The right hand side of (4.16) is called the boundary term.

Proof of the theorem : To prove the theorem, we show that the boundary term of (4.16) vanishes in the case of separated boundary conditions. From the first of the separated boundary conditions (4.12) we have

$$\alpha_1 u(a) + \alpha_2 u'(a) = 0 \text{ and } \alpha_1 v(a) + \alpha_2 v'(a) = 0$$

so that

$$p(a)[u(a)u'(a) - v(a)u'(a)] = \frac{\alpha_1}{\alpha_2}\, p(a)[u(a)v(a) - v(a)u(a)] = 0,$$

provided $\alpha_2 \neq 0$. If $\alpha_2 = 0$, the R.H.S. of (4.16) reduces similarly at $x = a$ to

$$\frac{\alpha_2}{\alpha_1}\, p(a)[u(a)v'(a) - v(a)u'(a)] = 0$$

Hence, unless $\alpha_1 = \alpha_2 = 0$, we have $p(a)[u(a)v'(a) - v(a)u'(a)] = 0$. Similar result holds when $x = b$, provided $\beta_1 \neq 0$, $\beta_2 \neq 0$. Thus excluding the possibilities $\alpha_1 = \alpha_2 = 0$, $\beta_1 = \beta_2 = 0$, the R.H.S. of (4.16) vanishes. Hence, noting that $\lambda \neq \mu$, we have from (4.16)

$$\int_a^b \rho(x)u(x)v(x)dx = 0$$

Corollary 4.2 : The results of Theorem 4.6 also holds for S-L systems with periodic boundary conditions. For, in this case, the R.H.S. of (4.16) for $x = a$ and $x = b$ are equal in magnitude but opposite in sign and, therefore, they cancel each other.

Now we know that any smooth periodic function $f(x)$ can be expanded into a Fourier series.

$$f(x) - a_0 + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx)$$

which is an infinite linear combination of the eigenfunctions of the S-L system, of Exampie 4.4, Moreover, we can easily find out a_k and b_k by the use of orthogonality of the eigenfunctions.

We can obtain the expressions for general $f(x)$ in the eigenfunctions of other S-L systems from the orthogonality relations obtained above; the resulting infinite series are called **Sturm-Liouville Series** and **this series converges to $f(x)$**.

4.8 Singular Systems : For an S-L equation, the interval I may be finite, semi-infinite or infinite. If I is finite, it may include neither, one or both the boundary points. If $\lim_{x\to a} p(a) = 0$, $\lim_{x\to a} \rho(a) = 0$ or if any one of the functions p, q or ρ is singular at $x = a$, then we exclude this boundary point a. When I is a closed, finite interval $a \leq x \leq b$, then an S-L equation is associated with a regular S-L system. If I is semi-infinite or infinite or if it is infinite and p or ρ vanishes at one or both boundary points or if q is discontinuous, then a regular S-L system cannot be obtained for S-L equation (4.10). In such cases, the S-L equation is called **singular**.

A **Singular S-L Systems** can be obtained from singular S-L equations by imposing suitable homogeneous linear boundary conditions, which can be described by formulas like (4.12). For example, the boundedness of u near a singuiar boundary point is a boundary condition defining a singular S-L system.

Definition : A real-valued function $f(x)$ is said to be **square-integrable** on the interval / relative to a given weight function $\rho(x) > 0$ if

$$\int_I f^2(x)\rho(x)dx < +\infty. \tag{4.17}$$

If the weight function $\rho(x) \equiv 1$, then we simply say that f **is sijuare-integrable on** ***I*****.**

The eigenfunctions of singular S-L systems are orthogonal provided they are square-integrable relative to the weight function ρ.

4.9 The Sequence of Eigenfunctions : The existence of an infinite sequence of eigenfunctions of a regular S-L system with the separated boundary-conditions can be shown with the help of the following theorem whicn we state runout proof :

Theorem 4.7 : A regular S-L system has an infinite sequence of real eigenvalues $\lambda_0 < \lambda_1 < \lambda_2 < \cdots$ with $\lim_{n\to\infty} \lambda_n = \infty$ and the corresponding eigenfunctions $u_n(x)$ of λ_n has exactly n zeros in the interval determined uniquely upto a constant factor.

4.10 Expansions in Eigenfunctions : All sufficiently smooth functions can be expanded in the form of an infinite series whose terms are constant multiples of the eigenfunctions of S-L system. The result will be proved for regular S-L systems. As an example, we have the expansion into eigenfunctions by the expansion in Fourier series. At first we recall two basic results in Fourier series :

Fourier's Convergence Theorem : If $f(x)$ be any continuously differentiable periodic function of period 2π and

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kx\, dx, b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin kx\, dx, \tag{4.18}$$

then the infinite series

$$\frac{1}{2}a_0 + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx) \tag{4.19}$$

converges uniformly to $f(x)$.

In this case, the non-zero terms $a_k \cos kx$, $b_k \sin kx (k = 1, 2, ...)$ are eigenfunctions of the periodic S-L system in question (Example 4.4). Hence $f(x)$ is represented as a sum of eigenfunctions.

In the sequel, we shall consider normalised cos. kx and sin kx in (4.19) as the eighenfunctions of the system. Although there exist continuous functions for which the Fourier series do not converge, but the following theorem applies to all continuous periodic functions.

Fejér's Convergence theorem : If $f(x)$ be any continuously differentiable periodic function of period 2π and

$$\sigma_N(x) = \frac{1}{N}\left\{\sum_{n=0}^{N-1}\left[\frac{1}{2}a_0 + \sum_{k=1}^{n}(a_k \cos kx + b_k \sin kx)\right]\right\}$$

$$= \frac{1}{2}a_0 + \sum_{k=1}^{N-1}\left(\alpha_k^N \cos kx + \beta_k^N \sin kx\right), \tag{4.20}$$

where $\alpha_k^N = \left(1 - \frac{k}{N}\right)a_k$, $\beta_k^N = \left(1 - \frac{k}{N}\right)b_k$ are the arithmetic mean of the first N paritial sums of the Fourier series of $f(x)$. Then the sequence of functions $\sigma_N(x)$ converges uniformly to $f(x)$.

Now we suppose that the function $f(x)$ is continuous on $0 \le x \le \pi$. We define a function $g(x)$ on $-\pi \le x \le \pi$ by $g(x) = f(|x|)$. Noting that $g(x)$ can be expanded to an even periodic function (since $g(-\pi) = g(\pi)$), of period 2π, by symmetry all $b_k (k = 1, 2, ...)$ are zero in the Fourier series of $g(x)$. Thus applying the above two convergence theorems we have the result :

Any continuous function on $0 \le x \le \pi$ can be approximated uniformly and arbitrarily by linear combinations of cosine functions. If the function is of class c^1 and $f'(0) = f'(\pi) = 0$, then it can be expanded into a uniformly convergent cosine series

$$f(x) = \frac{1}{2}a_0 + \sum_{k=1}^{\infty} a_k \cos kx$$

Similarly, we can obtain the result :

Any function of class c^1 on $0 \leq x \leq \pi$ satisfying $f(0) = f(\pi) = 0$ can be expanded into a uniformly convergent sine series

$$f(x) = \sum_{k=1}^{\infty} b_k \sin kx$$

The above results are examples of expansions of eigenfunctions of the regular S-L system $u'' + \lambda u = 0$ with separated boundary conditions $u'(0) = u'(\pi) = 0$ and $u(0) = u(\pi) = 0$ respectively. We row show that analogous expansions are possible for the eigenfunctions of any regular S-L system.

Orthogonal expansions : Let $\phi_k(x)(k = 1, 2, 3, ...)$ be any bounded square-integrable functions on an interval $I : a < x < b$, orthogonal with respect to a weight function $\rho(x) > 0$ so that

$$\int_I \phi_h(x)\phi_k(x)\rho(x)dx = 0 \text{ for } h \neq k, \tag{4.21}$$

and a function $f(x)$ can be expanded as the limit of a uniformly convergent series of multiples of the ϕ_k, i.e.

$$f(x) = \sum_{h=1}^{\infty} c_h \phi_h(x) \tag{4.22}$$

Multiplying both sides of (4.22) by $\phi_k(x)\rho(x)$ and integrating term-by-term, we get by the use of the orthogonality relations (4.21)

$$\int_I f(x)\phi_k(x)\rho(x)dx = \sum_{h=1}^{\infty}\int_I c_h\phi_h(x)\phi_k(x)\rho(x)dx = c_k\int_I \phi_k^2(x)\rho(x)dx$$

so that

$$c_n = \frac{\int_I f(x)\phi_n(x)\rho(x)dx}{\int_I \phi_n^2(x)\rho(x)dx} \tag{4.23}$$

Definition : The orthogonal functions ϕ_k are said to be **orthonormal** if $\int_I \phi_k^2(x)\rho(x)dx = 1$. Thus for a sequence of orthonormal functions, we have

$$c_k = \int_I f(x)\phi_k(x)\rho(x)dx$$

EXERCISES

1. Show that

$$a_k \cos kx + b_k \sin kx = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos[k(t-x)]dt$$

2. Show that

$$\frac{1}{2} + \sum_{k=1}^{n} \cos kx = \frac{\sin[(2n+1)x/2]}{2\sin(x/2)}$$

3. For the following regular S-L systems, find the eigenvalues and eigenfunctions and obtain an expansion formula for a function $f \in c^1$ into a series of eigenfunctions :

(i) $u'' + \lambda u = 0,\ u(0) = u'(\pi) = 0,\ 0 \le x \le \pi,$

(ii) $u'' + \lambda u = 0,\ u'(0) = u(\pi) = 0,\ 0 \le x \le \pi.$

4.8. Summary : The two important theorems, viz., Sturm's separation theorem and comparison theorem are discussed here wherefrom we can obtain the properties of the solutions of the second order linear ODE without having to solve the equation itself. Sturm-Liouville systems with specified separated boundary conditions are given as well as the notion of eigenunctions and their orthogonal properties are touched upon.

Unit : 5 ❑ Green's Function

5.0 Introduction : Suppose that we want to solve a non-homogeneous equation $Lu[x] = f(x)$, where L is a Sturm-Liouville operator, subject to some boundary conditions at the end points $x = a, b$ of the interval. If we can find the two linearly independent solutions of the homogeneous equation $L[u] = 0$, the solution of the non-homogeneous equation can be obtained in an integral form

$$u(x) = \int_a^b G(x,\xi) f(\xi) d\xi.$$

This function $G(x, \xi)$ is called the **Green's function** of the problem. This function has the symmetry property, some continuous and discontinuous properties and it satisfies the homogeneous equation $LG = 0$.

The Green's function can be formed from its properties and once it is found, the solution of the non-homogeneous equation is obtained in a compact integral form. The same method may be extended to non-homogeneous linear equations of higher order.

5.1 Introduction : Let us consider the non-homogeneous differential equation

$$Lu(x) = f(x) \tag{5.1}$$

where L is an ordinary differential operator, $f(x)$ is a known function and $u(x)$ is an unknown function. One method of solving the differential equation (5.1) is to find the operator L^{-1} in the form of an integral operator with a kernel $G(x, \xi)$ such that

$$u(x) = L^{-1} f(x) = \int G(x,\xi) f(\xi)\, d\xi \tag{5.2}$$

The kernel $G(x, \xi)$ of this integral operator is called **Green's function of the differential operator.**

Now applying the operator L on both sides of (5.2) we get

$$f(x) = LL^{-1} f(x) = \int LG(x,\xi) f(\xi)\, d\xi \tag{5.3}$$

This equation is satisfied if Green's function $G(x, \xi)$ is chosen in such a way that

$$LG(x,\xi) = \delta(x-\xi) \tag{5.4}$$

where $\delta(x-\xi)$ is Dirac δ-function (defined in Appendix).

Sturm-Liouville equation : Let us consider the Sturm-Liouville equation

$$L\,u(x) = f(x) \tag{5.5}$$

where $L = \frac{d}{dx}\left\{p(x)\frac{d}{dx}\right\} - q(x)$ is the Sturm-Liouville operator, $p(x)$ and $q(x)$ are real-valued functions of x and $p(x)$ is positive. [In fact, any second-order linear differential operator can be transformed into Sturm-Liouville operator after multiplying by an appropriate function.] Our object is to solve the equation (5.5) subject to certain boundary conditions satisfied by $u(x)$ at $x = a$ and $x = b$.

Let us introduce a function $U(x)$ which satisfies the homogeneous equation

$$LU(x) = \frac{d}{dx}\left\{p(x)\frac{d}{dx}\right\} - q(x)U(x) = 0 \tag{5.6}$$

Multiplying (5.5) by $U(x)$ and (5.6) by $u(x)$ and then subtracting we get

$$U(x)\frac{d}{dx}\left(p\frac{du}{dx}\right) - u(x)\frac{d}{dx}\left(p\frac{dU}{dx}\right) = U(x)f(x)$$

Integrating between the limits a and x, we obtain

$$\int_a^x\left[U(\xi)\frac{d}{d\xi}\left(p\frac{du}{d\xi}\right) - u(\xi)\frac{d}{d\xi}\left(p\frac{dU}{d\xi}\right)\right]d\xi = \int_a^x U(\xi)f(\xi)d\xi$$

which gives

$$p(x)[U(x)u'(x) - u(x)U'(x)] - p(a)[U(a)u'(a) - u(a)U'(a)]$$
$$= \int_a^x U(\xi)f(\xi)d\xi. \tag{5.7}$$

If we assume that on $x = a$

$$p(a)[U(a)u'(a) - u(a)U'(a)] = 0 \tag{5.8}$$

then

$$U(x)u'(x) - u(x)U'(x) = \frac{1}{p(x)}\int_a^x U(\xi)f(\xi)d\xi \tag{5.9}$$

Similarly choosing another function $V(x)$ satisfying the homogeneous equation $LV(x) = 0$ and the boundary condition

$$p(b)[V(b)u'(b) - u(b)V'(b)] = 0 \tag{5.10}$$

we get,

$$V(x)u'(x) - u(x)V'(x) = \frac{1}{p(x)}\int_b^x V(\xi)f(\xi)d\xi \tag{5.11}$$

Multiplying (5.9) $V(x)$ and (5.11) by $U(x)$ and then subtracting, we obtain

$$u(x)[U(x)V'(x) - U'(x)V(x)] = \frac{1}{p(x)}\Big[V(x)\int_a^x U(\xi)f(\xi)\,d\xi$$

$$+U(x)\int_x^b V(\xi)f(\xi)\,d\xi\Big] \tag{5.12}$$

which can be written in the form

$$u(x)\int_a^b G(x,\xi)f(\xi)\,d\xi$$

where

$$G(x,\xi) = \begin{cases} \dfrac{1}{p(x)W(x)} & V(x)U(\xi), \quad a \le \xi \le x \\ \dfrac{1}{p(x)W(x)} & U(x)V(\xi), \quad x \le \xi \le b \end{cases} \tag{5.13}$$

in the **Green's function of the problem** and

$$W(x) = U(x)V'(x) - U'(x)V(x) \tag{5.14}$$

is the **Wronskian.**

It is easy to verify that the product $p(x)W(x)$ is independent of x. For, we have

$$V(x)LU(x) = 0 \text{ and } U(x)LV(x) = 0$$

so that
$$V(x)\frac{d}{dx}(pU') = U(x)\frac{d}{dx}(pV')$$

$$\Rightarrow \frac{d}{dx}[p(x)\{U(x)V'(x) - U'(x)V(x)\}] = 0$$

$$\Rightarrow \frac{d}{dx}[p(x)W(x)] = 0$$

$$\Rightarrow [p(x)W(x)] = \text{const., independent of } x. \tag{5.15}$$

Some Properties of Green's function :

(a) Since $p(x)W(x)$ is independent of x, it follows from (5.13) that $G(x, \xi)$ is symmetric in x and ξ i.e.,

$$G(x, \xi) = G(\xi, x) \tag{5.16}$$

Thus we may rewrite (5.13) in the form

$$G(x,\xi) = \begin{cases} \frac{1}{p(\xi)W(\xi)} & V(\xi)U(x), \quad a \le x \le \xi \\ \frac{1}{p(\xi)W(\xi)} & U(\xi)V(x), \quad \xi \le x \le b \end{cases} \tag{5.17}$$

(b) The function $G(x, \xi)$ is continuous everywhere including the point $x = \xi$.

(c) $\frac{dG}{dx}$ and $p\frac{dG}{dx}$ are discontinuous at $x = \xi$. For, we have

$$\left(\frac{dG}{dx}\right)_{x=\xi-0} = \frac{1}{p(\xi)W'(\xi)}U'(\xi)V(\xi) \text{ and } \left(\frac{dG}{dx}\right)_{x=\xi+0} = \frac{1}{p(\xi)W'(\xi)}U(\xi)V'(\xi)$$

so that

$$\left(\frac{dG}{dx}\right)_{x=\xi+0} - \left(\frac{dG}{dx}\right)_{x=\xi-0} = \frac{1}{p(\xi)} \tag{5.18}$$

Also $\left(p(x)\frac{dG}{dx}\right)_{x=\xi-0} = \frac{1}{W(\xi)}U'(\xi)V(\xi)$ and $\left(p(x)\frac{dG}{dx}\right)_{x=\xi+0} = \frac{1}{W(\xi)}U(\xi)V'(\xi)$

so that

$$\left(p(x)\frac{dG}{dx}\right)_{x=\xi+0} - \left(p(x)\frac{dG}{dx}\right)_{x=\xi-0} = +1 \tag{5.19}$$

Hence the functions $\frac{dG}{dx}$ and $p(x)\frac{dG}{dx}$ have jumps $\frac{1}{p(\xi)}$ and +1 respectively at $x = \xi$. Therefore at $x = \xi$ we have

$$\frac{d}{dx}\left\{p(x)\frac{dG}{dx}\right\} = \delta(x-\xi) \tag{5.20}$$

(d) Noting that $G(x, \xi)$ is proportional to $U(x)$ in the domain $a \leq x \leq \xi$ and $LU = 0$, we have $LG = 0$ for $a \leq x \leq \xi$. Similarly, $LG = 0$ for $\xi \leq x \leq b$. Hence combining (5.13) and (5.20) we derive the result

$$LG(x,\xi) = \delta(x-\xi),\ a \leq x \leq b \tag{5.21}$$

(e) All the analyses above for the determination of the Green's function remain valid provided

$$W(x) = U(x)V'(x) - U'(x)V(x) \neq 0 \tag{5.22}$$

If $W(x) = 0$, then $\dfrac{U'(x)}{U(x)} = \dfrac{V'(x)}{V(x)} \Rightarrow V(x) = C_1U(x)$, C_1 being constant. Thus for the existence of Green's function, $V(x)$ must not be a multiple of $U(x)$. In such cases, the Green's function and, therefore, the solution of the inhomogeneous equation (5.5) does not exist for an arbitrary function $f(x)$.

Boundary Conditions : We have seen that Green's function can be determined if the two linearly independent solutions $U(x)$ and $V(x)$ of the homogeneous equation $Lu(x) = 0$ can be determined. These two solutions $U(x)$ and $V(x)$ respectively satisfy the conditions (5.8) and (5.10) at the boundaries $x = a$ and $x = b$. Now, if $p(a) \neq 0$ and $p(b) \neq 0$ then $u(x)$ satisfies the homogeneous boundary conditions given by

$$\begin{aligned} \alpha_1 u(a) + \alpha_2 u'(a) = 0 \\ \beta_1 u(b) + \beta_2 u'(b) = 0 \end{aligned} \tag{5.23}$$

If, in addition, $U(x)$ and $V(x)$ satisfy the boundary conditions

$$\alpha_1 U(a) + \alpha_2 U'(a) = 0, \quad \beta_1 V(b) + \beta_2 V'(b) = 0 \tag{5.24}$$

then the equations (5.8) and (5.10) are automatically satisfied. Thus $U(x)$ and $u(x)$ should satisfy the same condition at $x = a$. Similarly, $V(x)$ and $u(x)$ should satisfy the same condition at $x = b$.

On the other hand, if either $p(a)$ or $p(b)$ is zero, then $u(x)$ need not satisfy either of the equations (5.23) at the boundary of which $p(x) = 0$. If $p(a) = 0$, it is sufficient that $u(a)$ and $U(a)$ are finite. Similarly, if $p(b) = 0$, then $u(b)$ and $V(b)$ should be finite.

For some problems, it may be required to solve the equation (5.1) with inhomogeneous boundary conditions, i.e.,

$$\begin{aligned}\alpha_1 u(a)+\alpha_2 u'(a)&=\gamma_1,\\ \beta_1 u(b)+\beta_2 u'(b)&=\gamma_2.\end{aligned} \tag{5.25}$$

To solve such type of problems, we assume,

$$u(x)=u_1(x)+u_2(x) \tag{5.26}$$

where

$$Lu_1(x)=0 \text{ and } Lu_2(x)=f(x) \tag{5.27}$$

and $u_1(x)$ satisfies the inhomogeneous boundary conditions

$$\begin{aligned}\alpha_1 u_1(a)+\alpha_2 u_1'(a)&=\gamma_1,\\ \beta_1 u_1(b)+\beta_2 u_1'(b)&=\gamma_2\end{aligned} \tag{5.28}$$

while $u_2(x)$ satisfies the homogeneous boundary conditions

$$\begin{aligned}\alpha_1 u_2(a)+\alpha_2 u_2'(a)&=0\\ \beta_1 u_2(b)+\beta_2 u_2'(b)&=0\end{aligned} \tag{5.29}$$

Thus a given problem can be split into two parts : (i) to obtain the solution of the homogeneous equation $Lu_1(x)=0$ with inhomogeneous boundary conditions (5.28) and (ii) to find the solution of the inhomogeneous equation $Lu_2(x)=f(x)$ with homogeneous boundary conditions (5.29) by the use of Green's function method.

Example 5.1 : Solve the equation $\dfrac{d^2u}{dx^2}=f(x)$ subject to the boundary conditions $u(0) = u(1) = 0$.

Solution : We have $p(x) = 1$, $q(x) = 0$, $a = 0$, $b = 1$. The boundary conditions (5.24) give $U(0) = 0$, $V(1) = 0$. Since $LU=\dfrac{d^2U}{dx^2}=0$, we readily obtain $U(x) = Ax$ and similarly, $V(x) = B(1-x)$, where A and B are constants. The Wronskian is

$$W(x)=U(x)V'(x)-V(x)U'(x)=Ax(-B)-AB(1-x)=-AB, \text{ constant.}$$

Hence, using (5.17), the Green's function for the problem is

$$G(x,\xi) = \begin{cases} -x(1-\xi), & 0 \le x \le \xi, \\ -(1-x)\xi, & \xi \le x \le 1 \end{cases}$$

$$\text{i.e., } G(x,\xi) = -x(1-\xi) + (x-\xi)H(x-\xi),\ 0 < x, \xi < 1$$

where $H(x-\xi)$ is the unit step function defined by

$$H(x-\xi) = 1, \quad x \ge \xi$$
$$= 0, \quad x < \xi$$

Thus the solution of the given equation is

$$u(x) = -x\int_0^1 (1-\xi) f(\xi)\, d\xi + \int_0^x (x-\xi) f(\xi)\, d\xi$$

Example 5.2 : Solve the equation $\dfrac{d^2u}{dx^2} = f(x)$ subject to the non-homogeneous boundary conditions $u(0) = \alpha$ and $u(1) = \beta$.

Solution : Let $u(x) = u_1(x) + u_2(x)$, where $u_1(x)$ is the solution of the non-homogeneous equation $\dfrac{d^2u_1}{dx^2} = f(x)$ subject to the homogeneous boundary conditions $u_1(0) = 0$, $u_1(1) = 0$ and $u_2(x)$ is the solution of the equation $\dfrac{d^2u_2}{dx^2} = 0$ with inhomogeneous boundary conditions $u_2(0) = \alpha$ and $u_2(1) = \beta$.

Obviously, $u_2(x) = \alpha + (\beta - \alpha)x$ and $u_1(x)$ is obtained as in Example 5.1. Thus the complete solution of the given equation is

$$u(x) = -x\int_0^1 (1-\xi) f(\xi)\, d\xi + \int_0^x (x-\xi) f(\xi)\, d\xi + (\beta - \alpha)x$$

Example 5.3 : Solve the equation $\dfrac{d^2u}{dx^2} = f(x)$, $0 \le x \le 1$ subject to the boundary conditions $u(0) = \alpha$, $u'(1) = \beta$.

Solution : Let $u(x) = u_1(x) + u_2(x)$, where $u_1(x)$ is the solution of the inhomogeneous equation $u_1'' = f(x)$ with homogeneous boundary conditions $u_1(0) = 0$, $u_1'(1) = 0$

and $u_2(x)$ is the solution of the homogeneous equation $\frac{d^2u_2}{dx^2} = 0$ with inhomogeneous boundary conditions $u_2(0) = \alpha$, $u_2'(1) = \beta$.

Obviously $u_2(x) = \alpha + \beta x$. To determine $u_1(x)$ we note that $U(0) = 0$, $V'(1) = 0$. Thus $U(x) = Ax$, $V(x) = B$, $W = -AB$. Hence

$$G(x,\xi) = \begin{cases} -x, & 0 \le x \le \xi, \\ -\xi, & \xi \le x \le 1. \end{cases}$$

The complete solution is therefore

$$u(x) = -\int_0^x \xi \; f(\xi) \; d\xi - x\int_x^1 f(\xi) \; d\xi + \alpha + \beta x.$$

Example 5.4 : Solve the equation $x^2u'' + 2xu' = x^2$, $0 \le x \le 1$, with the boundary conditions $u(0)$ is finite and $u(1) + u'(1) = 0$.

Solution : The given equation is Sturm-Liouville type with $p(x) = x^2$. Thus $p(0) = 0$ and we require $u(0)$ [and hence $U(0)$] to be finite. Since $LU(x) = x^2U'' + 2xU' = 0$, we readily obtain $U(x) = A$ and similarly $V(x) = B(x)$ where A and B are arbitrary constants. Thus using (5.13), the Green's function of the problem is given by

$$G(x,\xi) = \begin{cases} -\dfrac{1}{x}, & 0 \le \xi \le x \\ -\dfrac{1}{\xi}, & x \le \xi \le 1 \end{cases}$$

Hence the solution of the given equation is

$$u(x) = -\frac{1}{x}\int_0^x \xi^2 d\xi - \int_x^1 \xi \, d\xi = \frac{x^2}{6} - \frac{1}{2}$$

Example 5.5 : Solve the equation $u'' + k^2u = f(x)$ subject to the boundary conditions $u(0) = 0$, $u(L) = 0$.

Solution : Here $p(x) = 1$, $q(x) = -k^2$, $a = 0$, $b = L$. Using (5.23) to (5.25), we obtain $U(0) = 0$, $V(L) = 0$.

Since $LU = \frac{d^2U}{dx^2} + k^2U = 0$, we have $U(x) = A_1 \sin kx + A_2 \cos kx$. The condition $U(0) = 0$ gives $A_2 = 0$ so that $U(x) = A_1 \sin kx$.

Similarly, the boundary condition $V(L) = 0$ gives $V(x) = B_1 \sin k(L-x)$.

The Wronskian is given by

$W = A_1 \sin kx[-B_1 k \cos k(L-x)] - A_1 k \cos kx[B_1 \sin k(L-x)] = -A_1 B_1 k \sin kL$

Hence the Green's function is

$$G(x,\xi) = \begin{cases} -\dfrac{\sin kx \sin k(L-\xi)}{k \sin kL}, & 0 \le x \le \xi \\ -\dfrac{\sin k(L-x) \sin k\xi}{k \sin kL}, & \xi \le x \le L \end{cases}$$

Thus the solution of the given equation is

$$u(x) = \int_0^L G(x,\xi) f(\xi)\, d\xi$$

$$= -\frac{\sin k(L-x)}{k \sin kL} \int_0^x \sin k\xi\, f(\xi)\, d\xi - \frac{\sin kx}{k \sin kL} \int_x^L \sin k(L-\xi)\, f(\xi)\, d\xi.$$

Example 5.6 : Solve the equation $u'' + k^2 u = f(x)$, $(k \neq \pi)$ subject to the boundary conditions $u(0) = \alpha$, $u'(1) = \beta$.

Solution : Let $u(x) = u_1(x) + u_2(x)$, where $u_1(x)$ is the solution of the inhomogeneous equation $u_1'' + k^2 u_1 = f(x)$ subject to the homogeneous boundary conditions $u_1(0) = 0$, $u_1'(1) = 0$, while $u_2(x)$ is the solution of the homogeneous equation $u_2'' + k^2 u_2 = 0$ subject to the boundary conditions $u_2(0) = \alpha$, $u_2'(1) = \beta$.

Obviously, $u_2(x) = \frac{1}{k \cos k}\{\alpha k \cos k(1-x) + \beta \sin kx\}$

To obtain $u_1(x)$, we note that $U(0) = 0$ and $V'(1) = 0$ so that $U(x) = A \sin kx$, $V(x) = B \cos k(1-x)$ and $W(x) = -ABk \cos k$. Hence the Green's function is

$$G(x,\xi) = \begin{cases} -\dfrac{\cos k(1-x) \sin k\xi}{k \cos k}, & 0 \le \xi \le x \\ -\dfrac{\sin kx \cos k(1-\xi)}{k \cos k}, & x \le \xi \le 1 \end{cases}$$

Thus the complete solution of the given equation is

$$u(x) = -\frac{1}{\cos k}\left[\cos k(1-x)\int_0^x \sin k\xi \cdot f(\xi)\, d\xi + \sin kx \int_x^1 \cos k(1-\xi) f(\xi) d\xi\right]$$

$$+\frac{1}{k\cos k}\{\alpha k \cos k(1-x) + \beta \sin kx\}$$

EXERCISES

1. Show that the solution of the differential equation $\frac{d^2u}{dx^2} = f(x)$ subject to the boundary conditions $u(0) = u(a) = 0$ is given by

$$u(x) = \int_0^a G(x,\xi) f(\xi)\, d\xi$$

where

$$G(x,\xi) = \begin{cases} -\dfrac{x(a-\xi)}{a}, & 0 \le x \le \xi \\ -\dfrac{\xi(a-x)}{a}, & \xi \le x \le a. \end{cases}$$

2. Show that ordinary Green's function does not exist for the problem $\frac{d^2u}{dx^2} = f(x), -1 \le x \le 1$, with the boundary conditions $u'(-1) = u'(1) = 0$.

3. Show that the Green's function for the equation $\frac{d^2u}{dx^2} = f(x),\ 0 \le x \le 1$ subject to the boundary conditions $u(0) = u'(0)$ and $u(1) = -u'(1)$ is given by

$$G(x,\xi) = \begin{cases} -\dfrac{1}{3}(x+1)(2-\xi), & 0 < x < \xi \\ -\dfrac{1}{3}(\xi+1)(2-x), & \xi < x < 1 \end{cases}$$

and the complete solution is given by

$$u(x) = -\frac{1}{3}\left[(2-x)\int_0^x (\xi+1) f(\xi)\, d\xi + (x+1)\int_x^1 (2-\xi) f(\xi)\, d\xi\right].$$

4. Show that for the equation $u'' + u = f(x)$, $0 \le x \le \pi$ with $u(0) = \alpha$ and $u(\pi) = \beta$, the Green's function does not exist for any arbitrary function $f(x)$.

5. Show that the Green's function for the equation $\frac{d^2u}{dx^2} - \alpha^2 u = f(x)$, $0 \le x \le 1$ subject to the boundary conditions $u(0) = 0$, $u(1) = 1$ is given by

$$G(x,\xi) = \begin{cases} -\dfrac{\sinh\alpha(\xi-1)\sinh\alpha x}{\alpha\sinh\alpha}, & 0 \le x \le \xi \\ -\dfrac{\sinh\alpha\xi\sinh\alpha(x-1)}{\alpha\sinh\alpha}, & \xi \le x \le 1. \end{cases}$$

Hence write the complete solution.

Appendix

Dirac-delta function : Let us define a function $\delta_\varepsilon(x)$ by the relation

$$\delta_\varepsilon(x) = \begin{cases} 0, & x < 0 \\ \dfrac{1}{\varepsilon}, & 0 < x < \varepsilon \\ 0, & x > \varepsilon \end{cases}$$

Noting that $\int_{-\infty}^{\infty} \delta_\varepsilon(x)dx = \int_0^\varepsilon \frac{dx}{\varepsilon} = 1$ we may regard $\delta(x)$ as the limiting form of $\delta_\varepsilon(x)$ as ε becomes zero, i.e., $\lim_{\varepsilon\to 0}\delta_\varepsilon(x) = \delta(x)$. This function $\delta(x)$ is known as **Dirac-delta function**.

We list below some properties of Dirac-delta function $\delta(x)$ (without proof)

1. $\delta(x - x_0) = 0$, if $x \ne x_0$ and undefined at $x = x_0$.

2. $\int_{x_0-\varepsilon}^{x_0+\varepsilon} \delta(x - x_0)\, dx = 1$ if $\varepsilon > 0$.

3. $\int_a^b f(x)\delta(x-x_0)\,dx = f(x_0)$, when $a < x < b$

$= 0,$ when $x < a$ or $x > b$ with $a < b$.

4. $\int_a^b f(x)\delta(x)\,dx = f(0)$ if $a < 0$ and $b > 0$.

5. $\delta(-x) = \delta(x)$

6. $x\delta(x) = 0$.

7. $\delta(ax) = \dfrac{\delta(x)}{|a|}$.

8. $\int \delta(a-x)\delta(x-b)\,dx = \delta(a-b)$.

5.2 Summary : The solution for the Sturm-Liouville equation

$$Lu(x) = f(x)$$

with the boundary conditions (homogeneous or non-homogeneous) are obtained in the form

$$u(x) = \int_a^b G(x,\xi) f(\xi)\,d\xi + v(x)$$

where $v(x)$ is the part of the solution for the non-homogeneous boundary conditions.

Unit : 6 □ Plane Autonomous Systems

6.0 Introduction : Given a differential equation it may not be possible to find the form of the solution even if we know that the solution does exist. This is due to the fact that the well-known functions such as polynomals, exponentials, sines, cosines, etc., are so limited in number that we cannot express the solutions of large varieties of differential equations in terms of these functions. Actually the differential equations define new type of function which are their solutions. It is better to consider if we can find the important properties of the solutions without actually solving them. In the present chapter we introduce a geometrical device called the phase plane method by which many properties of the solutions like equilibrium, periodicity, stability, etc., may be deduced directly from the differential equations of the type

$$x = P(x,y),\ \dot{y} = Q(x,y).$$

These equations are called the **phane autonomous system** in the plane of xy. The important technique is to find first the critical points in the plane of xy and linearise the system in the neighbourhood of such points to the form $\dot{x} = ax + by$, $\dot{y} = cx + dy$ (a, b, c, d constants). The nature of the critical points such as nodes, spirals, saddle points, etc., gives us a complete picture of the nature of the solutions in the neighbourhood of these critical points in the phase plane.

6.1 Autonomous and Non-autonomous Systems : In this chapter we shall be concerned with first-order ordinary differential equations in normal form, i.e., equations of the type

$$\frac{dx_i}{dt} = f_i(x_1, x_2, \cdots, x_n, t) \qquad (i = 1,\ 2,\ ...,\ n) \tag{6.1}$$

where f_i are given functions of the $n + 1$ variables $x_1, x_2, \cdots, x_n, t$. The solutions $x_1(t), x_2(t), \cdots, x_n(t)$ of (6.1) are of class c^1 and f_i are assumed to be continuous and real-valued in an $(n + 1)$ dimensional space R of the independent variables $x_1, x_2, \cdots, x_n, t$. The equation (6.1) can be written in vector notation as

$$\frac{dx}{dt} = f(x,t) \tag{6.2}$$

where $f = (f_1, f_2, \cdots, f_n)$ and $x = (x_1, x_2, \cdots, x_n)$. The equation (6.2) is called a **normal first-order differential equation.** It can be shown that the solutions of the system (6.1) or (6.2) exist and are unique and continuous.

If the functions f_i occurring in (6.1) depend on $x_1, x_2, \cdots, x_n$ and not on t, then the system given by

$$\frac{dx_i}{dt} = f_i(x_1, x_2, \cdots, x_n) \tag{6.3}$$

or, equivalently in vector form

$$\frac{dx}{dt} = f(x) \tag{6.4}$$

is said to be **autonomous.** On the other hand, if the time t occurs explicitly in (6.1) or (6.2), the system is said to be **non-autonomous.**

6.2 Plane Autonomous Systems : Let us consider autonomous system of two first-order differential equations in the xy-plane given by

$$\dot{x} = P(x, y),\ \dot{y} = Q(x, y) \tag{6.5}$$

where $P(x, y)$ and $Q(x, y)$ are continuous and have continuous first partial derivatives throughout the xy-plane (except possibly at some points). Elimination of y (or x) shows that the system of equations (6.5) is equivalent to one second-order ordinary differential equation with one degree of freedom.

The solutions $x(t)$, $y(t)$ of (6.5) may be represented on the xy-plane which we call the **phase plane.** As t increase, $(x(t), y(t))$ traces out a directed curve on the phase plane called the **phase path** or **phase trajectory** or simply the **path of the system.**

It follows from our assumption of the functions $P(x, y)$ and $Q(x, y)$ that if (x_0, y_0) is any point on the phase path and t_0 is any number such that $x(t_0) = x_0$, $y(t_0) = y_0$, then there exists a unique solutions $x = x(t)$, $y = y(t)$ of (6.5). If $x = x(t)$, $y = y(t)$ is a solution of (6.5), then $x = x(t + c), y = y(t + c)$ is also a solution of (6.5) for any constant c. Thus each phase path is represented by many solutions each differing only by a translation of the parameter. Hence there exist an infinity of motions (or solutions) corresponding to a given phase path.

Now eliminating dt between the equations (6.5), we have

$$\frac{dy}{dx}=\frac{Q(x,y)}{P(x,y)};\quad P(x,y)\neq 0 \tag{6.6}$$

The solution $y = y(x)$ of (6.6) defines a one-parameter family of curves called the **integral curves** or **solution curves**. If $P(x,y)=0,\ Q(x,y)\neq 0,$ we just interchange the roles of x and y and the integral curves are $x = x(y)$. A point, where $P(x, y)$ and $Q(x, y)$ are not simultaneously zero is called an **ordinary point** and at such point the phase path of (6.5) and the integral curves of (6.6) coincide. A point (x_0^*, y_0^*) for which $P(x_0^*,y_0^*)=Q(x_0^*,y_0^*)=0$ is called a **critical point** or a **singular point** or an **equilibrium point**. Through an ordinary point there passes one and only one solution curve, but this is not true for a critical point. At a critical point, $x=x_0^*, y=y_0^*$ is the constant solution which does not define a phase path and, therefore, no path goes through a critical point. The critical point is sometimes called **degenerate phase path**. We always assume that the critical point is isolated by a circle with centre at the point and containing no other critical point.

The signs of P and Q at a point determine the direction of the phase path at the point and the directions of all other points are settled by continuity. The diagram showing the phase paths in the phase plane is called the **phase diagram** and the point (x, y) is called the **state of the system.**

6.3 Linear Plane Autonomous Systems : The plane autonomous system (6.5) is said to be linear if it can be written in the form

$$\dot{x}=ax+by,\quad \dot{y}=cx+dy \tag{6.7}$$

where a, b, c, d are constants. It is obvious that the origin (0, 0) is the critical point of the system (6.7).

Let us now consider the system (6.5) and assume that $P(x, y)$ and $Q(x, y)$ vanish at the origin which is, therefore, the critical point of the system. By Taylor's expansion, we have

$$P(x,y)=ax+by+P_1(x,y),\ Q(x,y)=cx+dy+Q_1(x,y)$$

where $P_1(x,y), Q_1(x,y)=0(r^2)$ as $r=\sqrt{x^2+y^2}\rightarrow 0$ and

$$a = \left(\frac{\partial P}{\partial x}\right)_0,\ b = \left(\frac{\partial P}{\partial y}\right)_0,\ c = \left(\frac{\partial Q}{\partial x}\right)_0,\ d = \left(\frac{\partial Q}{\partial y}\right)_0, \tag{6.8}$$

the suffix zero indicates that the functions are evaluated at $x = 0, y = 0$. We also assume that $ad - bc \neq 0$, otherwise the point (0, 0) will be a non-elementary or higher order singularity. Thus the linear approximation of (6.5) in the neighbourhood of the origin (critical point) is given by (6.7).

A non-trivial solution of (6.7) is given by

$$x = re^{\lambda t},\ y = se^{\lambda t} \tag{6.9}$$

where r and s are related constants and λ is another constant. All these constants may be real or complex. A substitution of (6.9) into (6.7) leads to

$$\begin{aligned}(a-\lambda)r + bs &= 0,\\ cr + (d-\lambda)s &= 0\end{aligned} \tag{6.10}$$

which has non-trivial solutions $(r \neq 0,\ s \neq 0)$ iff

$$\begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = 0$$

$$\text{i.e., } \lambda^2 - p\lambda + q = 0 \tag{6.11}$$

whose roots are

$$\lambda_1, \lambda_2 = \frac{1}{2}\left(p \pm \sqrt{\Delta}\right), \tag{6.12}$$

where $p = a + d$, $q = ad - bc$ and $\Delta = p^2 - 4q$. The equation (6.11) is known as **secular equation** or **characteristic equation.**

If $\Delta \neq 0$, then because of the linearity of the homogeneous system (6.5), its general solution is of the form

$$\begin{aligned}x(t) &= c_1 r_1 e^{\lambda_1 t} + c_2 r_2 e^{\lambda_2 t},\\ y(t) &= c_1 s_1 e^{\lambda_1 t} + c_2 s_2 e^{\lambda_2 t}\end{aligned} \tag{6.13}$$

where c_1 and c_2 are arbitrary constants. If λ_1, λ_2 be complex, r_1, s_1, r_2, s_2 are complex; and, therefore, the solutions x, y to be real, c_1, c_2 are to be complex in general. On the phase plane, we have

$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{\lambda_1 c_1 s_1 e^{\lambda_1 t} + \lambda_2 c_2 s_2 e^{\lambda_2 t}}{\lambda_1 c_1 r_1 e^{\lambda_1 t} + \lambda_2 c_2 r_2 e^{\lambda_2 t}}. \tag{6.14}$$

We now consider several cases :

(i) λ_1, λ_2 are real, unequal and of same sign (node)

Let $\lambda_2 < \lambda_1 < 0$. Then from (6.14), we have

$$\frac{dy}{dx} = \frac{\lambda_1 c_1 s_1 + \lambda_2 c_2 s_2 e^{-(\lambda_1-\lambda_2)t}}{\lambda_1 c_1 r_1 + \lambda_2 c_2 r_2 e^{-(\lambda_1-\lambda_2)t}} \tag{6.15}$$

If $c_1 = 0$, the solution $(x, y) \to (0,0)$ as $t \to +\infty$ along the line $\frac{y}{x} = \frac{s_2}{r_2}$ (const.) from the two opposite directions while for $c_2 = 0$, solutions give another pair or straight lines into the origin in opposite directions along $\frac{y}{x} = \frac{s_1}{r_1}$. If $c_1 \neq 0$, $c_2 \neq 0$, the paths have slopes $\frac{dy}{dx} \to \frac{s_2}{r_2}$ as $t \to -\infty$ and $\frac{dy}{dx} \to \frac{s_1}{r_1}$ as $t \to +\infty$. The paths are therefore tangential to the straight line $\frac{y}{x} = \frac{s_1}{r_1}$ at the origin as $t \to +\infty$ and parallel to the straight line as $t \to -\infty$. Here the critical point at the origin is called a node and is shown in Figure 6.1. when both λ_1, λ_2 are negative. In this case the node is stable. The node is unstable when λ_1, $\lambda_2 > 0$, unequal and real and the figure is of the same type with the arrows reversed. These cases occur when

$\Delta > 0$, $q > 0$ and $p < 0$ (stable node) while $p > 0$ (unstable node) (6.16)

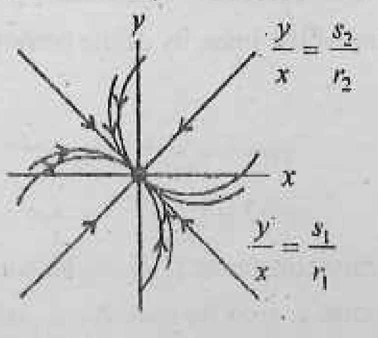

Fig. 6.1 : Stable node

In particular, if one root is zero, say λ_2, then $\frac{dy}{dx}$ = const. for all solutions and, therefore, describes a family of straight lines through the origin. These paths converge to the origin if $\lambda_1 < 0$ or radiate from the origin if $\lambda_1 < 0$. The critical point (0, 0) in this case is a particular type of node, called **proper node** or **star** (Figure 6.2). The proper node is stable if $\lambda_1 < 0$ and unstable if $\lambda_1 < 0$.

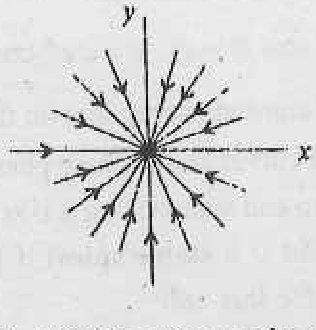

Fig. 6.2 : Stable proper node or star

(ii) λ_1, λ_2 are real, unequal and of different signs (saddle point)

Let $\lambda_1 < 0$, $\lambda_2 < 0$. Referring to (6.13), it is seen that for $c_1 = 0$, the solutions approach the origin as $t \to +\infty$ and (const.) while $(x, y) \to (0,0)$ as $t \to -\infty$ if $c_2 = 0$ and $\frac{y}{x} = \frac{s_1}{r_1}$ (const.). When $c_1 \neq 0$, $c_2 \neq 0$, $\frac{dy}{dx} \to \frac{s_1}{r_1}$ as $t \to +\infty$ approaching the straight line $\frac{y}{x} = \frac{s_1}{r_1}$ and $\frac{dy}{dx} \to \frac{s_2}{r_2}$ as $t \to -\infty$ departing from the straight line $\frac{y}{x} = \frac{s_2}{r_2}$. The straight line paths $\frac{y}{x} = \frac{s_1}{r_1}$ and $\frac{y}{x} = \frac{s_2}{r_2}$ are asymptotes for other solutions and are terned **separatrices.** In this case, the critical point (0, 0) is a **saddle point** (Figure 6.3) which is unstable. The conditions for the critical point to be a saddle point are.

$$\Delta > 0, \; q < 0 \tag{6.17}$$

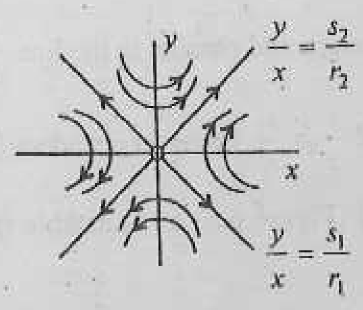

Fig. 6.3 : Saddle point

(iii) λ_1, λ_2 are complex with non-zero real part (spiral or focus)

We take $\lambda_1 = \bar{\lambda}_2 = \alpha + i\beta, \alpha$ and β being real, $r_1 = \bar{r}_2$, $s_1 = \bar{s}_2$ and suppose $c_1 = \bar{c}_2$ so that (6.13) represent real solutions of the form

$$x = ce^{\alpha t}\cos(\beta t + \varepsilon),\ y = c'e^{\alpha t}\cos(\beta t + \varepsilon')$$

where $c, c', \varepsilon, \varepsilon'$ are arbitrary constans depending on the coefficients of the system. It is easy to see that the family of curves on the phase plane is one parameter and consists of **spirals** surroundng the origin and approaching it if $\alpha < 0$ (Figure 6.4) and expanding if $\alpha > 0$. Thus the critical point is a **stable spiral** if Re $\lambda < 0$ and **unstable spiral** if Re $\lambda > 0$. The conditions for this are

$$\Delta < 0,\ p < 0 \text{ (stable) and } p > 0 \text{ (unstable)} \tag{6.18}$$

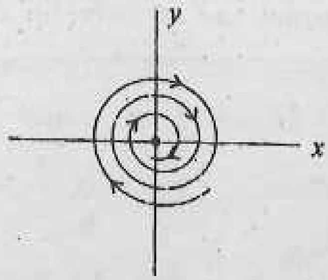

Fig. 6.4 : Stable spiral or focus

(iii) λ_1, λ_2 are real and equal (inflected node)

This is a degenerate case when there is only one family of solutions of the form (6.9). When λ_1 is very close to λ_2, the solutions r_1, s_1 and r_2, s_2 are nearly the same. We have seen in the analysis of node that the straight line $\frac{y}{x} = \frac{s_1}{r_1}$ is a path and all paths are tangential at the origin and parallel to the line $\frac{y}{x} = \frac{s_2}{r_2}$ at infinity. In this case, the straight lines $\frac{y}{x} = \frac{s_1}{r_1}$ and $\frac{y}{x} = \frac{s_2}{r_2}$ become coincident. The critical point is an **inflected node**, stable if $\lambda_1 = \lambda_2 < 0$ (Figure 6.5) and unstable if $\lambda_1 = \lambda_2 > 0$. The conditions for this are

$$\Delta = 0,\ p < 0 \text{ (stable) and } p > 0 \text{ (unstable)} \tag{6.19}$$

In particular, if $a = d \neq 0$, $b = c = 0$, the origin is a **proper node** or **star** (stable if $\lambda_1 < 0$, unstable if $\lambda_1 > 0$).

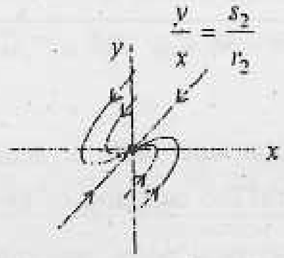

Fig. 6.5 : Stable inflected node

(v) λ_1, λ_2 are purely imaginary (centre)

Let $\lambda_1 = i\beta$ and $\lambda_2 = -i\beta$, β being real. Then as in the case (iii), we have

$$x = c\cos(\beta t + \varepsilon),\ y = c'\cos(\beta t + \varepsilon')$$

where $c, c', \varepsilon, \varepsilon'$ are arbitrary. These represent closed curves surrounding the origin. The critical point is called a **centre** (Figure 6.6) and the paths are ellipses. The conditions for this case are

$$p = 0,\ q < 0 \tag{6.20}$$

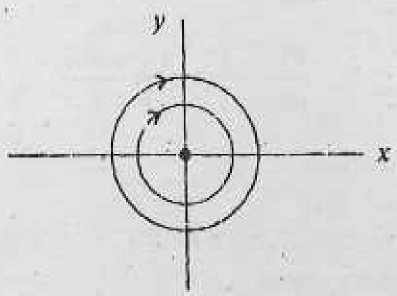

Fig. 6.6 : Centre

The critical point is **stable but not asymptotically stable.**

The above cases can be listed in a table as following :

The critical point is

(a) node, if λ_1, λ_2 real, unequal and of same sign ($\Delta > 0$, $q > 0$)

(b) saddle point, if λ_1, λ_2 real, unequal and of different sign ($\Delta > 0$, $q < 0$)

(c) spiral or focus, if λ_1, λ_2 complex with non-zero real part ($\Delta < 0$, $p \neq 0$)

(d) inflected node, if λ_1, λ_2 real and equal ($\Delta = 0$, $p \neq 0$)

(e) proper node or star, if one root is zero

(f) centre, if λ_1, λ_2 purely imaginary ($p = 0$, $q > 0$)

(g) parallel lines, if one root is zero and $q = 0$ (6.21)

Examples :

1. Locate the critical point and find its nature for the system $\dot{x} = x + y$, $\dot{y} = x - y + 1$. Also find the equation of the phase paths.

Solution : To locate the critical point, we solve the system $x + y = 0$ and $x - y + 1 = 0$ and get $x = -\frac{1}{2}$, $y = \frac{1}{2}$ so that $\left(-\frac{1}{2}, \frac{1}{2}\right)$ is the critical point.

Transferring the origin at $\left(-\frac{1}{2}, \frac{1}{2}\right)$ by the substitution $x = \xi - \frac{1}{2}$ and $y = \eta + \frac{1}{2}$, the given system reduces to $\dot{\xi} = \xi + \eta$, $\dot{\eta} = \xi - \eta$ whence $a = 1$, $b = 1$, $c = 1$, $d = -1$ and, therefore, $p = a + d = 0$, $q = ad - bc = -2 < 0$, $\Delta = p^2 = 4q = 8 > 0$. Thus the critical point is a saddle point which is unstable.

For the phase paths, we have

$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{x - y + 1}{x + y}$$

so that $d(xy) = x\,dx - y\,dy + dx$

Integrating, we get $2xy = x^2 - y^2 + 2x + c$, c being integration constant, is the required equation of the phase paths.

2. Show that the inflected node for the system $\dot{x} = ax + by$, $\dot{y} = cx + dy$ becomes star-shaped if $a = d$, $b = c = 0$.

Solution : Here $p = a + d = 2a \neq 0$, $q = ad - bc = a^2$, $\Delta = p^2 - 4q = 0$. Hence the critical point (0, 0) is an inflected node. Since $a = d \neq 0$ $b = c = 0$, the inflected node is a star.

6.4 Autonomous Equations in the Phase Plane : Let us consider second-order autonomous equation of the form

$$\ddot{x} = f(x, \dot{x}) \tag{6.22}$$

The constant solutions, i.e., solutions of $f(x, 0) = 0$ represent equilibrium states. The state of the system at a time $t = t_0$ consists of the pair of numbers $(x(t_0), y(t_0))$ i.e., $(x(t_0), \dot{x}(t_0))$ on the phase plane-xy with $\dot{x} = y$. This pair can also be treated as initial conditions and therefore determines subsequent (also earlier) states.

Let $\dot{x} = y$. Then (6.22) is equivalent to two first-order equations

$$\dot{x} = y, \dot{y} = f(x, y) \tag{6.23}$$

The phase paths are given by the solution of

$$\frac{dy}{dx} = \frac{f(x, y)}{y} \tag{6.24}$$

The constant solutions are obtained by putting $\dot{x} = 0$, $\ddot{x} = 0$, i.e., $\dot{x} = 0$, $\dot{y} = 0$ in (6.23). The critical point (x_0, y_0) is the solution of the pair

$$y = 0, f(x, y) = 0 \tag{6.25}$$

We note that

(i) the critical points are always situated on the x-axis,

(ii) the phase paths cut the x-axis at right angles (by (6.23) and (6.25)) except at critical points,

(iii) since the original state is returned on completing a circuit and the motion simply repeats itself indefinitey, closed paths always represent periodic solutions.

Next we consider the time taken between two points on a phase path. Let C be a segment of the phase path between two points A and B (Figure 6.7) and P represent any intermediate state. The representative point P moves along C with velocity $(\dot{x}, \dot{y})$ or $(\dot{x}, \ddot{x})$ and the time T_{AB} taken is given by

$$T_{AB} = \int_c dt = \int_c \left(\frac{dx}{dt}\right)^{-1} \left(\frac{dx}{dt}\right) dt = \int_c \frac{dx}{y} \tag{6.26}$$

which can be calculated if c is given.

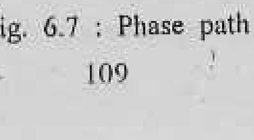

Fig. 6.7 : Phase path

6.5 The Damped Linear Oscillator : Let us consider the autonomous system $\ddot{x} = f(x, \dot{x})$. The simplest of such a system is a linear oscillator with linear damping given by the equation

$$\ddot{x} + \alpha\dot{x} + \beta x = 0 \tag{6.27}$$

where $\alpha > 0$, $\beta > 0$. As for example, we may consider a spring mass system with a dashpot or a circuit containing inductance, resistance and capacitance. The nature of the solutions of (6.27) depends on the roots of the auxiliary equation $m^2 + \alpha m + \beta = 0$. The roots are

$$m_1, m_2 = \frac{1}{2}\left(-\alpha \pm \sqrt{\Delta}\right) \text{ where } \Delta = \alpha^2 - 4\beta \tag{6.28}$$

(i) Strong damping ($\Delta > 0$) : Here the solution is $x(t) = Ae^{m_1 t} + Be^{m_2 t}$, m_1 and m_2 being negative and A, B are constants. Figure 6.8 represents two typical solutions. There is no oscillation and the t-axis is cut atmost once.

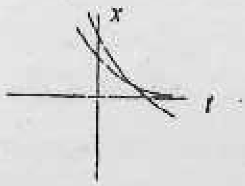

Fig. 6.8 : Two typical solutions

For the phase path, we put $\dot{x} = y$ and $\dot{y} = -\alpha y = \beta x$ so that the origin is the critical point. The equation of the phase path is obtained from $\frac{dy}{dx} = -\alpha - \beta\frac{x}{y}$ which is too complicated for solution. We, therefore, set

$$x = Ae^{m_1 t} + Be^{m_2 t}, \; y = Am_1 e^{m_1 t} + Bm_2 e^{m_2 t} \tag{6.29}$$

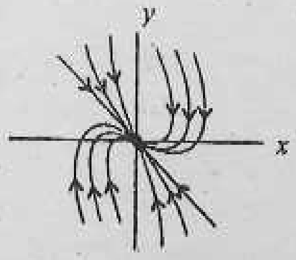

Fig. 6.9 : Phase paths

This set constitutes a parametric respresentation of the phase path for fixed values of A and B and the phase paths are represented in Figure 6.9. The critical point is a **node**. Since a slight displacement from $x = 0,\ \dot{x} = 0$, the state returns to the critical point, the **node is stable** and all phase paths terminate at the origin as $t \to \infty$.

(ii) Weak damping ($\Delta < 0$) : Here the roots (6.28) are complex with negative real part and the solution is $x(t) = Ae^{-\frac{1}{2}\alpha t}\cos\left(\frac{1}{2}\sqrt{-\Delta t} + \varepsilon\right)$, where A and ε are arbitrary constants. A typical solution is represented in Figure 6.10, which represents oscillations with decreasing amplitude and the oscillation decays more rapidly for larger α. Its image is plotted on the phase plane parametrically in Figure 6.11. The critical point at the origin is a **stable spiral.**

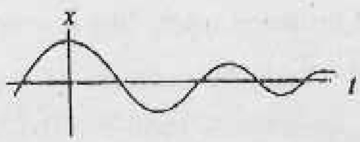

Fig. 6.10 : Typical solution

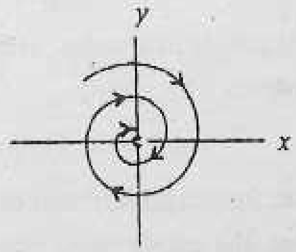

Fig. 6.11 : Phase paths

(iii) Critical damping ($\Delta = 0$) : Here $m_1 = m_2 = -\frac{1}{2}\alpha$ and the solution is $x(t) = (A + Bt)e^{-\frac{1}{2}\alpha t}$. The solutions resemble those of strong damping and the phase diagram shows a **stable node.**

EXERCISES

1. Determine the nature of the critical points for the following systems :

(i) $\dot{x} = -x - 2y,\ \dot{y} = 4x - 5y$ (Ans. Stable spiral)

(ii) $\dot{x} = 5x + 2y,\ \dot{y} = -17x - 5y$ (Ans. Centre)

(iii) $\dot{x} = -3x + 4y,\ \dot{y} = -2x + 3y$ (Ans. Saddle point)

(iv) $\dot{x} = -4x - y,\ \dot{y} = x - 2y$ (Ans. Stable inflected node)

(v) $\dot{x} = 4x - 3y,\ \dot{y} = 8x - 6y$ (Ans. Infinite number of critical points not isolated)

2. For the system $\dot{x} = ax + by,\ \dot{y} = cx + dy$ with $ad - bc = 0$, show that the system has infinitely many critical points not isolated. Also determine the phase paths.
3. Determine the nature of the critical point for the system $\dot{x} = \sin y,\ \dot{y} = \cos x$ and find the equation of the phase path.
4. Determine the nature and stability property of the critical point of the system $\dot{x} = x,\ \dot{y} = ky$ for $k > 0$ and $k < 0$. [Ans. (i) $k > 0$ and $k \neq 1$, unstable node, (ii) $k = 1$, star-shaped inflected node, (iii) $k < 0$, saddle point]
5. Find the nature and the stability property of the critical point of the system $\dot{x} = -ax + y,\ \dot{y} = -x - ay$ for $a < 0$ and $a > 0$. [Ans. $a < 0$, unstable spiral and $a > 0$, stable spiral]
6. Construct the phase diagram for the equation (i) $\ddot{x} + \omega^2 x = 0$ and (ii) $\ddot{x} - \omega^2 x = 0$.

6.6 Summary : The critical points like nodes, saddle points, spiral points, etc., are introduced for autonomous systems of the type

$$\dot{x} = P(x, y),\ \dot{y} = Q(x, y)$$

and the pattern of the solutions in the neighbourhood of these critical points in the phase plane are depicted. In particular, the phase plane characteristics of the second-order autonomous equations of the form

$$\ddot{x} = f(x, \dot{x})$$

in the neighbourhood of the critical points are touched upon.

Unit : 7 ❑ Special Functions

7.0 Introduction : So far we have considered the differential equations where the independent and dependent variables are real variables. Now we suppose that both the independent and dependent variables are complex variables and the coefficients involved in the equations are also complex analytic functions. A typical linear equation of the second order in the complex domain is

$$\frac{d^2w}{dz^2} + p(z)\frac{dw}{dz} + q(z)w = 0$$

where the functions $p(z)$ and $q(z)$ are both regular for all finite z or $p(z)$ and $q(z)$ have poles of order one or two atmost at a point of the complex z-plane. The solutions of this class of second-order linear equations in the neighbourhood of ordinary points or regular singular points define important new type of functions. Thus polynomials like Hermite polynomial, Laguerre polynomial and important frunctions like Bessel function, Hypergeometric function, Legendre function are obtained as solutions of the second-order linear equations of particular type. Many important properties of these polynomials and functions are discussed as well as their recurrence relations. These functions are called **special functions.**

7.1 Homogeneous Linear Differential Equations : Before we discuss the properties of several imp ɔrtant analytic functions defined by differential equations, it is desirable to consider whether there exists an analytic function $w(z)$ which satisfies the homogeneous linear differential equation

$$\frac{d^nw}{dz^n} + p_1(z)\frac{d^{n-1}w}{dz^{n-1}} + \cdots + p_{n-1}(z)\frac{dw}{dz} + p_n(z)w = 0$$

where the independent variable z is real or complex and the coefficients $p_1(z), p_2(z), \cdots, p_n(z)$ are analytic functions whose singularities of finite affix are poles, and, further, if such a solution does exist what effect the singularities of the coefficients have on the nature of the solution.

The simplest of this type $\frac{dw}{dz} + p(z)w = 0$ presents little difficulty. The variables are separable, and the solution is $w = \exp\{-\int p(z)dz\}$. On the other hand, the equation of order two

$$\frac{d^2w}{dz^2} + p(z)\frac{dw}{dz} + q(z)w = 0 \qquad (7.1)$$

has no such simple solution. We restrict our attention to this equation for two reasons; firstly, the analysis in this case is easily extended to the more general case, and, secondly, the particular functions with which we deal in the sequel do, in fact, satisfy second-order equations.

A point z_0 is said to be an **ordinary point** of the differential equation (7.1) if the functions $p(z)$ and $q(z)$ are regular in a neighbourhood of z_0; all other points are called **singular points** or singularities of the differential equation.

7.2 Solution Near an Ordinary Point : It can be shown that **if z_1 is an ordinary point of the equation (7.1) and if a_0 and a_1 are two arbitrary constants, then there exists a unique function $w(z)$ which is regular and satisfies the differential equation in a certain neighbourhood of z_0, and which also satisfies the initial conditions** $w(z_0) = a_0$, $w'(z_0) = a_1$. This theorem, which is due to Fuch, shows that the only possible singularities of the function defined by the differential equation are the poles of the coefficients $p(z)$ and $q(z)$.

For simplicity, we suppose that z_0 is zero. Then since $p(z)$ and $q(z)$ are regular in a neighbourhood $|z| < R$ of the origin, they are expansible as Taylor's series of the form

$$p(z) = \sum_{k=0}^{\infty} p_k z^k,\ q(z) = \sum_{k=0}^{\infty} q_k z^k, \qquad (7.2)$$

the radius of convergence of each series being not less than R. We now try to find a formal solution by substituting

$$w(z) = a_0 + a_1 z + a_2 z^2 + \cdots$$

in the equation

$$\frac{d^2w}{dz^2}+\sum_0^{\infty} p_k z^k \frac{dw}{dz}+\sum_0^{\infty} q_k z^k w=0$$

and equate coefficients. This gives

$$-2a_2 = a_1p_0 + a_0q_0,$$

$$-2\cdot 3a_3 = 2a_2p_0 + a_1p_1 + a_1q_0 + a_0q_1,$$

and, generally,

$$-(n-1)na_n = (n-1)a_{n-1}p_0 + (n-2)a_{n-2}p_1 + \cdots + a_1p_{n-2}$$

$$+a_{n-2}q_0 + a_{n-3}q_1 + \cdots + a_1q_{n-3} + a_0q_{n-2}.$$

These equations determine the coefficients a_n successively as linear combinations of a_0 and a_1. We can show that this power series has a radius of convergence which is not less than R.

The function

$$w(z)=\sum_{k=0}^{\infty} a_k z^k \tag{7.3}$$

is, therefore, regular when $|z| < R$ and satisfies the prescribed conditions at the origin. The formal process of term-by-term differentiation, multiplication, and rearrangement of power series by which this function was made to satisfy the differential equation are now seen to be completely justified, since all the series involved converge uniformly and absolutely in every closed domain within $|z| = R$.

Since a_n is a linear combination of a_0 and a_1, we can express the solution in the form $w(z)=a_0w_0(z)+a_1w_1(z)$. Each of the functions $w_0(z)$ and $w_1(z)$ is a solution of the differential equation and satisfies the initial conditions $w_0(0) = 1$, $w_0'(0)=0$, $w_1(0)=0$, $w_1'(0)=1$. Every solution of the differential equation regular in the neighbourhood of the origin is, therefore, a linear combination of the solutions $w_0(z)$ and $w_1(z)$ which we call a fundamental pair of solutions. Clearly $w_0(z)$ and $w_1(z)$ are linearly independent.

So far, the functions $w_0(z)$ and $w_1(z)$ are defined only in a neighbourhood of the origin. When we continue these functions analytically, they remain linearly independent solutions of the differential equation. The continuation can be carried out along any path which does not pass through a singular point of the differential equation and so the solution will ultimately be defined all over the z-plane.

7.3 The Nature of the Solution Near a Regular Singularity : The point z_0 is a singularity of the differential equation $w'' + p(z)w' + q(z)w = 0$ if it is a pole of one or both of the functions $p(z)$ and $q(z)$. We call it a **regular singularity** if it is not a singularity of either of the functions $(z - z_0)p(z)$ and $(z - z_0)^2 q(z)$; otherwise, it is called an **irregular singularity.**

If the origin is a regular singularity of the differential equation under consideration, the functions $zp(z)$ and $z^2q(z)$ are regular in a neighbourhood $|z| < R$ of the origin and so possess convergent Taylor's expansions of the form

$$zp(z) = \sum_0^{\infty} p_r z^r,\ z^2 q(z) = \sum_0^{\infty} q_r z^r \tag{7.4}$$

where the coefficients p_0, q_0 and q_1 are not all zero. We now show that, in general, the equation possesses two linerly independent solutions of the form

$$w(z) = z^{\alpha} \sum_0^{\infty} a_r z^r \tag{7.5}$$

where α is a root of a certain quadratic equation. When we substitute these power series in the differential equation and equate coefficients, we find that this expression is a formal solution of the equation if α and the coefficients a_r satisfy the conditions

$$a_0 F(\alpha) = 0 \text{ and } a_n F(\alpha + n) = -\sum_{s=0}^{n-1} a_s \{(\alpha + s)p_{n-s} + q_{n-s}\},\ (n \geq 1)$$

where $F(\alpha)$ denotes the quadratic $\alpha(\alpha - 1)p_0\alpha + q_0$. The first equation is satisfied by choosing a_0 arbitrarily and making α a root of the quadratic equation $F(\alpha) = 0$. This equation is called the **indicial equation** and its roots the **exponents** of the regular singular singularity under consideration. The remaining equations determine successively the coefficients a_s provided that $F(\alpha + n)$ does not vanish for any positive integral value of n. Hence if the indicial equation has distinct roots which do not differ by an integer, this process gives two formal solutions, one corresponding to each root of the indicial equation.

If, however, the roots of the indicial equation are equal, or differ by an integer, we may obtain only one formal solution. We leave this case as it represents much difficulties.

Now it can be shown that the series $w(z) = z^{\alpha}\sum_{0}^{\infty} a_r z^r$ does represent a solution of the equation (7.1) provided that the series $\Sigma a_r z^r$ terminates or else that it has a non-zero radius of convergence.

7.4 Solutions Valid for Large Values of | z | : To discuss the nature of the solution in the neighbourhood of the point at infinity, we make the transformation $z = 1/t$. Then the differential equation (7.1) becomes

$$\frac{d^2w}{dt^2} + \left\{\frac{2}{t} - \frac{1}{t^2}p\left(\frac{1}{t}\right)\right\}\frac{dw}{dt} + \frac{1}{t^4}q\left(\frac{1}{t}\right)w = 0 \tag{7.6}$$

The behaviour of the solution for large values of $|z|$ is determined by solving the transformed equation in the neighbourhood of the origin.

Accordingly we say that the point at infinity is an ordinary point if $\frac{2}{t} - \frac{1}{t^2}p\left(\frac{1}{t}\right)$ and $\frac{1}{t^4}q\left(\frac{1}{t}\right)$ are regular at the origin, i.e., if $2z - z^2p(z)$ and $z^4q(z)$ are regular at infinity. The complete solution of the equation, valid in the neighbourhood of the point at infinity, is in this case of the form $a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots$, where a_0, a_1, a_2, ... are arbitrary constants.

Again, the point $t = 0$ is a regular singularity of the transformed equation if $\frac{1}{t}p\left(\frac{1}{t}\right)$ and $\frac{1}{t^2}q\left(\frac{1}{t}\right)$ are regular there; we say, therefore, that the point at infinity is a regular singularity if $zp(z)$ and $z^2q(z)$ are regular there. In this case, $p(z)$ and $q(z)$ are expansible, by Laurent's theorem in series of the form

$$p(z) = \frac{p_0}{z} + \frac{p_1}{z^2} + \frac{p_2}{z^3} + \cdots, \quad q(z) = \frac{q_0}{z^2} + \frac{q_1}{z^3} + \frac{q_2}{z^4} + \cdots$$

convergent in a neighbourhood $|z| > R$ of the point at infinity

It may be shown as in 7.3 that there exist in the neighbourhood two linearly independent solutions

$$w(z) = z^{-\alpha}\Sigma a_k z^{-k}, \quad w(z) = z^{-\alpha'}\Sigma a'_k z^{-k}$$

where α and α' are the roots of the indicial equation $\alpha^2 - (p_0 - 1)\alpha + q_0 = 0$, provided that these roots do not differ by an integer or zero.

7.5 The Solution When the Exponent-difference is an Integer or Zero : When $\alpha - \alpha' = s$, where s is a positive integer or zero, the solution of (7.3) fails. For if $s = 0$, the two solutions become identical, whilst if $s > 0$, all the coefficients in one of the solutions from some point onwards are either infinite or indeterminate. It is, however, well-known that a knowledge of one solution of a linear differential equation of order n enables us to depress the order to $n - 1$. In our case, we obtain in this way a linear equation of the first order which can be integrated immediately.

To effect this depression of order, we make according to the usual rule, the change of the independent variable $w = w_0(z)v(z)$ where $w_0(z)$ is the known solution of exponent α. The function $v(z)$ is found to satisfy the equation

$$\frac{d^2v}{dz^2} + \left(\frac{2w_0'}{w_0} + p\right)\frac{dv}{dz} = 0$$

whose solution is

$$v(z) = A + B\int^z \frac{1}{\{w_0(z)\}^2} \exp\left\{-\int^z p(z)dz\right\} dz$$

where A and B are arbitrary constants. Hence the required second solution, valid near the origin, is

$$w(z) = w_0(z)\int^z \frac{1}{\{w_0(z)\}^2} \exp\left\{-\int^z p(z)dz\right\} dz \tag{7.7}$$

Now α and $\alpha - s$ are the roots of the indicial equation $\alpha^2 + (p_0 - 1)\alpha + q_0 - 0$ so that $p_0 = 1 + s - 2\alpha$. Hence we have

$$\frac{1}{\{w_0(z)\}^2}\exp\left\{-\int^z p(z)dz\right\} = \frac{1}{z^{2\alpha}(a_0 + a_1 z + \cdots)^2}.$$

$$\exp\left\{\int^z\left(\frac{2\alpha - 1 - s}{z} - p_1 - p_2 z\right)dz\right\}$$

$$= \frac{z^{-1-s}}{(a_0 + a_1 z + \cdots)^2}\exp\left\{-\int^z (p_1 + p_2 z)\,dz\right\}$$

$$= z^{-1-s}g(z)$$

where $g(0) = \frac{1}{a_0^2}$. Since $a_0 \neq 0$, the function $(a_0 + a_1 z + \cdots)^{-2}$ is regular in the neighbourhood of the origin. Hence $g(z)$ is also regular there and can be expanded as a convergent Taylor series $g(z) = \sum g_r z^r$. Substituting this series for $g(z)$, we find that the second solution is

$$w(z) = w_0(z)\int^z z^{1-s}\sum_0^\infty g_r z^r dz$$

$$= w_0(z)\left\{\sum_{r=0}^{s-1}\frac{g_r z^{r-s}}{r-s} + g_s \log z + \sum_{r=s+1}^{\infty}\frac{g_r z^{r-s}}{r-s}\right\} \tag{7.8}$$

In particular, when the exponent-difference s is zero, this solution can be written as

$$w(z) = g_0 w_0(z)\log z + z^{\alpha+1}\sum_{r=0}^{\infty} b_r z^r \tag{7.9}$$

As $g_0 \neq 0$, this solution possesses a logarithmic branch point at the origin. When the exponent-different s is a positive integer, the second solution takes the form

$$w(z) = g_0 w_0(z) \log z + z^{\alpha'} \sum_{r=0}^{\infty} c_r z^r \tag{7.10}$$

If it happens, as may be case, that g_0 is zero, the second solution does not involve a logarithmic term.

7.6 Hermite's equation : The differential equation defined by

$$\frac{d^2 w}{dz^2} - 2z\frac{dw}{dz} + 2\gamma w = 0 \tag{7.11}$$

where γ is a constant, is known as **Hermite's equation**. Evidently, $z = 0$ is an ordinary point of the equation (7.11). So there exists a unique solution $w(z)$ which is regular in a certain neighbourhood of $z = 0$ and which satisfies the conditions $w(0) = a_0$ and $w'(0) = a_1$ where a_0 and a_1 are arbitrary constants.

Let $w(z) = \sum_0^{\infty} a_k z^k$ and substitute this in (7.11) to get

$$\sum_2^{\infty} k(k-1)a_k z^{k-2} - 2z\sum_1^{\infty} k a_k z^{k-1} + 2\gamma \sum_0^{\infty} a_k z^k = 0$$

Equating the coefficient of z^k to zero we get

$$(k+2)(k+1)a_{k+2} - 2k a_k + 2\gamma a_k = 0$$

so that

$$a_{k+2} = -\frac{2(\gamma - k)}{(k+1)(k+2)} a_k$$

Putting $k = 0, 1, 2, \ldots$ we have

$$a_2 = -\frac{2\gamma}{1.2} a_0$$

$$a_3 = -\frac{2(\gamma-1)}{2.3} a_1$$

$$a_4 = -\frac{2(\gamma-2)}{3.4} a_2 = (-1)^2 . 2^2 \frac{\gamma(\gamma-2)}{4!} a_0$$

$$a_5 = -\frac{2(\gamma-3)}{4.5} a_3 = (-1)^2 \cdot \frac{2^2(\gamma-1)(\gamma-3)}{5!} a_1$$

$$\cdots \qquad\qquad \cdots$$

Hence

$$w(z) = a_0 \left[1 - \frac{2\gamma}{2!} z^2 + \frac{2^2 \gamma(\gamma-2)}{4!} z^4 - \cdots \right]$$

$$+ a_1 \left[z - \frac{2(\gamma-1)}{3!} z^3 + \frac{2^2(\gamma-1)(\gamma-3)}{5!} z^5 - \cdots \right]$$

$$\text{i.e.,} \quad w(z) = a_0 w_0(z) + a_1 w_1(z) \text{ (say)} \tag{7.12}$$

Obviously, $w_0(z)$ and $w_1(z)$ are two linearly independent solutions of (7.11) and so (7.12) is the general solution.

It may be noted that one of the two solutions $w_0(z)$ and $w_1(z)$ reduces to a polynomial if γ is a positive integer.

Hermite's polynomial : Let γ be an even integer and $\gamma = n$. Then clearly $w_0(z)$ reduces to a polynomial of degree n. Let us choose $a_0 = (-1)^{n/2} \cdot \frac{n!}{(n/2)!}$. Then the term containing z^n in $w_0(z)$ is

$$(-1)^{n/2} \frac{n!}{(n/2)!} (-2)^{n/2} \frac{n(n-2)\cdots(n-n+2)}{n!} z^n, \quad \text{i.e., } (2z)^n.$$

Similarly, the coefficient of z^{n-2} in the same solution is

$$(-1)^{n/2} \frac{n!}{(n/2)!} (-2)^{n/2} \cdot (-2)^{(n-2)/2} \frac{n(n-2)\cdots(n-n-2+2+2)}{(n-2)!} z^{n-2}$$

$$\text{i.e., } \frac{n(n-1)}{1!} (2z)^{n-2}$$

and so on.

Hence the solution is

$$w(z) = (2z)^n - \frac{n(n-1)}{1!}(2z)^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2!}(2z)^{n-4} - \cdots$$

$$+(-1)^{n/2}\frac{n!}{(n/2)!} \qquad (7.13)$$

Similarly, w1(z) reduces to a polynomial if $\gamma = m$, an odd integer. In this case we have

$$w_1(z) = a_1 z\left\{1 - \frac{2(m-1)}{3!}z^2 + \frac{2^2(m-1)(m-3)}{5!}z^4 - \cdots\right.$$

$$\left.+(-2)^{(m-1)/2}\frac{(m-1)(M-3)\cdots 2}{m!}z^{m-1}\right\}$$

Taking $a_1 = (-1)^{(m-1)/2}\dfrac{(m+1)!}{\{(m+1)/2\}!}$, the term containing z^m is

$$2^{(m-1)/2}\frac{(m+1)(m-1)(m-3)\cdots 2}{\{(m+1)/2\}!}z^m$$

$$= 2^m \frac{\frac{m+1}{2}\cdot\frac{m-1}{2}\cdot\frac{m-3}{2}\cdots 1}{\{(m+1)/2\}!}z^m = (2z)^m$$

and the last term is $(-1)^{(m-1)/2}\dfrac{(m+1)!}{\{(m+1)/2\}!}z$. We have therefore

$$w(z) = (2z)^m - \frac{m(m-1)}{1!}(2z)^{m-2} + \cdots + (-1)^{(m-1)/2}\cdot\frac{(m+1)!}{\{(m+1)/2\}!}z \qquad (7.14)$$

The polynomials defined by (7.13) and (7.14) or alternately, the polynomial defined by

$$H_n(z) = \sum_{k=0}^{[n/2]}\frac{(-1)^k n!}{k!(n-2k)!}(2z)^{n-2k} \qquad (7.15)$$

where

$$[n/2] = n/2 \qquad \text{if } n \text{ is even}$$
$$= (n-1)/2 \text{ if } n \text{ is odd}$$

is known as **Hermite's polynomial of degree *n*.**

Generating function for the Hermite polynomial : It is interesting to note that the Hermite polynomials are obtainale from the coefficients of $\frac{t^n}{n!}$ on the expansion of e^{2tz-t^2}.

For, we have

$$e^{2tz-t^2} = e^{2tz} \cdot e^{-t^2} = \left\{1 + 2tz + \frac{(2z)^2 t^2}{2!} + \cdots + \frac{(2z)^{n-2k} t^{n-2k}}{(n-2k)!} + \cdots\right\}$$

$$\times \left\{1 + \frac{(-t^2)}{2!} + \frac{(-t^2)^2}{2!} + \cdots + \frac{(-t^2)^k}{k!} + \cdots\right\}$$

so that the coefficient of t^n in this expression is $= \sum_{k=0}^{[n/2]} \frac{(-1)^k}{k!(n-2k)!}(2z)^{n-2k}$

Hence $H_n(z) = \sum_{k=0}^{\infty} \frac{(-1)^k n!(2z)^{n-2k}}{k!(n-2k)!}$ and we can write

$$e^{2tz-t^2} = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(z) \qquad (7.16)$$

The Rodrigues formula for $H_n(z)$: We shall show that

$$H_n(z) = (-1)^n e^{z^2} \frac{d^n}{dz^n}(e^{-z^2}) \qquad (7.17)$$

which is known as **Rodrigues formula.**

To prove the formula, we note that

$$e^{2tz-t^2} = \sum_{n=0}^{\infty} \frac{H_n(z)}{n!} t^n$$

$$\text{or, } e^{z^2-(t-z)^2} = H_0(z) + \frac{H_1(z)}{1!}t + \frac{H_2(z)}{2!}t^2$$

$$+\cdots+\frac{H_n(z)}{n!}t^n + \frac{H_{n+1}(z)}{(n+1)!}t^{n+1}+\cdots$$

Differentiating both sides partially w.r.t. t n times and then putting $t = 0$, we have

$$\left[\frac{\partial^n}{\partial t^n}e^{-(t-z)^2}\right]_{t=0} e^{z^2} = \frac{H_n(z)}{n!}\cdot n! = H_n(z)$$

Now let $z - t = u$ so that $\frac{\partial}{\partial t} = -\frac{\partial}{\partial u}$ and at $t = 0$, $z = u$. Hence

$$H_n(z) = \left[(-1)^n \frac{\partial^n}{\partial u^n} e^{-u^2}\right]_{u=z} e^{z^2}$$

$$\text{i.e., } H_n(z) = (-1)^n e^{z^2} \frac{d^n}{dz^n}\left(e^{-z^2}\right)$$

Recurrence formulae for Hermite polynomials :

1. $H_n'(z) = 2nH_{n-1}(z),\ n \geq 1$

Proof : We have

$$\sum_{n=0}^{\infty} \frac{H_n(z)}{n!}t^n = e^{-t^2+2tz}$$

Differentiating both sides w.r.t z, we get

$$\sum_{n=0}^{\infty} \frac{H_n'(z)}{n!}t^n = e^{-t^2+2tz}\cdot 2t = 2t\sum_{n=0}^{\infty}\frac{H_n(z)}{n!}t^n$$

$$= 2\sum_{n=0}^{\infty}\frac{H_n(z)}{n!}t^{n+1} = 2\sum_{n=1}^{\infty}\frac{H_{n-1}(z)}{(n-1)!}t^n$$

Equating the coefficient of t^n from both sides, we have

$$\frac{H_n'(z)}{n!} = 2\frac{H_{n-1}(z)}{(n-1)!} \text{ i.e., } H_n'(z) = 2nH_{n-1}(z)$$

II. $\mathbf{2zH_n(z) = 2nH_{n-1}(z) + H_{n+1}(z)}$

Proof : We have $\sum_{n=0}^{\infty}\frac{H_n(z)}{n!}t^n = e^{-t^2+2tz}$

Differentiating both sides w.r.t. t we get

$$\sum_{n=0}^{\infty}\frac{H_n(z)}{(n-1)!}t^{n-1} = -2te^{-t^2+2tz} + 2ze^{-t^2+2tz}$$

Since term of L.H.S. corresponding to $n = 0$ is zero, so

$$\sum_{n=1}^{\infty}\frac{H_n(z)}{(n-1)!}t^{n-1} = -2t\sum_{n=0}^{\infty}\frac{H_n(z)}{n!}t^n + 2z\sum_{n=0}^{\infty}\frac{H_n(z)}{n!}t^n$$

or, $$2z\sum_{n=0}^{\infty}\frac{H_n(z)}{n!}t^n = 2\sum_{n=0}^{\infty}\frac{H_n(z)}{n!}t^{n+1} + \sum_{n=1}^{\infty}\frac{H_n(z)}{(n-1)!}t^{n-1}$$

or, $$2z\sum_{n=0}^{\infty}\frac{H_n(z)}{n!}t^n = 2\sum_{n=1}^{\infty}\frac{H_{n-1}(z)}{(n-1)!}t^n + \sum_{n=0}^{\infty}\frac{H_{n+1}(z)}{n!}t^n$$

Equating the coefficients of t^n from both sides, we get

$$2z\frac{H_n(z)}{n!} = 2\frac{H_{n-1}(z)}{(n-1)!} + \frac{H_{n+1}(z)}{n!}$$

$$\text{i.e., } 2zH_n(z) = 2nH_{n-1}(z) + H_{n+1}(z)$$

Note : Equating the coefficients of t^0, we have $2zH_0(z) = H_1(z)$

III. $\mathbf{H_n'(z) = 2zH_{n-1}(z) - H_{n+1}(z)}$

Proof : From the recurrence formulae I and II, we get

$$H_n'(z) = 2zH_{n-1}(z)$$

$$\text{and } 2zH_n(z) = 2nH_{n-1}(z) + H_{n+1}(z)$$

Subtracting we get the required result.

IV. $\boldsymbol{H_n''(z) - 2zH_n'(z) + 2nH_n(z) = 0}$

Proof : Hermite's differential equation is

$$\frac{d^2w}{dz^2} - 2z\frac{dw}{dz} + 2nw = 0$$

Since $H_n(z)$ is a solution of this equation, we have the required result.

Orthogonal property of Hermite polynomial : To prove that

$$\int_{-\infty}^{\infty} e^{-z^2} H_m(z) H_n(z)\, dz = \begin{Bmatrix} \sqrt{\pi}\, 2^n n! & \text{if } m = n \\ 0 & \text{if } m \neq n \end{Bmatrix} = \sqrt{\pi}\, 2^n . n!\, \delta_{mn} \tag{7.18}$$

where δ_{mn} is the Kronecker delta.

Proof : We have $e^{-t^2+2tz} = \sum_{n=0}^{\infty} H_n(z)\frac{t^n}{n!}$ and $e^{-s^2+2sz} = \sum_{m=0}^{\infty} H_m(z)\frac{s^m}{m!}$

so that $e^{-t^2+2tz} \cdot e^{-s^2+2sz} = \sum_{n=0}^{\infty} H_n(z)\frac{t^n}{n!} \cdot \sum_{m=0}^{\infty} H_m(z)\frac{s^m}{m!}$

$\therefore\ \frac{1}{m!n!} H_m(z)H_n(z) =$ coefficient of $s^m t^n$ in the expansion of

$\therefore\ \int_{-\infty}^{\infty} e^{-z^2} H_m(z)H_n(z)\, dz = m!n!$ times the coefficient of $s^m t^n$ in the expansion of

$$\int_{-\infty}^{\infty} e^{-z^2} \cdot e^{-s^2+2sz} \cdot e^{-t^2+2tz} dz$$

Now $\int_{-\infty}^{\infty} e^{-z^2} \cdot e^{-s^2+2sz} \cdot e^{-t^2+2tz} dz = e^{-s^2-t^2} \int_{-\infty}^{\infty} e^{-z^2+2tz+2sz} dz$

$$= e^{-s^2-t^2} \int_{-\infty}^{\infty} e^{-[z-(t+s)]^2+(t+s)^2} dz$$

$$= e^{2ts}\int_{-\infty}^{\infty} e^{-u^2}\,du \qquad \text{where } z-(t+s)=u$$

$$= e^{2ts}\sqrt{\pi}$$

$$= \sqrt{\pi}\left[1+(2ts)+\frac{(2ts)^2}{2!}+\cdots+\frac{(2+s)^n}{n!}+\cdots\right]$$

Hence the coefficient of $s^m t^n$ in the expansion of

$$\int_{-\infty}^{\infty} e^{-z^2}\cdot e^{-s^2+2sz}\cdot e^{-t^2+2tz}\,dz$$

is 0 if $m \neq n$ and $\frac{2^n\sqrt{\pi}}{n!}$ if $m = n$

Thus

$$\int_{-\infty}^{\infty} e^{-z^2} H_m(z)H_n(z)\,dz = \begin{Bmatrix} \sqrt{\pi}\,2^n n! & \text{if } m=n \\ 0 & \text{if } m\neq n \end{Bmatrix} = \sqrt{\pi}\,2^n n!\,\delta_{mn}$$

7.7 Laguerre equation : The differential equation given by

$$z\frac{d^2w}{dz^2}+(1-z)\frac{dw}{dz}+\gamma w=0 \qquad (7.19)$$

is known as **Laguerre equation**.

Comparing this equation with (7.1) we have $p(z)=\frac{1-z}{z}$, $q(z)=\frac{\gamma}{z}$ so that $zp(z)=1-z$, $z^2q(z)=\gamma z$. The point $z=0$ is a pole of both $p(z)$ and $q(z)$ and that $zp(z)$ and $z^2q(z)$ are regular at the origin. So $z=0$ is a regular singular point of the differential equation (7.19).

The indicial equation of (7.19) corresponding to the singularity $z = 0$ is $\alpha(\alpha-1)+1\cdot\alpha+0=0$ i.e., $\alpha = 0, 0$. Hence the equation has a solution of the form

$w(z) = z^{\alpha}\sum_{k=0}^{\infty} a_k z^k \cdot a_0 \neq 0$, the power series being convergent in the neighbourhood of the origin. Substituting this value of $w(z)$ in (7.19) we get

$$z\sum_{k=2}^{\infty} k(k-1)\, a_k z^{k-2} + (1-z)\sum_{k=1}^{\infty} k\, a_k z^{k-1} + \gamma \sum_{k=0}^{\infty} a_k z^k = 0$$

Equating to zero the coefficient of z^k, we have

$$(k+1)ka_{k+1} + (k+1)a_{k+1} - ka_k + \gamma a_k = 0$$

$$\text{i.e.,} \quad a_{k+1} = \frac{k-\gamma}{(k+1)^2} a_k$$

Substituting $k = 0, 1, 2, \ldots$ we get

$$a_1 = \frac{(-\gamma)}{1^2} a_0,\; a_2 = \frac{1-\gamma}{2^2} a_1 = \frac{(-\gamma)(1-\gamma)}{(2!)^2} a_0,$$

$$a_3 = \frac{2-\gamma}{3^2} a_2 = \frac{(-\gamma)(1-\gamma)(2-\gamma)}{(3!)^2} a_0$$

$$\ldots$$

Hence

$$w(z) = a_0\left[1 - \frac{\gamma}{1^2} z + \frac{\gamma(\gamma-1)}{(2!)^2} z^2 - \frac{\gamma(\gamma-1)(\gamma-2)}{(3!)^2} z^3 + \cdots\right],\; a_0 \neq 0 \qquad (7.20)$$

Laguerre polynomial : If $\gamma = n$, a non-negative integer, then $w(z)$ is a polynomial of degree n. If $a_0 = n!$ and $\gamma = n$, then the solution given by

$$w(z) = \sum_{k=0}^{n} \frac{(-1)^k (n!)^2}{(k!)^2 (n-k)!} z^k$$

is called **Laguerre polynomial of degree n and is denoted by $L_n(z)$.** Thus

$$L_n(z) = \sum_{k=0}^{n} \frac{(-1)^k (n!)^2}{(k!)^2 (n-k)!} z^k \qquad (7.21)$$

Generating function for the Laguerre polynomials : The Laguerre polynomials $L_n(z)$ are defined by the following generating relation :

$$e^{-\frac{zt}{1-t}} = (1-t)\sum_{n=0}^{\infty}\frac{L_n(z)}{n!}t^n$$

For, we have

$$(1-t)^{-1}e^{-\frac{zt}{1-t}} = (1-t)^{-1}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\frac{z^k t^k}{(1-t)^k}$$

$$= \sum_{k=0}^{\infty}\frac{(-1)^k}{k!}z^k t^k (1-t)^{-(k+1)}$$

$$= \sum_{k=0}^{\infty}\frac{(-1)^k}{k!}z^k t^k \sum_{r=0}^{\infty}\frac{(k+r)!}{k!r!}t^r$$

$$= \sum_{k,r=0}^{\infty}(-1)^k\frac{(k+r)!}{(k!)^2 r!}z^r t^{k+r}$$

The coefficient of t^n in this expansion is $\sum_{k=0}^{n}\frac{(-1)^k n!}{(k!)^2(n-k)!}z^k$ and so

$$(1-t)^{-1}e^{-\frac{zt}{1-t}} = \sum_{n=0}^{\infty}\frac{t^n}{n!}L_n(z)$$

i.e., $$e^{-\frac{zt}{1-t}} = (1-t)\sum_{n=0}^{\infty}\frac{t^n}{n!}L_n(z)$$

The Rodrigues formula for $L_n(z)$: We shall show that

$$L_n(z) = e^z\frac{d^n}{dz^n}(z^n e^{-z}) \tag{7.22}$$

which is called the **Rodrigue's formula** for Laguerre polynomial of degree n. Since, by Leibnitz rule for the nth derivatives of the product of two functions, we have

$$\frac{d^n}{dz^n}(z^n e^{-z}) = \sum_{k=0}^{n}\binom{n}{k}\frac{d^{n-k}}{dz^{n-k}}(z^n)\frac{d^k}{dz^k}(e^{-z})$$

$$= \sum_{k=0}^{n}\frac{n!}{k!(n-k)!}\cdot\frac{n!z^k}{k!}(-1)^k e^{-z}$$

$$= e^{-z}\sum_{k=0}^{n}\frac{(-1)^k (n!)^2 z^k}{(k!)^2(n-k)!}$$

$$= e^{-z}L_n(z)$$

so that $L_n(z) = e^z \frac{d^n}{dz^n}(z^n e^{-z})$.

Recurrence formula for Laguerre polynomials :

I. $L_{n+1}(z) + (z - 2n - 1)L_n(z) + n^2 L_{n-1}(z) = 0$

Proof : We have $e^{\frac{zt}{1-t}} = (1-t)\sum_{n=0}^{\infty}\frac{L_n(z)}{n!}t^n$

Differentiating both sides w.r.t. t we get

$$\frac{z}{(1-t)^2}e^{\frac{zt}{1-t}} = (1-t)\sum_{n=0}^{\infty}\frac{L_n(z)t^{n-1}}{(n-1)!} - \sum_{n=0}^{\infty}\frac{L_n(z)}{n!}t^n$$

$$\text{or,}\quad -z\sum_{n=0}^{\infty}\frac{L_n(z)}{n!}t^n = (1-t)^2\sum_{n=0}^{\infty}\frac{L_n(z)t^{n-1}}{(n-1)!} - (1-t)\sum_{n=0}^{\infty}\frac{L_n(z)}{n!}t^n$$

Equating the coefficient of t^n from sides, we get

$$-z\frac{L_n(z)}{n!} = \frac{L_{n+1}(z)}{n!} - 2\frac{L_n(z)}{(n-1)!} + \frac{L_{n-1}(z)}{(n-2)!} - \frac{L_n(z)}{n!} + \frac{L_{n-1}(z)}{(n-1)!}$$

or, $-z\,L_n(z) = L_{n+1}(z) - 2nL_n(z) + n(n-1)\,L_{n-1}(z) - L_n(z) + n\,L_{n-1}(z)$

i.e., $L_{n+1}(z) + (z-2n-1)\,L_n(z) + n^2 L_{n-1}(z) = 0$

II. $\boldsymbol{L_n'(z) - nL_{n-1}'(z) + nL_{n-1}(z) = 0}$

Since $e^{-\frac{zt}{1-t}} = (1-t)\sum_{n=0}^{\infty}\frac{L_n(z)}{n!}t^n$, we get by differentiating w.r.t. z

$$-\frac{t}{1-t}e^{-\frac{zt}{1-t}} = (1-t)\sum_{n=0}^{\infty}\frac{L_n'(z)}{n!}t^n$$

$$\text{or,}\qquad -t\sum_{n=0}^{\infty}\frac{L_n(z)}{n!}t^n = (1-t)\sum_{n=0}^{\infty}\frac{L_n'(z)}{n!}t^n$$

Equating the coefficient of t^n from both sides, we get

$$-\frac{L_{n-1}(z)}{(n-1)!} = \frac{L_n'(z)}{n!} - \frac{L_{n-1}'(z)}{(n-1)!}$$

$$\text{i.e.,}\qquad L_n'(z) - n\,L_{n-1}'(z) + n\,L_{n-1}(z) = 0$$

III. $\boldsymbol{zL_n''(z) + (1-z)\,L_n'(z) + n\,L_n(z) = 0}$

We have $L_n'(z) = n\{L_{n-1}'(z) - L_{n-1}(z)\}$

Raising the index by 1, $L_{n+1}'(z) = (n+1)\{L_n'(z) - L_n(z)\}$

Differentiating w.r.t. z, $L_{n+1}''(z) = (n+1)\{L_n''(z) - L_n'(z)\}$

Again raising the index by 1,

$$L_{n+2}''(z) = (n+2)\{L_{n+1}''(z) - L_{n+1}'(z)\} \qquad \text{(A)}$$

Now differentiating both sides w.r.t. z of the recurrence relation

$$L_{n+1}(z) + (z-2n-1)L_n(z) + n^2 L_{n-1}(z) = 0$$

we get

$$L'_{n+1}(z) + L_n(z) + (z - 2n - 1) L'_n(z) + n^2 L'_{n-1}(z) = 0$$

Raising the index by 1,

$$L'_{n+2}(z) + L_{n+1}(z) + (z - 2n - 3) L'_{n+1}(z) + (n+1)^2 L'_n(z) = 0$$

Differentiating w.r.t. z

$$L''_{n+2}(z) + 2L'_{n+1}(z) + (z - 2n - 3) L''_{n+1}(z) + (n+1)^2 L''_n(z) = 0$$

Using the relation (A), we have

$$(n+2)\{L''_{n+1}(z) - L'_{n+1}(z)\} + 2L'_{n+1}(z) + (z - 2n - 3) L''_{n+1}(z) + (n+1)^2 L''_n(z) = 0$$

$$\text{or,} \quad (z - n - 1) L''_{n+1}(z) - nL'_{n+1}(z) + (n+1)^2 L''_n(z) = 0$$

Using (A), $(z - n - 1)(n+1)\{L''_n(z) - L'_n(z)\} - n(n+1)\{L'_n(z) - L_n(z)\}$

$$+ (n+1)^2 L''_n(z) = 0$$

$$\text{i.e.,} \quad zL''_n(z) + (1 - z) L'_n(z) + nL_n(z) = 0$$

This shows that $w = L_n(z)$ is the solution of the Laguerre equation (7.19).

Orthogonal property : We have

$$\sum_{n=0}^{\infty} \frac{1}{n!} L_n(z) t^n = \frac{1}{1-t} e^{-\frac{tz}{1-t}} \text{ and } \sum_{m=0}^{\infty} \frac{1}{m!} L_m(z) s^m = \frac{1}{1-s} e^{-\frac{sz}{1-s}}$$

$$\therefore \sum_{n,m=0}^{\infty} e^{-z} \cdot \frac{t^n}{n!} \cdot \frac{s^m}{m!} L_n(z) \cdot L_m(z) = e^{-z} \frac{1}{(1-t)(1-s)} e^{-\frac{tz}{1-t}} \cdot e^{-\frac{sz}{1-t}}$$

Thus

$$\int_0^{\infty} e^{-z} \cdot \frac{1}{n!m!} L_n(z) L_m(z)\, dz = \text{coefficient of } t^n s^m \text{ in the expansion of}$$

$$\int_0^\infty e^{-z} \frac{1}{(1-t)(1-s)} e^{-\frac{tz}{1-t}} \cdot e^{-\frac{sz}{1-t}} dz$$

Now $\int_0^\infty e^{-z} \frac{1}{(1-t)(1-s)} e^{-\frac{tz}{1-t}} \cdot e^{-\frac{sz}{1-t}} dz$

$$= \frac{1}{(1-t)(1-s)} \int_0^\infty e^{-z\left\{1+\frac{t}{1-t}+\frac{s}{1-s}\right\}} dz$$

$$= \frac{1}{(1-t)(1-s)} \cdot \frac{1}{1+\frac{t}{1-t}+\frac{s}{1-s}} \left[e^{-z\left\{1+\frac{t}{1-t}+\frac{s}{1-s}\right\}} \right]_0^\infty$$

$$= \frac{1}{1-st} = (1-st)^{-1} = 1 + st + s^2t^2 + \cdots + (st)^n + \cdots$$

in which the coefficient of $t^n s^m$ is zero if $m \neq n$ and 1 if $m = n$.

Hence $\int_0^\infty e^{-z} \cdot \frac{1}{n!m!} L_n(z) L_m(z) dz = \begin{cases} 0 & \text{if} \quad m \neq n \\ 1 & \text{if} \quad m = n \end{cases}$

Thus $\int_0^\infty e^{-z} L_n(z) L_m(z) dz = \begin{cases} 0 & \text{if} \quad m \neq n \\ (n!)^2 & \text{if} \quad m = n \end{cases}$

which is the

7.8 Bessel equation : The differential equation

$$z^2 \frac{d^2w}{dz^2} + z\frac{dw}{dz} + (z^2 - \gamma^2)w = 0 \qquad (7.23)$$

where γ is constant (real or complex) is known as **Bessel equation**.

Comparing this equation with (7.1) we have $p(z) = \frac{1}{z}$, i.e., $zp(z) = 1$ and $q(z) = \frac{z^2-\gamma^2}{z^2}$, i.e., $z^2 q(z) = z^2 - \gamma^2$. Thus $z = 0$ is a pole of $p(z)$ and also of $q(z)$

and that both $zp(z)$ and $z^2q(z)$ are analytic at $z = 0$. So $z = 0$ is a regular singular point of the differential equation (7.23). The indicial equation of (7.23) corresponding to the singularity $z = 0$ is $\alpha(\alpha - 1) + \alpha - \gamma^2 = 0$, i.e., $\alpha = \pm\gamma$.

Case 1 : First we suppose that v is neither zero nor an integer and Re $\gamma \geq 0$. Then the solution of (7.23) can be written in the form

$$w(z) = z^{\gamma} \sum_{k=0}^{\infty} a_k z^k = \sum_{k=0}^{\infty} a_k z^{\gamma+k}, \quad a_0 \neq 0$$

Substituting this in (7.23) we get

$$z^2 \sum_{k=0}^{\infty} (\gamma + k)(\gamma + k - 1) a_k z^{\gamma+k-2} + z \sum_{k=0}^{\infty} (\gamma + k) a_k z^{\gamma+k-1}$$

$$+ (z^2 - \gamma^2) \sum_{k=0}^{\infty} a_k z^{k+\gamma} = 0$$

Equating the coefficient of $z^{\gamma+k}$ to zero, we have

$$(\gamma + k)(\gamma + k - 1) a_k + (\gamma + k) a_k + a_{k-2} - \gamma^2 a_k = 0$$

$$\text{or,} \quad a_k = -\frac{a_{k-2}}{k(2\gamma + k)}$$

Since $a_{-1} = 0$, it follows that $a_1 = 0$ and by repeated application $a_k = 0$ for every odd integral value of k. Putting $k = 2, 4, 6, \ldots$ we get

$$a_2 = -\frac{a_0}{2(2\gamma + 2)} = -\frac{a_0}{2^2(\gamma + 1)}, \; a_4 = -\frac{a_2}{4(2\gamma + 4)} = (-1)^2 \cdot \frac{a_0}{2^4 . 2!(\gamma + 1)(\gamma + 2)},$$

$$a_6 = -\frac{a_0}{6(2\gamma + 6)} = (-1)^3 \cdot \frac{a_0}{2^6 . 3!(\gamma + 1)(\gamma + 2)(\gamma + 2)}, \ldots$$

In general,

$$a_{2k} = (-1)^k \cdot \frac{a_0}{2^{2k} \cdot k!(\gamma + 1)(\gamma + 2) \cdots (\gamma + k)}$$

Thus a solution of (7.23) is given by

$$w(z) = a_0 z^\gamma \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{2^{2k} \cdot k!(\gamma+1)(\gamma+2)\cdots(\gamma+k)}$$

Taking $a_0 = \dfrac{1}{2^\gamma \Gamma(\gamma+1)}$, the solution becomes

$$J_\gamma(z) = \frac{1}{\Gamma(\gamma+1)} \left(\frac{z}{2}\right)^\gamma \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{2^{2k} k!(\gamma+1)(\gamma+2)\cdots(\gamma+k)}$$

$$\text{i.e., } J_\gamma(z) = \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{\gamma+2k}}{\Gamma(\gamma+k+1)\Gamma(k+1)}$$

This series with $-\gamma$ in place of γ defines a second linearly independent solution of the Bessel equation given by

$$J_{-\gamma}(z) = \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{-\gamma+2k}}{\Gamma(-\gamma+k+1)\Gamma(k+1)}$$

Hence the general solution of the Bessel equation (7.23) is

$$w(z) = AJ_\gamma(z) + BJ_{-\gamma}(z)$$

where A and B are arbitrary constants.

Case II : When the exponent difference is either zero or a positive integer, the second independent solution of (7.23) is $v(z)J_\gamma(z)$ where

$$\frac{dv}{dz} = \frac{1}{\{J_\gamma(z)\}^2} e^{-\int \frac{1}{z} dz} = \frac{1}{z\{J_\gamma(z)\}^2}$$

so that the solution is $J_\gamma(z) \int \dfrac{dz}{z\{J_\gamma(z)\}^2}$

Bessel function : The Bessel function $J_n(z)$ of first kind and of order n (real or complex) is defined by

$$J_n(z) = \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{n+2k}}{\Gamma(n+k+1)\Gamma(k+1)}$$

If n is a positive integer, then

$$J_{-n}(z) = \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{-n+2k}}{\Gamma(-n+k+1)\Gamma(k+1)} = \sum_{k=n}^{\infty} \frac{(-1)^k (z/2)^{-n+2k}}{\Gamma(-n+k+1)\Gamma(k+1)}$$

($\because$ if k is an integer, $\Gamma(-k)$ is infinity of $k \geq 0$, so we get terms in J_{-n} equal to zero till $-n+k+1 \leq 0$ i.e., $k \leq n-1$)

$$= \sum_{s=1}^{\infty} \frac{(-1)^{n+s} (z/2)^{n+2s}}{\Gamma(s+1)\Gamma(n+s+1)} \qquad \text{(Putting } -n+k=s\text{)}$$

$$= (-1)^n \sum_{s=0}^{\infty} \frac{(-1)^s (z/2)^{n+2s}}{\Gamma(n+s+1)\Gamma(s+1)} = (-1)^n J_n(z)$$

Hence $J_{-n}(z) = (-1)^n J_n(z)$

Recurrence relation for Bessel function :

I. $zJ_n(z) = zJ_{n-1}(z) - nJ_n(z)$

Proof : We have

$$\frac{d}{dz}\left[z^n J_n(z)\right] = \sum_{k=0}^{\infty} \frac{d}{dz}\left[\frac{(-1)^k z^{2n+2k}}{2^{n+2k}\Gamma(n+k+1)\Gamma(k+1)}\right]$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k 2(n+k) z^{2n+2k-1}}{2^{n+2k}\Gamma(n+k+1)\Gamma(k+1)}$$

$$= z^n \sum_{k=0}^{\infty} \frac{(-1)^k (z/2) z^{n+2k-1}}{\Gamma(n+k)\Gamma(k+1)} \qquad (\because \Gamma(n+1) = n\Gamma(n))$$

$$= z^n J_{n-1}(z)$$

Thus $\frac{d}{dz}\left[z^n J_n(z)\right] = z^n J_{n-1}(z)$, i.e., $z^n J_n'(z) + nz^{n-1} J_n(z) = z^n J_{n-1}(z)$

so that

$$z J_n'(z) = z J_{n-1}(z) - n J_n(z) \qquad \text{(A)}$$

II. $zJ_n'(z) = -zJ_{n+1}(z) + nJ_n(z)$

Proof : We have

$$\frac{d}{dz}\left[z^{-n}J_n(z)\right] = \frac{d}{dz}\left[z^{-n}\sum_{k=0}^{\infty}\frac{(-1)^k(z/2)^{n+2k}}{\Gamma(n+k+1)\Gamma(k+1)}\right]$$

$$= \sum_{k=0}^{\infty}\frac{(-1)^k\cdot 2k\cdot z^{2k-1}}{2^{n+2k}\Gamma(n+k+1)\Gamma(k+1)}$$

$$= z^{-n}\sum_{k=1}^{\infty}\frac{(-1)^k(z/2)^{n-1+2k}}{\Gamma(n+k+1)\Gamma(k)}$$

$$= z^{-n}\sum_{s=1}^{\infty}\frac{(-1)^{1+s}(z/2)^{(n+1)+2s}}{\Gamma(n+1+s+1)\Gamma(s+1)} \quad \text{(Putting } k = s+1\text{)}$$

$$= -z^{-n}J_{n+1}(z)$$

Hence $\qquad z^{-n}J_n'(z) - nz^{n-1}J_n(z) = -z^{-n}J_{n+1}(z)$ so that

$$z\,J_n'(z) = -z\,J_{n+1}(z) + n\,J_n(z) \qquad \text{(B)}$$

III. $2J_n'(z) = J_{n-1}(z) - J_{n+1}(z)$

Proof : Adding (A) and (B) we get the required result.

IV. $2nJ_n(z) = z[J_{n-1}(z) + J_{n+1}(z)]$

Proof : Subtracting (B) from (A) and rearranging, we get the required result.

Generating function for Bessel function of integral order : We shall show that when n is an integer, $J_n(z)$ may be defined by means of the expansion of

$$e^{z\left(t-\frac{1}{t}\right)/2}$$

Proof : We have

$$e^{z\left(t-\frac{1}{t}\right)/2} = e^{\frac{1}{2}zt}\cdot e^{-\frac{1}{2}zt^{-1}} = \sum_{k=0}^{\infty}\frac{\left(\frac{1}{2}z\right)^k t^k}{k!}\sum_{s=0}^{\infty}\frac{\left(-\frac{1}{2}z\right)^s t^{-s}}{s!}$$

$$= \sum_{k,s=0}^{\infty} \frac{(-1)^s \left(\frac{1}{2}z\right)^{k+s} t^{k-s}}{k!s!}$$

if n is a non-negative integer, and if $k = n + s$, we have

$$e^{z\left(t-\frac{1}{t}\right)/2} = \sum_{n,s=0}^{\infty} \frac{(-1)^s \left(\frac{1}{2}z\right)^{n+2s} t^n}{(n+s)!s!}$$

$$= t^n \sum_{n,s=0}^{\infty} \frac{(-1)^s \left(\frac{1}{2}z\right)^{n+2s} t^n}{\Gamma(n+s+1)\Gamma(s+1)} = \sum_{n=0}^{\infty} t^n J_n(z)$$

Hence the coefficient of t^n in the above expansion is $J_n(z)$.

Again, if $k - s = n$ where n is a positive integer, we have

$$e^{z\left(t-\frac{1}{t}\right)/2} = \sum_{n,k=0}^{\infty} \frac{(-1)^{k+n}(z/2)^{n+2k}}{(k+n)!k!} t^n = \sum_{n=0}^{\infty} (-1)^n J_n(z)\, t^n$$

$$= \sum_{n=0}^{\infty} J_{-n}(z)\, t^n$$

Hence the coefficient of t^n in the above expansion is $J_{-n}(z)$.

Some trigonometric expansions involving Bessel's functions : We put $t = e^{i\theta}$ in the relation

$$e^{z\left(t-\frac{1}{t}\right)/2} = \sum_{n=-\infty}^{\infty} J_n(z)\, t^n$$

We get $e^{iz\sin\theta} = \sum_{-\infty}^{\infty} J_n(z)(\cos n\theta + i\sin n\theta)$

Equating the real and imaginary parts, we have for real z

$$\cos(z\sin\theta) = \sum_{-\infty}^{\infty} J_n(z)\cos n\theta \tag{C}$$

and
$$\sin(z\sin\theta) = \sum_{-\infty}^{\infty} J_n(z)\sin n\theta \tag{D}$$

From (C) we obtain

$$\cos(z\sin\theta) = J_0(z) + \sum_{n=1}^{\infty}[J_n(z)\cos n\theta + J_{-n}(z)\cos(-n\theta)]$$

$$= J_0(z) + \sum_{n=1}^{\infty}[J_n(z)\cos n\theta + (-1)^n J_n(z)\cos n\theta]$$

$$= J_0(z) + 2\sum_{n=1}^{\infty} J_{2n}(z)\cos(2n\theta) \tag{E}$$

Putting $\theta = \pi/2$ we get

$$\cos z = J_0(z) + 2\sum_{n=1}^{\infty} J_{2n}(z)(-1)^n$$

i.e.,
$$\cos z = J_0(z) - 2J_2(z) + 2J_4(z) - \cdots$$

Also putting $\theta = 0$ in (E) we get

$$1 = J_0(z) + 2J_2(z) + 2J_4(z) + \cdots$$

Again from (D), we obtain

$$\sin(z\sin\theta) = \sum_{n=1}^{\infty}\left[J_n(z)\sin n\theta + (-1)^n J_n(z)\sin(-n\theta)\right]$$

$$= \sum_{n=1}^{\infty}\left[J_n(z)\sin n\theta + (-1)^{n+1} J_n(z)\sin n\theta\right]$$

$$= 2\sum_{n=1}^{\infty} J_{2n-1}(z)\sin(2n-1)\theta$$

Putting $\theta = \pi/2$, we have

$$\sin z = 2J_1(z) - 2J_3(z) + 2J_5(z) - \cdots$$

Bessel's integral formula for integral order : Let m be any integer including zero. Multiplying (C) be $\cos m\theta$ and (D) by $\sin m\theta$ and then adding we obtain

$$\cos(m\theta - z\sin\theta) = \sum_{n=-\infty}^{\infty} J_n(z)\cos(m-n)\theta$$

and so
$$\int_0^{\pi}\cos(m\theta - z\sin\theta)d\theta = \sum_{n=-\infty}^{\infty} J_n(z)\int_0^{\pi}\cos(m-n)\theta\, d\theta$$

$$= J_m(z)\pi$$

Hence $\frac{1}{\pi}\int_0^{\pi}\cos(n\theta - z\sin\theta) = J_n(z)$, which is called the **Bessel's integral formula** of order n.

7.9 Hypergeometric equation : The differential equation

$$z(1-z)\frac{d^2w}{dz^2} + \{c-(a+b+1)z\}\frac{dw}{dz} - abw = 0 \qquad (7.24)$$

where a, b, c are constants, is known as **Gauss' hypergeometric equation.** Comparing equation (7.24) with (7.1) we find

$$p(z) = \frac{c-(a+b+1)z}{z(1-z)},\ q(z) = -\frac{ab}{z(1-z)}$$

Hence $z = 0$ is a singularity of (7.24). Further

$$zp(z) = \frac{c-(a+b+1)z}{1-z} \text{ and } z^2q(z) = -\frac{abz}{1-z}$$

Thus $z = 0$ is a regular singularity of (7.24). The indicial equation of (7.24) corresponding to the singularity $z = 0$ is $\alpha(\alpha-1)+c\alpha = 0$ so that $\alpha = 0, 1-c$

We first find out the solution corresponding to the exponent $a = 0$. If $1 - c$ is not a positive integer, i.e., c is not zero or a negative integer, the equation (7.24) has a solution of the form $w(z)=\sum_{k=0}^{\infty}a_k z^k$, $a_0 \neq 0$. Substituting this in (7.24) we obtain

$$z(z-1)\sum_{k=0}^{\infty}k(k-1)a_k z^{k-2}+[c-(a+b+1)z]\sum_{k=0}^{\infty}k_{a_k}z^{k-1}$$

$$-ab\sum_{k=0}^{\infty}a_k z^k=0$$

Equating the coefficient of z^k from both sides, we obtain

$$(k+1)ka_{k+1}-k(k-1)a_k+c(k+1)a_{k+1}-(a+b+1)ka_k-aba_k=0$$

$$\text{i.e.,}\quad a_{k+1}=\frac{(a+k)(b+k)}{(k+1)(c+k)}a_k.$$

Taking $a_0=1$, we calculate the other a_r's in succession :

$$a_1=\frac{a\cdot b}{1\cdot c},\ a_2=\frac{(a+1)(b+1)}{2\cdot(c+1)}a_1=\frac{a(a+1)\cdot b(b+1)}{2!c(c+1)}$$

$$a_3=\frac{(a+2)(b+2)}{3(c+2)}a_2=\frac{a(a+1)(a+2)\cdot b(b+1)(b+2)}{3!c(c+1)(c+2)}$$

$$\ldots \qquad \ldots$$

Thus the solution of the hypergeometric equation (7.24) is given by

$$w(z)=\sum_{k=0}^{\infty}\frac{a(a+1)(a+2)\cdots(a+k-1)b(b+1)(b+2)\cdots(b+k-1)}{k!c(c+1)(c+2)\cdots(c+k-1)}z^k$$

$$\text{i.e.,}\qquad w(z)=\sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{k!(c)_k}z^k \tag{7.25}$$

where $(a)_k=a(a+1)\cdots(a+k-1)$, etc. The infinite series in (7.25) is known as hypergeometric series and is denoted by $F(a,\ b,\ c;\ z)$.

We now find out the solution of (7.24) corresponding to the exponent $1-c$. If c is not zero or a negative integer, there is a second independent solution of the equation (7.24) near $z=0$. We find the solution by substituting $w=z^{1-c}u$ in the equation (7.24). We get

$$z(1-z)u''+[(2-c)-\{(a+1-c)+(b+1-c)+1\}z]u'-(a+1-c)(b+1-c)u=0$$

which is a hypergeometric equation of the type (7.24) with the constants a, b, c replaced respectively by $a+1-c$, $b+1-c$ and $2-c$. So the equation has the power series solution $u=F(a+1-c,\ b+1-c,\ 2-c;z)$ and, therefore, the second independent solution of (7.24) is

$$w(z)=z^{1-c}F(a+1-c,\ b+1-c,\ 2-c;z)$$

Hence, if c is neither an integer nor zero, the solution of the hypergeometric equation (7.24) is

$$w(z)=AF(a,b,c;z)+Bz^{1-c}F(a+1-c,\ b+1-c,\ 2-c;z)$$

Let us now proceed to find out the solution of (7.24) valid in a neighbourhood of $z=1$ (we note that $z=1$ is also a singularity of the equation (7.24) and that it is a regular singularity). For this, we introduce a new independent variable t by $t=1-z$. This makes $z=1$ corresponding to $t=0$ and the equation (7.24) is transformed to

$$t(1-t)w''+[c-(a+b+1)(1-t)](-w')-abw=0$$

i.e., $$t(1-t)w''+[(a+b+1-c)-(a+b+1)t]w'-abw=0 \qquad (7.25)$$

where the prime now denotes derivatives w.r.t. t. The equation (7.25) is a hypergeometric equation of the type (7.24) with c replaced by $a+b+1-c$. The general solution of (7.25) near $t=0$ is, therefore,

$$w=AF(a,b,a+b+1-c;t)+Bt^{1-(a+b+1-c)}\times$$
$$F[a+1-(a+b+1-c),\ b+1-(a+b+1-c),\ 2-(a+b+1-c);t]$$

Hence the general solution of (7.24) near $z=1$ is

$$w(z)=AF(a,b,a+b+1-c;1-z)$$
$$+B(1-z)^{c-a-b}F(c-b,\ c-a,c-a-b+1;\ 1-z)$$

In this case, it is necessary to assume that $c - a - b$ is not an integer or zero.

Solution near the point at infinity : Putting $z = \frac{1}{t}$ in (7.24), the equation is transformed to

$$t(1-t)\frac{d^2w}{dt^2} + [(1-a-b)-(2-c)t]\frac{dw}{dt} + \frac{ab}{t}w = 0 \tag{7.26}$$

Here $t = 0$ is a singular point and it is a regular singularity. The indicial equation of (7.26) corresponding to $t = 0$ is $\alpha(\alpha-1)+(1-a+b)\alpha+ab=0$, i.e., $\alpha = a, b$ which are the exponents of (7.26) at the singularity $t = 0$.

We first find the solution corresponding to the exponent a. Changing the independent variable w by the substitution $w = t^a \cdot u$, the equation (7.26) is transformed to

$$t(1-t)\frac{d^2u}{dt^2} + [(1+a-b)-\{a+(1+a-c)+1\}t]\frac{du}{dt} - a(1+a-c)u = 0 \tag{7.27}$$

which is a hypergeometric equation of the type (7.24) and so one solution of (7.27) near $t = 0$ is

$$u = F(a, 1+a-c, 1+a-b; t)$$

Thus one solution of (7.24) near the point at infinity is

$$w(z) = z^{-a}F(a, 1+a-c, 1+a-b; \tfrac{1}{z})$$

Similarly, the other solution of (7.24) corresponding to the exponent b is

$$w(z) = z^{-b}F(b, 1+b-c, 1+b-a, \tfrac{1}{z})$$

Hence, if $a \neq b$ and if a and b do not differ by an integer, the general solution is

$$w(z) = Az^{-a}F(a, 1+a-c, 1+a-b; \tfrac{1}{z}) + Bz^{-b}F(b, 1+b-c, 1+b-a; \tfrac{1}{z})$$

Some properties of hypergeometric function :

1. Symmetry property : We show that $F(a,b,c;z) = F(b,a,c;z)$

Proof : We have $F(a,b,c;z) = \sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{k!(c)_k}z^k = \sum_{k=0}^{\infty}\frac{(b)_k(a)_k}{k!(c)_k} = F(b,a,c;z)$

II. Differentiation of hypergeometric function :

We have $F(a,b,c;z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{k!(c)_k} z^k$

$$\therefore \frac{d}{dz} F(a,b,c;z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k z^{k-1}}{(k-1)!(c)_k} = \sum_{k=1}^{\infty} \frac{(a)_k (b)_k z^{k-1}}{(k-1)!(c)_k} = \sum_{j=0}^{\infty} \frac{(a)_{j+1} (b)_{j+1}}{j!(c)_{j+1}} z^j$$

where $j = k - 1$.

Now $(a)_{j+1} = a(a+1)(a+2)\cdots(a+j) = a[(a+1)(a+2)\cdots(a+1+j-1)]$

$$= a(a+1)_j$$

Similarly, $(b)_{j+1} = b(b+1)_j$ and $(c)_{j+1} = c(c+1)_j$

Hence $\frac{d}{dz} F(a,b,c;z) = \frac{ab}{c} \sum_{j=0}^{\infty} \frac{(a+1)_j (b+1)_j}{j!(c+1)_j} z^j = \frac{ab}{c} F(a+1, b+1, c+1; z)$

Similarly, $\frac{d^2}{dz^2} F(a,b,c;z) = \frac{a(a+1)b(b+1)}{c(c+1)} F(a+2, b+2, c+2; z)$

By repeating the process m times we get

$$\frac{d^m}{dz^m} F(a,b,c;z) = \frac{a(a+1)\cdots(a+m-1)b(b+1)\cdots(b+m-1)}{c(c+1)\cdots(c+m-1)} \times$$
$$F(a+m, b+m, c+m; z)$$

i.e., $\frac{d^m}{dz^m} F(a,b;c;z) = \frac{(a)_m (b)_m}{(c)_m} F(a+m, b+m, c+m; z)$

In particular, for $z = 0$ we have since

$$F(a,b,c,0) = \lim_{z \to 0} \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{k!(c)_k} z^k$$

$$= \lim_{z \to 0} \left[1 + \frac{ab}{c} z + \frac{a(a+1)b(b+1)}{2!c(c+1)} z^2 + \cdots \right]$$

$$= 1.$$

So, $\lim_{z\to 0} F(a+1, b+1, c+1; z) = 1$ and hence

$$\left[\frac{d}{dz}\{F(a,b,c;z)\}\right]_{z=0} = \frac{ab}{c}$$

III. If a is a negative integer, then the series stops after certain terms :

For example, let $a = -n$. Then

$$F(-n,b,c;z) = \sum_{k=0}^{\infty}\frac{(-n)_k(b)_k}{k!(c)_k}z^k = \sum_{k=0}^{\infty}\frac{(-n)_k(b)_k}{k!(c)_k},$$

since $(n+1)$th term is zero.

Similarly, when b is a negative integer then also the series stops after a finite number of terms.

But the series which stops at a certain term for a or b, a negative integer, may be made to start again after certain terms when c is also a negative integer of specified value. For example, if $a = -n$ and $c = -(n+m)$, then

$$F(-n,b,-n-m;z) = \sum_{k=0}^{\infty}\frac{(-n)_k(b)_k}{k!(-n-m)_k}z^k$$

But $(-n)_k = (-n)(-n+1)\cdots(-n+k-1) = (-1)^k\dfrac{n!}{(n-k)!}$

and $(-n-m)_k = (-1)^k\dfrac{(n+m)!}{(n+m-k)!}$

Hence $\dfrac{(-n)_k}{(-n-m)_k} = \dfrac{n!}{(n+m)!}\cdot\dfrac{(n+m-k)!}{(n-k)!}$

$$= \frac{n!}{(n+m)!}\left[(n+m-k)(n+m-k-1)\cdots(n-k+1)\right]$$

$$= \frac{(n+m-k)(n+m-k-1)\cdots(n-k+1)}{(n+m)(n+m-1)\cdots(n+1)}$$

$$= \left(1-\frac{k}{n+m}\right)\left(1-\frac{k}{n+m-1}\right)\cdots\left(1-\frac{k}{n+1}\right)$$

Thus $F(-n,b,-n-m;z)=\sum_{k=0}^{\infty}\left(1-\frac{k}{n+m}\right)\left(1-\frac{k}{n+m-1}\right)\cdots\left(1-\frac{k}{n+1}\right)\frac{(b)_k}{k!}z^k$

The terms in the summation are non-zero for k = 0, 1, ..., n; for $k = n + 1$, $n + 2$, ..., $n + m$, all terms are zero. But the terms corresponding to $n + m + 1$ and following are not zero. So for a or $b = -n$ and $c = -n - m$, the series stops at nth term and starts again at $(n + m + 1)$-th term.

Integral formula for the hypergeometric function : We know that

$$F(a,b,c;z)=\sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{k!(c)_k}z^k$$

Now $\frac{(b)_k}{(c)_k}=\frac{b(b+1)\cdots(b+k-1)}{c(c+1)\cdots(c+k-1)}=\frac{1\cdot2\cdots b(b+1)\cdots(b+k-1)}{1\cdot2\cdots c(c+1)\cdots(c+k-1)}\cdot\frac{\Gamma(c)}{\Gamma(b)}$

$$=\frac{\Gamma(b+k)}{\Gamma(c+k)}\cdot\frac{\Gamma(c)}{\Gamma(b)}=\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\cdot\frac{\Gamma(c-b)\Gamma(b+k)}{\Gamma(c+k)}$$

$$=\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}B(c-b,\,b+k)\left(\because \text{Beta function } B(m,n)=\frac{\Gamma(m)\Gamma(b)}{\Gamma(m+n)}\right)$$

$$=\frac{1}{B(b,c-b)}\int_0^1(1-t)^{c-b-1}\cdot t^{b+k-1}dt \text{ (by the definition of beta function)}$$

Hence $F(a,b,c;z)=\frac{1}{B(b,c-b)}\sum_{k=0}^{\infty}\frac{(a)_k}{k!}z^k\int_0^1(1-t)^{c-b-1}\cdot t^{b+k-1}dt$

Interchanging the order of summation and integration, we get

$$F(a,b,c;z)=\frac{1}{B(b,c-b)}\int_0^1(1-t)^{c-b-1}\cdot t^{b-1}\left\{\sum_{k=0}^{\infty}\frac{(a)_k(t)^t}{k!}\right\}dt$$

$$=\frac{1}{B(b,c-b)}\int_0^1(1-t)^{c-b-1}\cdot t^{b-1}\left\{1+\frac{a(t)}{1!}+\frac{a(a+1)(t)^2}{2!}+\cdots\right\}dt$$

i.e., $F(a,b,c;z)=\frac{1}{B(b,c-b)}\int_0^1(1-t)^{c-b-1}\cdot t^{b-1}(1-zt)^{-a}dt$

which is called the **integral formula for the hypergeometric function.**

In particular, we have for $z = 1$

$$F(a,b,c;1) = \frac{1}{B(b,c-b)}\int_0^1 (1-t)^{c-b-1}t^{b-1}(1-t)^{-a}dt$$

$$= \frac{1}{B(b,c-b)}\int_0^1 (1-t)^{c-a-b-1}t^{b-1}dt$$

$$= \frac{B(b,c-a-b)}{B(b,c-b)}$$

Replacing beta functions by gamma functions we get

$$F(a,b,c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}$$

This is known as **Gauss' formula.**

7.10 Legendre equation : The differential equation

$$(1-z^2)\frac{d^2w}{dz^2} - 2z\frac{dw}{dz} + n(n+1)w = 0 \tag{7.28}$$

where n is a constant, is known as Legendre equation. Comparing (7.28) with (7.1) we see that $p(z) = \frac{-2z}{1-z^2}$ and $q(z) = \frac{n(n+1)}{1-z^2}$ both of which are regular at the origin. So $z = 0$ is an ordinary point of the differential equation (7.28). Hence there exists a unique solution $w(z)$ which is regular in a certain neighbourhood of $z = 0$ and which satisfies the conditions $w(0) = a_0$ and $w'(0) = a_1$ where a_0 and a_1 are arbitrary constants.

Let $w(z) = \sum_{k=0}^{\infty} a_k z^k$ and substitute this in (7.28) to get

$$(1-z^2)\sum_{k=2}^{\infty} k(k-1)a_k z^{k-2} - 2z\sum_{k=1}^{\infty} ka_k z^{k-1} + n(n+1)\sum_{k=0}^{\infty} a_k z^k = 0$$

Equating the coefficient of z^k to zero we get

$$(k+2)(k+1)a_{k+2} - k(k-1)a_k - 2ka_k + n(n+1)a_k = 0$$

i.e. $a_{k+2} = -\frac{(n-k)(n+k+1)}{(k+1)(k+2)} a_k, \ (k = 0, 1, 2, \cdots)$

Putting $k = 0, 1, 2, \ldots$ we obtain

$$a_2 = -\frac{n(n+1)}{2!} a_0,$$

$$a_3 = -\frac{(n-1)(n+2)}{3!} a_1,$$

$$a_4 = -\frac{(n-2)(n+3)}{3 \cdot 4} a_2 = \frac{n(n-2)(n+1)(n+3)}{4!} a_0,$$

$$a_5 = -\frac{(n-3)(n+4)}{4 \cdot 5} a_3 = \frac{(n-1)(n-3)(n+2)(n+4)}{5!} a_0,$$

$$\cdots \qquad \cdots$$

Thus, the solution of the equation (7.28) is

$$w(z) = a_0\left[1 - \frac{n(n+1)}{2!} z^2 + \frac{n(n-2)(n+1)(n+3)}{4!} - \cdots\right]$$

$$a_1\left[z - \frac{(n-1)(n+2)}{3!} z^3 + \frac{(n-1)(n-3)(n+2)(n+4)}{5!} z^5 - \cdots\right]$$

i.e., $w(z) = a_0 w_0(z) + a_1 w_1(z)$ (say) (7.29)

It follows that $w_0(z)$ and $w_1(z)$ are two particular solutions of the Legendre differential equation valid in a certain neighbourhood of $z = 0$ and so (7.29) is the general solution of (7.28).

It may be noted that if n is a positive integer, then either $w_0(z)$ or $w_1(z)$ reduces to a polynomial.

Solution near the point at infinity : Let n be a positive integer and $z = \frac{1}{t}$. Then the Legendre equation (7.28) is transformed to

$$t^2(t^2-1)\frac{d^2w}{dt^2} + 2t^3\frac{dw}{dt} + n(n+1)w = 0 \qquad (7.30)$$

Comparing this with $\frac{d^2w}{dt^2}+p(t)\frac{dw}{dt}+q(t)w=0$ we have $p(t)=\frac{2t^3}{t^2(t^2-1)}$

i.e., $tp(t)=-2t^2(1-t^2)^{-1}$ and $q(t)=\frac{n(n+1)}{t^2(t^2-1)}$ i.e., $t^2q(t)=-n(n+1)(1-t^2)^{-1}$

Since $tp(t)$ and $t^2q(t)$ are analytic at $t=0$ it follows that $t=0$, is a regular singularity of (7.30) and so the point at infinity is a regular singularity of the equation (7.28). The indicial equation of (7.30) at $t=0$ is $\alpha(\alpha-1)-n(n+1)=0$ so that $\alpha=-n, n+1$, which are the exponents of (7.30) at the singularity. We now find the solution corresponding to the exponent α, where α is either $-n$ or $n+1$. Then (7.30) has a solution of the form $w(t)=t^{\alpha}\sum_{k=0}^{\infty}a_kt^k=\sum_{k=0}^{\infty}a_kt^{\alpha+k}$, $a_0\neq 0$. Substituting this in (7.30) we get

$$(t^4-t^2)\sum_{k=0}^{\infty}(\alpha+k)(\alpha+k-1)a_kt^{\alpha+k-2}+2t^3\sum_{k=0}^{\infty}(\alpha+k)t^{\alpha+k-1}$$

$$+n(n+1)\sum_{k=0}^{\infty}a_kt^{\alpha+k}=0$$

Equating the coefficient of $t^{\alpha+k}$ to zero we have

$$(\alpha+k-2)(\alpha+k-3)a_{k-2}-(\alpha+k)(\alpha+k-1)a_k+2(\alpha+k-2)a_{k-2}$$
$$+n(n+1)a_k=0$$

or, $$\{(\alpha+k)(\alpha+k-1)-n(n+1)\}a_k=\{(\alpha+k-2)(\alpha+k-3)$$
$$+2(\alpha+k-2)\}a_{k-2}$$

i.e., $$\{(\alpha+k)(\alpha+k-1)-n(n+1)\}a_k=(\alpha+k-2)(\alpha+k-1)a_{k-2} \quad (7.31)$$

Putting $\alpha=-n$ in (7.31) we get

$$\{(-n+k)(-n+k-1)-n(n+1)\}a_k=(-n+k-2)(-n+k-1)a_{k-2} \quad (7.32)$$

or, $$a_k=-\frac{(n-k+2)(n-k+1)}{k(2n-k+1)}a_{k-2} \quad (7.33)$$

Since $a_{-1} = 0$, we have $a_1 = 0 = a_3 = \cdots = a_{2n-1}$. For the L.H.S. of (7.32) is zero and (7.32) reduces to an identity. We may then take for a_{2n+1} an arbitrary constant B (say).

Again, taking $k = 2, 4, \ldots$ in succession

$$a_2 = -\frac{n(n-1)}{2(2n-1)}a_0,\ a_4 = -\frac{(n-2)(n-3)}{4.(2n-3)}a_2 = \frac{n(n-1)(n-2)(n-3)}{2.4.(2n-1)(2n-3)}a_0,$$

$$\ldots \qquad \ldots \qquad \ldots$$

We observe that all a_k's with even suffix for $k \geq n+1$ are zero, since $a_k = 0$ for $k = n + 1$ and $k = n + 2$. So the solution takes the form

$$w(t) = t^{-n}[(a_0 + a_2t^2 + a_4t^4 + \cdots) + (a_{2n+1}t^{2n+1} + a_{2n+3}t^{2n+3} + \cdots)]$$

So, a solution of (7.28) in a neighbourhood of the point at infinity corresponding to the exponent $-n$ is

$$w(z) = z^n a_0\left[1 - \frac{n(n-1)}{2(2n-1)}z^{-2} + \frac{n(n-1)(n-2)(n-3)}{2.4.(2n-1)(2n-3)}z^{-4} + \cdots\right]$$

$$+z^{-n-1}\left[a_{2n+1} + a_{2n+3}\cdot z^{-2} + a_{2n+5}\cdot z^{-4} + \cdots\right] \qquad (7.34)$$

$$a_{2n+3} = -\frac{(-n-1)(-n-2)}{(2n+3)(-2)}B = \frac{(n+1)(n+2)}{2(2n+3)}B \qquad \text{where } B = a_{2n+1},$$

$$a_{2n+5} = -\frac{(n-2n-5+2)(n-2n-5+1)}{(2n-2n-5+1)(2n+5)}a_{2n+3} = \frac{(n+1)(n+2)(n+3)(n+4)}{2.4.(2n+3)(2n+5)}B,$$

So, one required solution becomes (on putting $a_0 = A$)

$$w(z) = Az^n\left\{1 - \frac{n(n-1)}{2(2n-1)}z^{-2} + \frac{n(n-1)(n-2)(n-3)}{2.4.(2n-1)(2n-3)}z^{-4} - \cdots\right\}$$

$$+Bz^{-n-1}\left\{1 + \frac{(n+1)(n+2)}{2(2n+3)}z^{-2} + \frac{(n+1)(n+2)(n+3)(n+4)}{2.4.(2n+3)(2n+5)}z^{-4} + \cdots\right\} \qquad (7.35)$$

$$= Aw_1(z) + Bw_2(z), \text{ (say)}.$$

Again, for $\alpha = n + 1$, we get another solution of (7.28) where a_k are given from (7.31) as

Since $a_{-1} = 0$, it follows that $a_{k=0}$ for all odd suffix k. Putting $k = 2, 4, \ldots$ we have

$$a_2 = \frac{(n+1)(n+2)}{2(2n+3)} a_0,$$

$$a_4 = \frac{(n+3)(n+4)}{4(2n+5)} a_2 = \frac{(n+1)(n+2)(n+3)(n+4)}{2.4.(2n+3)(2n+5)} a_0,$$

$$\cdots \qquad\qquad \cdots$$

So, the solution becomes

$$w(t) = t^{n+1}(a_0 + a_2 t^2 + a_4 t^4 + \cdots)$$

and so another solution of (7.28) near the point at infinity corresponding to the exponent $n + 1$ is

$$w(z) = a_0 z^{-n-1}\left[1 + \frac{(n+1)(n+2)}{2(2n+3)} \cdot z^{-2} + \frac{n(n+1)(n+3)(n+4)}{2.4.(2n+3)(2n+5)} \cdot z^{-4} + \cdots\right]$$

$$= cw_2(z) \qquad \text{where } c = a_0$$

Thus there are two linearly independent solutions of (7.28), viz., $w = Aw_1$ and $w = cw_2$.

Legendre's polynomial : Legendre's polynomial $P_n(z)$ may be defined by the generating relation

$$(1 - 2zt + t^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} P_n(z) t^n \qquad (7.36)$$

In fact, we have

$$(1 - 2zt + t^2)^{-\frac{1}{2}} = \{1 - t(2z - t)\}^{-\frac{1}{2}}$$

$$= 1 + \frac{\frac{1}{2}t(2z-t)}{1!} + \frac{\frac{1}{2}\left(\frac{1}{2}+1\right)t^2(2z-t)^2}{2!} + \cdots$$

$$+ \frac{\frac{1}{2}\left(\frac{1}{2}+1\right)\cdots\left(\frac{1}{2}+n-k-1\right)t^{n-k}(2z-t)^{n-k}}{(n-k)!} + \cdots \quad (7.37)$$

Also, ..

$$(2z-t)^{n-k} = (2z)^{n-k} + \binom{n-k}{1}(2z)^{n-k-1}(-1)t + \binom{n-k}{2}(2z)^{n-k-2}(-1)^2 t^2 +$$

$$\cdots + \binom{n-k}{k}(2z)^{n-k-k}(-1)^k t^k + \text{ finite number of terms.}$$

Thus the coefficient of t^n in the expansion (7.37) is

$$\sum_{k=0}^{[n/2]} \frac{(-1)^k (n-k)!}{k!(n-2k)!}(2z)^{n-2k} \cdot \frac{1\cdot 3\cdots(2n-2k-1)}{2^{n-k}(n-k)!}$$

$$= \sum_{k=0}^{[n/2]} \frac{(-1)^k 2^{-k}}{k!(n-2k)!} \cdot z^{n-2k} \frac{(2n-2k)!}{2\cdot 4\cdots(2n-2k)}$$

$$= \sum_{k=0}^{[n/2]} \frac{(-1)^k (2n-2k)!}{2^n k!(n-2k)!(n-k)!} z^{n-2k} = P_n(z), \text{ (say)}$$

where $[n/2] = n/2$ if n is even

$= (n-1)/2$ if n is odd

Rodrigues formula : We show that $P_n(z) = \dfrac{1}{2^n \cdot n!}\dfrac{d^n}{dz^n}(z^2-1)^n$ (7.38)

which is called **Rodrigues formula**.

Proof : We have

$$\frac{1}{2^n \cdot n!}\frac{d^n}{dz^n}(z^2-1)^n = \frac{1}{2^n \cdot n!}\frac{d^n}{dz^n}\sum_{k=0}^{n}\binom{n}{k} z^{2n-2k}(-1)^k$$

$$= \frac{1}{2^n \cdot n!} \frac{d^n}{dz^n} \sum_{k=0}^{[n/2]} \binom{n}{k} z^{2n-2k} (-1)^k$$

$$= \frac{1}{2^n \cdot n!} \sum_{k=0}^{[n/2]} \frac{n!(-1)^k}{k!(n-k)!} \frac{d^n}{dz^n} (z^{2n-2k})$$

$$= \sum_{k=0}^{[n/2]} \frac{(-1)^k}{2^n \cdot k!(n-k)!} \{(2n-2k)(2n-2k-1)\cdots(2n-2k-n+1)\} \cdot z^{n-2k}$$

$$= \sum_{k=0}^{[n/2]} \frac{(-1)^k (2n-2k)!}{2^n \cdot k!(n-k)!(n-2k)!} z^{n-2k}$$

$$= P_n(z)$$

[We require the following result of complex variables to get the next property of $P_n(z)$

Cauchy's integral formula : If $f(t)$ is an analytic function of a complex variable t regular in a region bounded by a closed contour c and continuous within and on c, it possesses derivatives of all orders which are regular within c, the nth derivative being given by

$$f^n(z) = \frac{n!}{2\pi i} \int_c \frac{f(t)}{(t-z)^{n+1}} dt,$$

z being a point within c.]

Schläfli's integral formula : To show that

$$P_n(z) = \frac{1}{2\pi i} \int_c \frac{(t^2-1)^n}{2^n (t-z)^{n+1}} dt \qquad (7.39)$$

where c is a closed contour surrounding the point $t = z$.

Proof : In Cauchy's integral formula, we put $f(t) = (t^2-1)^n$. Then

$$\frac{d^n}{dz^n} (z^2-1)^n = \frac{n!}{2\pi i} \int_c \frac{(t^2-1)^n}{(t-z)^{n+1}} dt$$

and by using Rodrigue's formula (7.38) we get

$$P_n(z) = \frac{1}{2^n n!} \cdot \frac{n!}{2\pi i} \int_c \frac{(t^2-1)^n}{(t-z)^{n+1}} dt = \frac{1}{2\pi i} \int_c \frac{(t^2-1)^n}{2^n (t-z)^{n+1}} dt$$

Laplace's integral formula : To show that

$$P_n(z) = \frac{1}{\pi} \int_0^{\pi} \left\{ z + (z^2-1)^{\frac{1}{2}} \cos\phi \right\}^n d\phi \qquad (7.40)$$

Proof : In Schläfli's integral formula (7.39) we take the contour c to be a circle as $c: |t - z| = \sqrt{|z^2 - 1|}$. We prove the theorem when $|z| \geq 1$. On c, $t = z + (z^2-1)^{1/2} e^{i\phi}$, $-\pi \leq \phi \leq \pi$. Then

$$t^2 - 1 = z^2 + (z^2-1)e^{2i\phi} + 2z(z^2-1)^{\frac{1}{2}} e^{i\phi} - 1$$

$$= 2(z^2-1)e^{i\phi}\left(\frac{e^{-i\phi} + e^{i\phi}}{2}\right) + 2z(z^2-1)^{\frac{1}{2}} e^{i\phi}$$

$$= 2(z^2-1)^{\frac{1}{2}} e^{i\phi}\left[(z^2-1)^{\frac{1}{2}} \cos\phi + z\right]$$

Again, $dt = i(z^2-1)^{\frac{1}{2}} e^{i\phi} d\phi$.

$$\text{So, } P_n(z) = \frac{1}{2\pi i} \int_{-\pi}^{\pi} \frac{2^n (z^2-1)^{n/2} e^{in\phi} \left[(z^2-1)^{\frac{1}{2}} \cos\phi + z\right]^n i(z^2-1)^{\frac{1}{2}} e^{i\phi} d\phi}{2^n (z^2-1)^{(n+1)/2} e^{i(n+1)\phi}}$$

$$\text{i.e., } P_n(z) = \frac{1}{\pi} \int_0^{\pi} \left\{ z + (z^2-1)^{\frac{1}{2}} \cos\phi \right\}^n d\phi$$

Recurrence relations :

1. $(n+1)P_{n+1}(z) - (2n+1)zP_n(z) + nP_{n-1}(z) = 0$

Proof : We have

$$(1 - 2zt + t^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} P_n(z) t^n \qquad (7.36)$$

Differentiating both sides w.r.t. t, we get

$$-\frac{1}{2}(1-2zt+t^2)^{-\frac{3}{2}}(-2z+2t)=\sum_{n=0}^{\infty}P_n(z)\cdot nt^{n-1} \tag{7.41}$$

or, $$-(1-2zt+t^2)^{-\frac{1}{2}}(t-z)=(1-2zt+t^2)\sum_{n=1}^{\infty}P_n(z)\cdot nt^{n-1}$$

or, $$-(t-z)\sum_{n=0}^{\infty}P_n(z)t^n=(1-2zt+t^2)\sum_{n=1}^{\infty}P_n(z)\cdot nt^{n-1}$$

Equating coefficient of t^n from both sides, we get

$$-P_{n-1}(z)+zP_n(z)=(n+1)P_{n+1}(z)-2znP_n(z)+(n-1)P_{n-1}(z)$$

$$\text{i.e., } (n+1)P_{n+1}(z)-(2n+1)_n P_n(z)+nP_{n-1}(z)=0$$

II. $\boldsymbol{zP_n'(z)-P_{n-1}'(z)=nP_n(z)}$

Proof : Differentiating both sides of (7.36) w.r.t. z, we get

$$-\frac{1}{2}(-2t)(1-2zt+t^2)^{-\frac{3}{2}}=\sum_{n=0}^{\infty}P_n'(z)t^n$$

or, $$t(z-t)(1-2zt+t^2)^{-\frac{3}{2}}=(z-t)\sum_{n=0}^{\infty}P_n'(z)t^n$$

or, $$t\sum_{n=1}^{\infty}nP_n(z)t^{n-1}=(z-t)\sum_{n=0}^{\infty}P_n'(z)t^n \qquad \text{(by using (7.41))}$$

Equating coefficient of t^n we have

$$nP_n(z)=zP_n'(z)-P_{n-1}'(z)$$

$$\text{i.e., } nP_n'(z)-P_{n-1}'(z)=nP_n(z)$$

III. $\boldsymbol{P_{n+1}'(z)-P_{n-1}'(z)=(2n+1)P_n(z)}$

Proof : Differentiating both sides of the recurrence relation I w.r.t. z, we get

$$(n+1)P_{n+1}'(z)-(2n+1)[zP_n'(z)+P_n(z)]+nP_{n-1}'(z)=0$$

$$\text{or, } (n+1)P'_{n+1}(z) - (2n+1)[P'_{n-1}(z) + (n+1)P_n(z)] + nP'_{n-1}(z)$$

(using the recurrence relation II)

$$\text{or, } (n+1)P'_{n+1}(z) - (n+1)P'_{n-1}(z) - (2n+1)(n+1)P_n(z) = 0$$

$$\text{or, } P'_{n+1}(z) - P'_{n-1}(z) = (2n+1)P_n(z)$$

IV. $\boldsymbol{P'_{n+1}(z) - zP'_n(z) = (n+1)P_n(z)}$

Proof : Subtracting the recurrence relation II from that of III we get the result.

V. $\boldsymbol{(1-z^2)P'_n(z) = nP_{n-1}(z) - nzP_n(z)}$

Proof : Replacing n by $n-1$ in the recurrence relation IV, we have

$$P'_n(z) - zP'_{n-1}(z) = nP_{n-1}(z)$$

$$\text{or, } P'_n(z) - z[zP'_n(z) - nP_n(z)] = nP_{n-1}(z)$$

(by recurrence relation II)

$$\text{or, } (1-z^2)P'_n(z) = nP_{n-1}(z) - nzP_n(z)$$

$P_n(z)$ satisfies Legendre equation : Differentiating the recurrence relation V w.r.t. z, we get

$$(1-z^2)P''_n(z) - 2zP'_n(z) = nP'_{n-1}(z) - nP_n(z) - nzP'_n(z)$$

$$= n[zP'_n(z) - nP_n(z)] - nP_n(z) - nzP'_n(z)$$

(by using recurrence relation II)

$$\text{i.e., } (1-z^2)P''_n(z) - 2zP'_n(z) + n(n+1)P_n(z) = 0 \tag{7.42}$$

which shows that $P_n(z)$ satisfies Legendre equation (7.28).

Orthogonal properties : We shall show that

$$\int_{-1}^{1} P_m(z)P_n(z)dz = \begin{cases} 0 & \text{for } m \neq n \\ \dfrac{2}{2n+1} & \text{for } m = n \end{cases}$$

where m and n are positive integers.

Proof : We know that the Legendre polynomial satisfies the equation (7.42), i.e., the equation

$$\frac{d}{dz}\left\{(1-z^2)P_n'(z)\right\} + n(n+1)P_n(z) = 0$$

Similarly, $$\frac{d}{dz}\left\{(1-z^2)P_m'(z)\right\} + m(m+1)P_m(z) = 0$$

Multiplying the first equation by $P_m(z)$ and the second by $P_n(z)$ and then subtracting, we get

$$P_m(z)\frac{d}{dz}\left\{(1-z^2)P_n'(z)\right\} - P_n(z)\frac{d}{dz}\left\{(1-z^2)P_m'(z)\right\} = \{m(m+1)$$

$$-n(n+1)\}P_m(z)P_n(z)$$

or, $$(m-n)(m+n+1)P_m(z)P_n(z) = \frac{d}{dz}\left[(1-z^2)\{P_m(z)P_n'(z) - P_n(z)P_m'(z)\}\right]$$

Hence $$(m-n)(m+n+1)\int_{-1}^{1} P_m(z)P_n(z)dz = [(1-z^2)\{P_m(z)P_n'(z)$$

$$-P_n(z)P_m'(z)\}]_{-1}^{1} = 0$$

so that if $m \neq n$, then

$$\int_{-1}^{1} P_m(z)P_n(z)dz = 0$$

Again, using Rodrigues formula, we have

$$\int_{-1}^{1} P_n^2(z)dz = \frac{1}{2^n \cdot n!}\int_{-1}^{1} P_n(z)\frac{d^n}{dz^n}(z^2-1)^n dz$$

$$= \frac{1}{2^n \cdot n!}\left[\left\{P_n(z)\frac{d^{n-1}}{dz^{n-1}}(z^2-1)\right\}_{-1}^{1} - \int_{-1}^{1} P_n'(z)\frac{d^{n-1}}{dz^{n-1}}(z^2-1)^n dz\right]$$

$$= \frac{(-1)}{2^n \cdot n!}\int_{-1}^{1} P_n'(z)\frac{d^{n-1}}{dz^{n-1}}(z^2-1)^n dz$$

$$= \frac{(-1)^2}{2^n \cdot n!} \int_{-1}^{1} P_n''(z) \frac{d^{n-2}}{dz^{n-2}} (z^2 - 1)^n dz$$

$$\vdots$$

$$= \frac{(-1)^n}{2^n \cdot n!} \int_{-1}^{1} P_n^{(n)}(z) \left[(z^2 - 1)^n\right] dz$$

$$= \frac{(-1)^n}{2^n \cdot n!} \int_{-1}^{1} \frac{d^n}{dz^n} \left\{\frac{(2n)!z^n}{2^n n! n!}\right\} (z^2 - 1)^n dz$$

$$\left[\because P_n(z) = \sum_{k=0}^{[n/2]} \frac{(-1)^k (2n-2k)! z^{n-2k}}{2^n (n-k)!(n-2k)! k!}\right.$$

the term containing z^n is $\left.\frac{(2n)! z^n}{2^n n! n!}\right]$

$$= (-1)^n \cdot \frac{(2n)!}{2^{2n}(n!)^3} \int_{-1}^{1} n!(-1)^n (1 - z^2)^n dz$$

$$= \frac{2(2n)!}{2^{2n}(n!)^2} \int_0^1 (1 - z^2)^n dz$$

$$= \frac{2(2n)!}{2^{2n}(n!)^2} \int_0^{\pi/2} \cos^{2n+1}\theta \, d\theta \quad [\text{Putting } z = \sin\theta]$$

$$= \frac{2(2n)!}{2^{2n}(n!)^2} \cdot 2^{2n} \frac{(n!)^2}{(2n+1)!}$$

$$\therefore \int_{-1}^{1} P_n^2(z) dz = \frac{2}{2n+1}$$

Note : We have seen in equation (7.35) that the solution of Legendre equation in descending power of z is

$$y = Az^n \left\{1 - \frac{n(n-1)}{2(2n-1)} z^{-2} + \frac{n(n-1)(n-2)(n-3)}{2.4 \cdot (2n-1)(2n-3)} z^{-4} + \cdots\right\}$$

$$+Bz^{-n-1}\left\{1+\frac{(n+1)(n+2)}{2(2n+3)}z^{-2}+\frac{(n+1)(n+2)(n+3)(n+4)}{2.4.(2n+3)(2n+5)}z^{-4}+\cdots\right\}$$

Writing

$$P_n(z)=\frac{(2n)!}{2^n.(n!)^2}z^n\left\{1-\frac{n(n-1)}{2(2n-1)}z^{-2}+\frac{n(n-1)(n-2)(n-3)}{2.4.(2n-1)(2n-3)}z^{-4}+\cdots\right\} \text{ and}$$

$$Q_n(z)=\frac{2^n(n!)^2}{(2n)!}z^{-n-1}\left\{1+\frac{(n+1)(n+2)}{2(2n+3)}z^{-2}+\frac{(n+1)(n+2)(n+3)(n+4)}{2.4.(2n+3)(2n+5)}z^{-4}+\cdots\right\}$$

we note that $P_n(z)$ and $Q_n(z)$ are two particular solutions of the Legendre equation. **The functions $P_n(z)$ and $Q_n(z)$ are called Legendre functions of the first and second kind respectively.**

EXAMPLES

1. Prove that $H_n''(z)=4n(n-1)H_{n-2}(z)$

Soln. From recurrence formula $H_n'(z)=2nH_{n-1}(z)$ we get by differentiation w.r.t. z, $H_n''(z)=2nH_{n-1}'(z)=2n[2(n-1)H_{n-2}(z)]$ (by replacing n by $n-1$ in the recurrence formula). Hence $H_n''(z)=4n(n-1)H_{n-2}(z)$.

2. Evaluate

$$\int_{-\infty}^{\infty} ze^{-z^2}H_n(z)H_m(z)dz$$

Soln. Noting the recurrence formula $2zH_n(z)=2nH_{n-1}(z)+H_{n+1}(z)$, we have

$$\int_{-\infty}^{\infty} ze^{-z^2}H_n(z)H_m(z)\,dz=\int_{-\infty}^{\infty} e^{-z^2}\left\{nH_{n-1}(z)+\frac{1}{2}H_{n+1}(z)\right\}H_m(z)dz$$

$$=n\int_{-\infty}^{\infty} e^{-z^2}H_{n-1}(z)H_m(z)dz+\frac{1}{2}\int_{-\infty}^{\infty} e^{-z^2}H_{n+1}(z)H_m(z)dz$$

$$=n\sqrt{\pi}\,2^{n-1}(n-1)!\,\delta_{n-1,m}+\frac{1}{2}\sqrt{\pi}\,2^{n+1}(n+1)!\,\delta_{n+1,m}$$

$$=\sqrt{\pi}\,2^{n-1}n!\{\delta_{n-1,m}+2(n+1)\delta_{n+1,m}\}$$

3. Prove that (i) $H_{2n}(0) = (-1)^n \cdot \frac{(2n)!}{n!}$ and (ii) $H_{2n+1}(0) = 0$

Soln. Since $\sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(z) = e^{-t^2+2tz}$, we have by putting $z = 0$

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(0) = e^{-t^2} = 1 - t^2 + \frac{(t^2)^2}{2!} - \frac{(t^2)^3}{3!} + \cdots + (-1)^n \frac{(t^2)^n}{n!} + \cdots$$

(i) Equating the coefficients of t^{2n} from both sides, we get

$$\frac{1}{(2n)!} H_{2n}(0) = (-1)^n \cdot \frac{1}{n!}, \text{ i.e., } H_{2n}(0) = (-1)^n \cdot \frac{(2n)!}{n!}$$

and (ii) equating the coefficients of t^{2n+1} from both sides, we get

$$\frac{1}{(2n+1)!} H_{2n+1}(0) = 0 \quad \text{i.e., } H_{2n+1}(0) = 0$$

4. Prove that if $m < n$

$$\frac{d^m}{dz^m}\{H_n(z)\} = \frac{2^m n!}{(n-m)!} H_{n-m}(z)$$

Soln. We have $\sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(z) = e^{-t^2+2tz}$

$$\therefore \sum_{n=0}^{\infty} \frac{t^n}{n!} \frac{d^m}{dz^m}\{H_n(z)\} = \frac{d^m}{dz^m}\{e^{-t^2+2tz}\}$$

$$= (2t)^m e^{-t^2+2tz}$$

$$= (2t)^m \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(z)$$

$$= 2^m \sum_{n=0}^{\infty} \frac{1}{n!} t^{n+m} H_n(z)$$

$$= 2^m \sum_{r=m}^{\infty} \frac{1}{(r-m)!} t^r H_{r-m}(z) \quad \text{(Putting } n + m = r\text{)}$$

Equating the coefficient of t^n from the two sides, we have

$$\frac{1}{n!}\frac{d^m}{dz^m}\{H_n(z)\} = 2^m \cdot \frac{1}{(n-m)!}H_{n-m}(z)$$

$$\text{i.e., } \frac{d^m}{dz^m}\{H_n(z)\} = \frac{2^m n!}{(n-m)!}H_{n-m}(z)$$

5. Prove that $P_n(z) = \frac{2}{\sqrt{\pi}n!}\int_0^\infty t^n e^{-t^2} H_n(zt)dt$

Soln. We have $H_n(z) = \sum_{k=0}^{[n/2]} \frac{(-1)^k n!}{k!(n-2k)!}(2z)^{n-2k}$

so that

$$\frac{2}{\sqrt{\pi}n!}\int_0^\infty t^n e^{-t^2} H_n(zt)\,dt$$

$$= \frac{2}{\sqrt{\pi}n!}\int_0^\infty t^n e^{-t^2} \sum_{k=0}^{[n/2]} \frac{(-1)^k n!}{k!(n-2k)!}(2zt)^{n-2k}\,dt$$

$$= \sum_{k=0}^{[n/2]} \frac{2^{n-2k+1}(-1)^k z^{n-2k}}{\sqrt{\pi}k!(n-2k)!}\int_0^\infty e^{-t^2} t^{2n-2k}dt$$

$$= \sum_{k=0}^{[n/2]} \frac{2^{n-2k+1}(-1)^k z^{n-2k}}{\sqrt{\pi}k!(n-2k)!}\int_0^\infty e^{-t^2} t^{2(n-k+\frac{1}{2})-1}dt$$

$$= \sum_{k=0}^{[n/2]} \frac{2^{n-2k+1}(-1)^k z^{n-2k}}{\sqrt{\pi}k!(n-2k)!} \cdot \frac{1}{2}\Gamma\left(n-k+\frac{1}{2}\right)\sqrt{\pi}$$

$$\left[\because 2\int_0^\infty e^{-t^2} \cdot t^{2n-1}dt = \Gamma(n)\right]$$

$$= \sum_{k=0}^{[n/2]} \frac{2^{n-2k}(-1)^k z^{n-2k}(2n-2k)!}{\sqrt{\pi}k!(n-2k)!2^{2n-2k} \cdot (n-k)!}\sqrt{\pi}$$

$$\left[\because \Gamma\left(n+\frac{1}{2}\right) = \frac{(2n)!}{2^{2n} \cdot n!}\sqrt{\pi}\right]$$

$$= \sum_{k=0}^{[n/2]} \frac{(-1)^k (2n-2k)!}{2^n k!(n-2k)!(n-k)!}z^{n-2k} = P_n(z)$$

Hence $P_n(z) = \frac{2}{\sqrt{\pi n!}} \int_0^\infty t^n e^{-t^2} H_n(zt)\, dt$

6. Prove that $L_n(0) = 1$

Soln. We have $\sum_{n=0}^{\infty} t^n L_n(z) = \frac{1}{1-t} e^{-tz/(1-t)}$

Putting $z = 0$ we have $\sum_{n=0}^{\infty} t^n L_n(0) = \frac{1}{1-t} (1-t)^{-1}$

$$= 1 + t + t^2 + \cdots = \sum_{n=0}^{\infty} t^n$$

Equating the coefficient of t^n we get $L_n(0) = 1$

7. Prove that

$$\int_0^\infty e^{-sz} L_n(z) dz = \frac{1}{s}\left(1 - \frac{1}{s}\right)^n$$

Soln. Since $\sum_{n=0}^{\infty} t^n L_n(z) = \frac{1}{1-t} e^{-tz/(1-t)}$

$$\therefore \sum_{n=0}^{\infty} t^n \int_0^\infty e^{-sz} L_n(z)\, dz = \frac{1}{1-t} \int_0^\infty e^{-\left(\frac{t}{1-t}+s\right)z} dz$$

$$= \frac{1}{1-t}\left[-\frac{1-t}{s+t(1-s)} e^{-\frac{s+t(1-s)}{1-t}}\right]_0^\infty$$

$$= \frac{1}{s+t(1-s)} = \frac{1}{s}\left[1 - t\left(1 - \frac{1}{s}\right)\right]^{-1}$$

$$= \frac{1}{s} \sum_{n=0}^{\infty} t^n \left(1 - \frac{1}{s}\right)^n$$

Equating the coefficient of t^n from both sides, we get

$$\int_0^\infty e^{-sz} L_n(z)dz = \frac{1}{s}\left(1-\frac{1}{s}\right)^n$$

5. Show that $J_n(-z) = (-1)^n J_n(z)$ when n is an integer

Soln. We have $J_n(z) = \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{n+2k}}{\Gamma(n+k+1)\Gamma(k+1)}$

Case I : Let n be a positive integer. Replacing z by $-z$, we get

$$J_n(-z) = \sum_{k=0}^{\infty} \frac{(-1)^k (-z/2)^{n+2k}}{\Gamma(n+k+1)\Gamma(k+1)} = (-1)^n \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{n+2k}}{\Gamma(n+k+1)\Gamma(k+1)}$$

$$= (-1)^n J_n(z)$$

$\therefore\ J_n(-z) = (-1)^n J_n(z)$

Case II. Let n be a negative integer, say $n = -m$ where m is a positive integer.

Then $J_n(z) = J_{-m}(z) = (-1)^m J_m(z)$

Replacing z by $-z$, we get

$$J_n(-z) = (-1)^m J_m(-z) = (-1)^m (-1)^m J_m(z) = J_m(z) = J_{-n}(z)$$

$$= (-1)^n J_n(z)$$

$\therefore\ J_n(-z) = (-1)^n J_n(z)$

Note : It may be noted that $J_n(z)$ is an even function of z if n is even $(\because J_n(-z) = J_n(z))$ and $J_n(z)$ is an odd function of z if n is odd $(\because J_n(-z) = -J_n(z))$.

9. Show that

$$\text{(i) } J_{-\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}} \cos z \text{ and (ii) } J_{\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}} \sin z$$

Soln. We know that

$$J_n(z) = \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{n+2k}}{\Gamma(n+k+1)\Gamma(k+1)}$$

(i) Put $n = -\frac{1}{2}$ and get

$$J_{-\frac{1}{2}}(z) = \sum_{k=0}^{\infty} \frac{(-1)^k z^{-\frac{1}{2}+2k}}{2^{-\frac{1}{2}+2k}\Gamma\left(k+\frac{1}{2}\right)\Gamma(k+1)} = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{2^{2k}\Gamma\left(k+\frac{1}{2}\right)\Gamma(k+1)}\sqrt{\frac{2}{z}}$$

In **Legendre's duplication formula** for Gamma function

$$\sqrt{\pi}\Gamma(2x) = 2^{2x-1}\Gamma(x)\Gamma\left(x+\frac{1}{2}\right)$$

we put $x = k + \frac{1}{2}$ and get

$$\sqrt{\pi}\Gamma(2k+1) = 2^{2k}\Gamma\left(k+\frac{1}{2}\right)\Gamma(k+1)$$

so that $J_{-\frac{1}{2}}(z) = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{\sqrt{\pi}\Gamma(2k+1)}\sqrt{\frac{2}{z}} = \sum_{k=0}^{\infty}\sqrt{\frac{2}{\pi z}}\frac{(-1)^k}{(2k)!}z^{2k}$

$$= \sqrt{\frac{2}{\pi z}}\left[1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots\right] = \sqrt{\frac{2}{\pi z}}\cos z$$

(ii) To prove (ii), put $n = \frac{1}{2}$ in Bassel function and $x = k+1$ in Legendre duplication formula and then proceed as above.

10. Prove that

$$J_{n-1} = \frac{2}{z}[nJ_n - (n+2)J_{n+2} + (n+4)J_{n+4} - \cdots]$$

and hence deduce that

$$\frac{1}{2}z\,J_n = (n+1)J_{n+1} - (n+3)J_{n+3} + (n+5)J_{n+5} - \cdots$$

Soln. From the recurrence formula $z[J_{n-1} + J_{n+1}] = 2nJ_n$, we have

$$J_{n+1} + J_{n-1} = \frac{2}{z}nJ_n \qquad (1)$$

Replacing n by $n + 2$ and changing sign, we have

$$-J_{n+3} - J_{n+1} = -\frac{2}{z}(n+2)J_{n+2} \tag{2}$$

Again replacing n by $n + 4$ in (1), we get

$$J_{n+5} + J_{n+3} = \frac{2}{z}(n+4)J_{n+4} \tag{3}$$

Replacing n by $n + 6$ in (1) and changing sign, we get

$$-J_{n+7} - J_{n+5} = -\frac{2}{z}(n+6)J_{n+6} \tag{4}$$

Adding (1), (2), (3), (4), ..., we have

$$J_{n-1} = \frac{2}{z}[nJ_n - (n+2)J_{n+2} + (n+4)J_{n+4} - (n+6)J_{n+6} + \cdots]$$

Replacing n by $n + 1$, we have

$$\frac{1}{2}zJ_n = (n+1)J_{n+1} - (n+3)J_{n+3} + (n+5)J_{n+5} - (n+7)J_{n+7} + \cdots$$

11. Show that

$$\sqrt{\frac{\pi z}{2}}J_{3/2}(z) = \frac{1}{z}\sin z - \cos z$$

Soln. We have

$$J_n(z) = \frac{z^n}{2^n\Gamma(n+1)}\left[1 - \frac{z^2}{2(2n+2)} + \frac{z^4}{2.4.(2n+2)(2n+4)} - \cdots\right]$$

Putting $n = \frac{3}{2}$, we have

$$J_{\frac{3}{2}}(z) = \frac{z^{3/2}}{2^{3/2}\Gamma(5/2)}\left[1 - \frac{z^2}{2.5} + \frac{z^4}{2.4.5.7} - \frac{z^6}{2.4.5.6.7.9} + \cdots\right]$$

$$= \frac{z\sqrt{z}}{2\sqrt{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}}\left[1 - \frac{z^2}{2.5} + \frac{z^4}{2.4.5.7} - \frac{z^6}{2.4.5.6.7.9} + \cdots\right]$$

$$\therefore \sqrt{\frac{\pi z}{2}}J_{\frac{3}{2}}(z) = \frac{1}{3}\left[z^2 - \frac{z^4}{2.5} + \frac{z^6}{2.4.5.7} - \frac{z^8}{2.4.5.6.7.9} + \cdots\right]$$

$$= \frac{2z^2}{3!} - \frac{4z^4}{5!} + \frac{6z^6}{7!} - \frac{8z^8}{9!} + \cdots$$

$$= \left(\frac{1}{2!} - \frac{1}{3!}\right)z^2 - \left(\frac{1}{4!} - \frac{1}{5!}\right)z^4 + \left(\frac{1}{6!} - \frac{1}{7!}\right)z^6 - \left(\frac{1}{8!} - \frac{1}{9!}\right)z^8 + \cdots$$

$$= \left(\frac{1}{2!}z^2 - \frac{1}{4!}z^4 + \frac{1}{6!}z^6 - \cdots\right) + \left(-\frac{z^2}{3!} + \frac{z^4}{5!} - \frac{z^6}{7!} + \cdots\right)$$

$$= -\left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots\right) + \frac{1}{z}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots\right)$$

$$= -\cos z + \frac{\sin z}{z}$$

Hence $\sqrt{\frac{\pi z}{2}} J_{\frac{3}{2}}(z) = \frac{1}{z}\sin z - \cos z$

12. Show that

$$(1+z)^n = F(-n,1,1;z)$$

Soln. We have $(1+z)^n = 1 + nz + \frac{n(n-1)}{2!}z^2 + \frac{n(n-1)(n-2)}{3!}z^3 + \cdots$

$$= 1 + \frac{(-n)\cdot(z)}{1!} + \frac{(-n)(-n+1)12}{2!1.2}(-z)^2 + \frac{(-n)(-n+1)(-n+2)1.2.3}{3!1.2.3}(-z)^3 + \cdots$$

$$= \sum_{k=0}^{\infty} \frac{(-n)_k (1)_k}{k!(1)_k}(-z)^k = F(-n,1,1;z)$$

13. Prove that $\log(1+z) = zF(1,1,2;-z)$

Soln. We have $\log(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \cdots$

$$= z[1 - \frac{1}{2}z + \frac{1}{3}z^2 - \frac{1}{4}z^3 + \cdots]$$

$$= z\left[1 + \frac{1.1}{1.2}(-z) + \frac{(1.2)(1.2)}{1.2.2.3}(-z)^2 + \frac{(1.2.3)(1.2.3)}{(1.2.3)(2.3)}(-z)^3 + \cdots\right]$$

$$= z\sum_{k=0}^{\infty}\frac{(1)_k(1)_k}{k!(2)_k}(-z)^k = zF(1,1,2;-z)$$

14. Show that

$$\tan^{-1} z = zF\left(\tfrac{1}{2},1,\tfrac{3}{2};-z^2\right)$$

Soln. We have

$$\tan^{-1} z = z - \frac{z^3}{3} + \frac{z^5}{5} - \frac{z^7}{7} + \cdots$$

$$= z\left[1 - \frac{z^2}{3} + \frac{z^4}{4} - \frac{z^6}{6} + \cdots\right]$$

$$= z\left[1 + \frac{\frac{1}{2}\cdot 1}{1\cdot\frac{3}{2}}(-z^2) + \frac{\frac{1}{2}\left(\frac{1}{2}+1\right)1.(1+1)}{1\cdot 2\cdot\frac{3}{2}\left(\frac{3}{2}+1\right)}(-z^2)^2 + \right.$$

$$\left.\frac{\frac{1}{2}\left(\frac{1}{2}+1\right)\left(\frac{1}{2}+2\right)1.(1+1)(1+2)}{1.2.3\frac{3}{2}\left(\frac{3}{2}+1\right)\left(\frac{3}{2}+2\right)}(-z^2)^3 + \cdots\right]$$

$$= z\cdot\sum_{k=0}^{\infty}\frac{\left(\frac{1}{2}\right)_k(1)_k}{k!\left(\frac{3}{2}\right)_k}(-z^2)^k = zF\left(\frac{1}{2},1,\frac{3}{2};-z^2\right)$$

15. Show that

$$P_0(z) = 1,\ P_1(z) = z,\ P_2(z) = \frac{1}{2}(3z^2-1),\ P_3(z) = \frac{1}{2}(5z^3-3z),$$

$$P_4(z) = \frac{1}{8}(35z^4 - 30z^2 + 3)$$

Soln. We know that

$$\sum_{n=0}^{\infty} t^n P_n(z) = (1 - 2zt + t^2)^{-\frac{1}{2}}$$

$$= \{1 - t(2z - t)\}^{-\frac{1}{2}}$$

$$= 1 + \frac{t}{2}(2z - t) + \frac{1\cdot 3}{2\cdot 4}t^2(2z-t)^2 + \frac{1.3.5}{2.4.6}t^3(2z-t)^3$$

$$+ \frac{1.3.5.7}{2.4.6.8}t^4(2z-t)^4 + \cdots$$

or, $P_0(z) + tP_1(z) + t^2P_2(z) + t^3P_3(z) + t^4P_4(z) + \cdots$

$$= 1 + tz + \frac{1}{2}(3z^2 - 1)t^2 + \frac{1}{2}(5z^3 - 3z)t^3 + \frac{1}{8}(35z^4 - 30z^2 + 3) + \cdots$$

Equating the coefficients of like powers of t, we have

$$P_0(z) = 1,\ P_1(z) = z,\ P_2(z) = \frac{1}{2}(3z^2 - 1),\ P_3(z) = \frac{1}{2}(5z^3 - 3z),$$

$$P_4(z) = \frac{1}{8}(35z^4 - 30z^2 + 3)$$

16. Express $P(z) = z^4 + 2z^3 + 2z^2 - z - 3$ in terms of Legendre polynomials.

Soln. From Ex. 15, we have

$$35z^4 = 8P_4(z) + 30z^2 - 3 \text{ and } 3z^2 = 2P_2(z) + 1 \text{ i.e., } z^2 = \frac{2}{3}P_2(z) + \frac{1}{3}$$

Hence $z^4 = \frac{8}{35}P_4(z) + \frac{30}{35}\left\{\frac{2}{3}P_2(z) + \frac{1}{3}\right\} - \frac{3}{35} = \frac{8}{35}P_4(z) + \frac{4}{7}P_2(z)\frac{1}{5}$

Also $5z^3 = 2P_3(z) + 3z = 2P_3(z) + 3P_1(z)$ so that $z^3 = \frac{2}{5}P_3(z) + \frac{3}{5}P_1(z)$

Hence $P(z) = \frac{8}{35}P_4(z) + \frac{1}{5} + \frac{4}{5}P_2(z) + \frac{4}{5}P_3(z) + \frac{6}{5}P_1(z) + \frac{4}{3}P_2(z)$

$$+ \frac{2}{3} - P_1(z) - 3$$

$$= \frac{8}{35}P_4(z) + \frac{4}{5}P_3(z) + \frac{40}{21}P_2(z) + \frac{1}{5}P_1(z) - \frac{32}{15}$$

i.e., $P(z) = \frac{8}{35}P_4(z) + \frac{4}{5}P_3(z) + \frac{40}{21}P_2(z) + \frac{1}{5}P_1(z) - \frac{32}{15}P_0(z)$

17. Show that (i) $P_n(1) = 1$, (ii) $P_n(-z) = (-1)^n P_n(z)$. Hence deduce that $P_n(-1) = (-1)^n$

Soln. (i) We have $\sum_{n=0}^{\infty} t^n P_n(z) = (1-2zt+t^2)^{-\frac{1}{2}}$

Putting $z = 1$, we get

$$\sum_{n=0}^{\infty} t^n P_n(1) = (1-2t+t^2)^{-\frac{1}{2}} = (1-t)^{-1} = 1+t+t^2+\cdots = \sum_{n=0}^{\infty} t^n$$

Equating the coefficient of t^n, we get $P_n(1) = 1$.

(ii) Since $(1-2zt+t^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} t^n P_n(z)$, we have

$$(1+2zt+t^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} (-t)^n P_n(z) = \sum_{n=0}^{\infty} (-1)^n P_n(z) t^n$$

Also $(1+2zt+t^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} t^n P_n(-z)$

Hence $\sum_{n=0}^{\infty} t^n P_n(-z) = \sum_{n=0}^{\infty} (-1)^n P_n(z) t^n$

Equating the coefficients of t^n from both sides, we get

$$P_n(-z) = (-1)^n P_n(z)$$

Deduction. Putting $z = 1$ in the above relation and noting that $P_n(1) = 1$, we have $P_n(-1) = (-1)^n$

18. Prove that

(i) $P_n(0) = 0$, for n odd and (ii) $P_n(0) = \dfrac{(-1)^{\frac{n}{2}} n!}{2^n \{(n/2)!\}^2}$, for n even.

Soln. (i) We know that

$$P_n(z) = \frac{1.3.5\cdots(2n-1)}{n!}\left[z^n - \frac{n(n-1)}{2(2n-1)} z^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2.4(2n-1)(2n-3)} z^{n-4} - \cdots\right]$$

when n is odd, the last term contains z. Hence $P_n(0) = 0$

(ii) Again, we have $\sum_{k=0}^{\infty} t^n P_n(z) = (1-2zt+t^2)^{-\frac{1}{2}}$ so that

$$\sum_{k=0}^{\infty} t^n P_n(0) = (1+1^2)^{-\frac{1}{2}} = \{1-(-1^2)\}^{-\frac{1}{2}}$$

$$= 1 + \frac{1}{2}(-t^2) + \frac{1\cdot 3}{24}(-t^2)^2 + \frac{1.3.5}{2.4.6}(-t^2)^3$$

$$+ \cdots + \frac{1.3.5\cdots(2r-1)}{2.4.6\cdots 2r}(-t^2)^r + \cdots$$

Here all powers of t on the R.H.S. are even. Equating the coefficient of t^{2m} on both sides, we have

$$P_{2m}(0) = \frac{1.3.5\cdots(2m-1)}{2.4.6\cdots 2m}(-1)^m = (-1)^m \frac{(2m)!}{2^{2m}\cdot(m!)^2}$$

i.e., when $n = 2m$, i.e., even, $P_n(0) = \dfrac{(-1)^{n/2} n!}{2^n\{(n/2)!\}^2}$

EXERCISES

1. For what value of n, $H_n(0) = 0$? [Ans. odd integer]

2. Prove that

$$H_n(z) = 2^{n+1} e^{z^2} \int_0^{\infty} e^{-t^2} t^{n+1} P_n(z/t)\, dt$$

[*Hints* : Proceed as in Ex. 5]

3. Show that for $n = 0, 1, 2, \ldots$

(i) $H_{2n}(z) = \dfrac{2^{n+1}(-1)^n}{\sqrt{\pi}} e^{z^2} \int_0^{\infty} e^{-t^2} \cdot t^{2n} \cos 2zt dt$

(ii) $H_{2n+1}(z) = \dfrac{2^{2n+2}(-1)^n}{\sqrt{\pi}} e^{z^2} \int_0^{\infty} e^{-t^2} \cdot t^{2n+1} \sin 2zt dt$

4. Show that

$$H_1(z) = 2zH_0(z)$$

5. Prove that

$$H_5(z) = 32z^5 - 160z^3 + 120z$$

[*Hints* : Use Rodrigues formula for Hermite polynomials.]

6. Prove that

$$\int_z^\infty e^{-t} L_n(t)dt = e^{-z}[L_n(z) - L_{n-1}(z)]$$

7. Find the values of

(i) $\int_0^\infty e^{-z} L_2(z) L_5(z) dz$

(ii) $\int_0^\infty e^{-z} L_4^2(z) dz$. [Ans. (i) 0, (ii) 1]

8. Expand $L(z) = z^3 + z^2 - 3z + 2$ in a series of Laguerre polynomials.

[Ans. $L(z) = -6L_3(z) + 20L_2(z) - 19L_1(z) + 7L_0(z)$]

9. Show that

$$L_n''(0) = \frac{1}{2} n(n-1)$$

10. Show that

$$J_{\frac{5}{2}}(z) = \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} \left[\frac{3-z^2}{z^2} \sin z - \frac{3}{z} \cos z\right]$$

11. Show that

$$\int_0^z z^n J_{n-1}(z) dz = z^n J_n(z)$$

12. Show that

$$\int_0^z z^{n+1} J_n(z) dz = z^{n+1} J_{n+1}(z)$$

13. Show that

(i) $J_{2n}(z) = (-1)^n \cdot \dfrac{2}{\pi} \int_0^{\pi/2} \cos 2n\phi \cos(z \sin \phi) d\phi$

(ii) $J_{2n-1}(z) = (-1)^{n+1} \cdot \dfrac{2}{\pi} \int_0^{\pi/2} \cos(2n+1)\phi \sin(z \sin \phi) d\phi$

14. Prove that

$$\int_0^\infty \frac{J_n(z)}{z}dz = \frac{1}{n}$$

15. Prove that

$$\int_0^b zJ_0(az)dz = \frac{b}{a}J_1(ab)$$

16. Prove that (i) $4\int J_{n+1}(z)dz = \int J_{n-1}(z)dz - 2J_n(z)$

(ii) $4\frac{d^2}{dz^2}[J_n(z)] = J_{n-2}(z) - 2J_n(z) + J_{n+2}(z)$

17. Show that

$$(1-z)^{-\alpha} = F(\alpha, \beta, \beta; z)$$

18. Show that

$$z^n = F(-n,1,1;1-z)$$

19. Express z^7 as a series in Legendre polynomials.

[Ans. $\frac{16}{529}P_7 + \frac{8}{39}P_5 + \frac{14}{33}P_3 + \frac{1}{3}P_1$]

20. Show that

$$\int_{-1}^{1} zP_n(z)P_{n-1}(z)dz = \frac{n-1}{2n+1}\int_{-1}^{1} P_{n+1}(z)P_{n-1}(z)dz + \frac{n}{2n+1}\int_{-1}^{1}\{P_{n-1}(z)\}^2 dz$$

21. Show that

$\int_{-1}^{1} f(z)P_n(z)dz = 0$ if n is even and f is odd.

22. Use Rodrigues formula $P_n(z) = \frac{1}{2^n n!}\frac{d^n}{dz^n}(z^2-1)^n$ to show that

$$\int_{-1}^{1} f(z)P_n(z)dz = \frac{(-1)^n}{2^n n!}\int_{-1}^{1} f^n(z)(z^2-1)^n dz$$

in which f is a continuous function in $(-1, 1)$ and $f^{n}(z)$ denotes the nth derivative of f. Hence show that

$$\int_{-1}^{1} z^m P_n(z)dz = 0 \ \text{ for } m < n$$

$$= \frac{2^{m+1}(n!)^2}{(2n+1)!}, \ m = n$$

23. Prove that

$$\int_z^1 P_n(z)dz = \frac{1}{2n+1}[P_{n-1}(z) - P_{n+1}(z)]$$

7.11 Summary : Various types of special functions are introduced as solutions of some second-order linear differential equations of the form

$$w'' + p(z)w' + q(z)w = 0$$

in the complex z-plane, in the neighbourhood of ordinary points or regular singular points. In particular the orthogonal properties of the polynomials like those of Hermite, Laguerre and Legendre are obtained. The recurrence relations of these polynomials and the functions such as the Hypergeometric function and Bessel function are deduced. These special functions satisfy the typical second-order equations.

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Paper : PG (MT) 03 :
Group B

UNIT 1 ❑ FUNDAMENTAL CONCEPTS

§ 1.1 Introduction

The basic concepts from solid geometry and some properties of ordinary differential equations (ODE) play important roles in the theory of partial differential equations (PDE). We, therefore, first discuss those before the sutdy of partial differential equations.

§ 1.2 Surfaces and Curves in Three Dimensions

Let P(x, y, z) be a point in three-dimentional space and the coordinates are connected by a relation of the form

$$F(x,\ y,\ z) = 0 \tag{1.1}$$

Then the equation (1.1) represents the euqation of a surface on which the point P lies. To demonstrate this, consider the increments $(\delta x, \delta y, \delta z)$ in (x, y, z) related by the equation

$$\frac{\partial F}{\partial x}\delta x + \frac{\partial F}{\partial y}\delta y + \frac{\partial F}{\partial z}\delta z = 0$$

in which any two can be chosen arbitrarily. Thus, in every neighbourhood of the point P(x,y,z), there exist points $P(x+\xi, y+\eta, z+\tau)$ satisfying the relation (1,1) for which we can choose any two of the variable $\xi; \eta, \tau$ orbitrarily and the third is given by

$$\xi\frac{\partial F}{\partial x} + \eta\frac{\partial F}{\partial y} + \gamma\frac{\partial F}{\partial z} = 0$$

The projection of the initial direction $\overline{PP'}$ on the xy plane can be chosen arbitrarly. Hence the equation (1.1) represents a relation satisfied by points lying on the surface.

Consider now a set of relations of the form

$$x = f_1(u,v), y = f_2(u,v), z = f_3(u,v) \tag{1.2}$$

so that to each pair of values u, v there corresponds a set of numbers and hence a point in space. However, every point in space does not correspond to a pair of values of u and v. Solving the first two equations in (1.2), u and v can be expressed as a function x and y of the form

$$u = F_1(x,y), v = F_2(x,y)$$

which when substituted into the third equation of (1,2). determines z as a function of x and y in the form

$$z = f(x,y)$$

Hence there exists a functional relation (1.1) between the coordinates x, y, z and the relation shows that the point (x, y, z) lies on a surface. The equation (1.2) expresses the fact that point (x, y, z) determined from them always lies on a fiexed surface.

Equations of the type (1.2) are called *parametric euqations* of the surface. However, parametric euqaions of surface are not uniuq. For example, the following two sets of parametric equations.

$$x = a\sin u\cos v,\ y = a\sin u\ \ \sin v,\ z = a\cos u$$

and

$$x = \frac{2av}{1+v^2}\sin u.\ y = \frac{2av}{1+v^2}\cos u,\ z = \frac{1-v^2}{1+v^2}$$

represent the same spherical surface $x^2+y^2+z^2=a^2$

A surface may be thought of being generated by a curve. If a point has coordinates (x, y, z) satisfying the equation (1.1) and lying on a plane $z = k$, then the coordinates satisfy equations.

$$z = k, F(x, y, k) = 0$$

Thus the point (x, y, z) lies on a curve $(x,y,z)\Gamma$, say in the plane $z=k$. As for example, if we consider a sphere S: $x^2+y^2+z^2=a^2$, then the equations $z=k$, $x^2+y^2=a^2-k^2$ shows that Γ is a circle of radus $(a^2-k^2)^{1/2}$, real if $k<a$. Since k varies from $-a$ to ⁂, each point of the sphere is covered by one such circle. We may, therefore, think the surface of the sphere of being generated by such circles. In general, we can say that the surface (1.1) is generated by the curves (1.3).

Alternatively, the curve represented by the pair of equations (1.3) may be thought of as the intersection of the surface (1.1) with the plance $z=k$, or, more generally, the intersection of two surfaces. For, if the point (x,y,z) lies on both the surfaces $S_1:F(x,y,z)=0$ and $S_2:G(x,y,z)=0$, then the points common to both S_1 and S_2 satisfy the pair of equations

$$F(x,y\,z)=0,\ G(z,y,z)=0 \qquad (1.4)$$

Thus the locus of a point whose coordinates satisfy a pair of equations (1.4) is a curve in space.

A curve may be specified by parametric equations of the form

$$x = f_1(t), y = f_2(t), z = f_3(t) \qquad (1.5)$$

in which t is a continuous variable. For, if the coordinates of Point P are given by (1.5) then P lies on a curve whose euqations are of the form

$$\Phi_1(x, y) = 0, \quad \Phi_2(x, z) = 0$$

obtained by elimination of t between the equations $x = f_1(t), y = f_2(t), z = f_3(t)$ respectively.

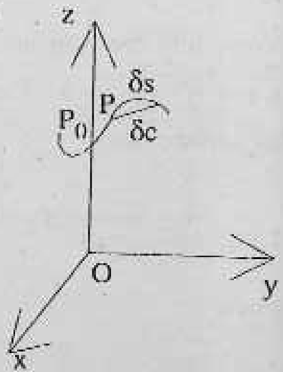
Let us now consider a point P on the curve

$$x = x(s), \ y = y(s), \ z = z(s) \qquad (1.6)$$

characterised by the arc length s, measured from some fixed point P_0 along the curve. Let $Q\{x(s+\delta s), y(s+\delta s), z(s+\delta s)\}$ be another point on the curve at a distance δs from P. Let the chord PQ = δc.

Then $\lim\limits_{\delta s \to 0} \frac{\delta c}{\delta s} = 1$. Now the direction cosines of the chord PQ are

$$\frac{x(s+\delta s) - x(s)}{\delta c}, \quad \frac{y(s+\delta s) - y(s)}{\delta c}, \quad \frac{z(s+\delta s) - z(s)}{\delta c}$$



i.e. $\frac{\delta s}{\delta c}\left\{\frac{dx}{ds} + o(\delta s)\right\}, \ \frac{\delta s}{\delta c}\left\{\frac{dy}{ds} + 0(\delta s)\right\}, \ \frac{\delta s}{\delta c}\left\{\frac{dz}{ds} + 0(\delta s)\right\}$

(By Maclaurin's theorem)

$$\to \frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds} \text{ as } \delta s \to 0, \text{ i.e. } Q \to P$$

and the chord PQ takes up the direction to the tangent to the curve at P. Hence the direction cosines of the tangent to the curve (1.6) at the point P are $\left(\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}\right)$. Here we have assumed that the curve (1.6) is completely arbitrary.

Now suppose that the curve Γ Given by the equation (1.6) lies on the surface S characterised by the equation F $(x, y, z) = 0$ so that the point $\{x(s), y(s), z(s)\}$ of Γ lies on the surface S

$$F\{x(s), y(s), z(s)\} = 0 \tag{1.7}$$

If the curve lies entirely on the surface S, then the equation (1.7) becomes an identity for all values of s. Differentiating (1.7) with respect to s, we get

s, we (1.7)

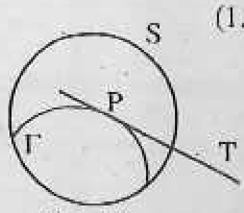


Fig 1.2

$$\frac{\partial F}{\partial x}\frac{dx}{ds} + \frac{\partial F}{\partial y}\frac{dy}{ds} + \frac{\partial F}{\partial s}\frac{dz}{ds} = 0 \tag{1.8}$$

so that the tangent to the curve Γ at the point P is perpendicular to the line whose direction ratios are

$\left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}\right)$.It may be noted that the curve Γ is arbitrary except that it passes through the point P and lies entirely on the surface S. Moreover, since the line with direction ratios $\left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}\right)$ is perpendicular to the tangent to every curve lying on S and passes through P, this ,must be normal to the surface S at the point P.

Now let the surface S has an equation of the form $z = f(x,y)$ so that $F(x, y, z) = f(x,y) - z = 0$. Then the direction cosines of the normal at any poont (x, y, z) of the surface are

$$\frac{1}{\sqrt{p^2 + y^2 + 1}}(p, q, -1) \tag{1.9}$$

where

$$p = \frac{\partial z}{\partial x},\ q = \frac{\partial z}{\partial y} \tag{1.10}$$

Now the equation of the tangent plane π_1 at the point $P(x,y,z)$ to the surface $S_1 : F(x, y, z) = 0$ is

$$(X-x)\frac{\partial F}{\partial x}+(Y-y)\frac{\partial F}{\partial y}+(Z-z)\frac{\partial F}{\partial z}=0 \tag{1.11}$$

Where (X,Y,Z) are the coordinates of any other point of the tangent plae. Similarly, the euqation of the tangent plane $\pi_2$at the point $P(x,y,z)$ to the surface $S_2:G(x,y,z)=0$ is

$$(X-x)\frac{\partial G}{\partial x}+(Y-y)\frac{\partial G}{\partial y}+(Z-z)\frac{\partial G}{\partial z}=0 \tag{1.12}$$

The intersection L of the planes π_1 and π_2 is the tangent at P to the curve Γ generated by the intersection of the two surface S_1 and S_2. From (1.11) and (1.12), it follows that the equations of the line L are

$$\frac{X-x}{\frac{\partial F}{\partial y}\frac{\partial G}{\partial z}-\frac{\partial F}{\partial z}\frac{\partial G}{\partial y}}=\frac{Y-y}{\frac{\partial F}{\partial z}\frac{\partial G}{\partial x}-\frac{\partial F}{\partial x}\frac{\partial G}{\partial y}}=\frac{Z-z}{\frac{\partial F}{\partial x}\frac{\partial G}{\partial y}-\frac{\partial F}{\partial y}\frac{\partial G}{\partial x}}$$

i.e. $$\frac{X-x}{\frac{\partial(F,G)}{\partial(y,z)}}=\frac{Y-y}{\frac{\partial(F,G)}{\partial(z,x)}}=\frac{Z-z}{\frac{\partial(F,G)}{(x,y)}} \tag{1.13}$$

Hence the direction ratios of the line L are

$$\left\{\frac{\partial(F,G)}{\partial(y,z)},\frac{\partial(F,G)}{\partial(z,x)},\frac{\partial(F,G)}{\partial(x,y)}\right\} \tag{1.14}$$

Example 1.1 : Show that the condition that the surface $F(x,y,z)=0$ and $G(x,y,z)=0$ should touch is that the eliminate of x, y, z from these equations and the euqations $F_x:G_x=F_y:G_y=F_z:G_z$ should hold.

Hence find the condition that the plane $lx+my+nz+p=0$ should touch the central conicoid $ax^2+by^2+cz^2=1$.

Solution. If the two surface tauch each other at some point $P(x,y,z)$, say, then they have the common tangent plane, i.e. the equations

$$(X-x)\frac{\partial F}{\partial x}+(Y-y)\frac{\partial F}{\partial y}+(Z-z)\frac{\partial F}{\partial z}=0,$$

and $$(X-x)\frac{\partial G}{\partial x}+(Y-y)\frac{\partial G}{\partial y}+(Z-z)\frac{\partial G}{\partial z}=0$$

must be the same, Thus we have

$$\frac{F_x}{G_x}=\frac{F_y}{G_y}=\frac{F_z}{G_z} \tag{1.15}$$

Hence the required condition is obtained by eliminating x, y, z from the given equations of surface and the equation (1.15).

Second part : Let

$$F(x,y,z)=lx+my+nz+p=0 \tag{1.16a}$$

and $$G(x,y,z)=ax^2+by^2+cz^2-1=0 \tag{1.16b}$$

If these two surface touch, then we have by using (1.15)

$$\frac{l}{2ax}=\frac{m}{2by}=\frac{n}{2cz} \text{ i.e. } \frac{l}{ax}=\frac{m}{by}=\frac{n}{cz}=\frac{1}{k} \quad \text{(say)}$$

where k is some constant. Hence

$$x=\frac{l}{ak},\ y=\frac{m}{bk},\ z=\frac{n}{ck} \tag{1.16c}$$

so that from (1.16a) we get $k=-\frac{1}{P}\left(\frac{l^2}{a}+\frac{m^2}{b}+\frac{b^2}{c}\right)$ (1.16d)

Again (1.16b) and (1.26c) give

$$k^2=\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}\Rightarrow p^2=\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}$$ (by using 1.16d)

This is required condition.

1.3. Simulatancous Differential Equations of the First order and the First Degree in Three Variables.

An ordinary differential euqation of an order higher than the first can always be reduced to a system of first order differential equation. For example, consiser a differential equation of the n th order

$$\frac{d^n y}{dx^n} = f\left(x, y, \frac{dy}{dx}, \ldots\ldots\ldots, \frac{d^{n-1}y}{dx^{n-1}}\right)$$

This may be replaced by the following system of n equations of the first order :

$$\frac{dy}{dx} = y_1, \frac{dy_1}{dx} = y_2, \ldots\ldots, \frac{dy_{n-1}}{dx} = f(x, y, y_1, y_2, \ldots\ldots y_{n-1}) \qquad (1.17)$$

Similar results hold for more than one dependent variable. If there are m dependent variables, the system contains m equations each of the form

where $y_1, y_2, \ldots y_m$ are the m dependent variables. Thus we may always assume the differential equations to be of the first order and of the first degree.

Let us now consider equations in three variables given by

$$P_1 dx + Q_1 dy + R_1 dz = 0$$

$$P_2 dx + Q_2 dy + R_2 dz = 0$$

where each of the functions $P_i Q_i R_i, (i = 1, 2)$ is a function of x, y, z. it follows that

$$\frac{dx}{Q_1 R_2 - Q_2 R_1} = \frac{dy}{R_1 P_2 - R_2 P_1} = \frac{dz}{P_1 Q_2 - P_2 Q_1}$$

which may be written as

$$\frac{dx}{p} = \frac{dy}{Q} = \frac{dz}{R} \qquad (1.18)$$

We take these equations as a set of simultaneous equations ofthe first order. The *existance* and *uniqueness* of equations (1.18) follow from the following theorem which we state without proof :

***Theorem* 1.1** : *Let the functions* $f_1(x,y,z)$ *and* $f_2(x,y,z)$ *be continuous in the region defined by* $|x - a|<k$, $|y - b|<l$, $|z - c|<m$ *where* l, m, n *are constants, and satisfy Lipschitz condition*

$$|f_1(x,y,z)-f_1(x,\eta,\tau)<A_1|y-\eta|+B_1||z-\tau|$$

$$|f_2(x,y,z)-f_2(x,\eta,\tau)<A_2|y-\eta|+B_2||z-\tau|$$

in the defined region, where A_1, B_1, A_2, B_2 *are some finite constansts. Then there exists a unique pair of functions* $y(x)$ *and* $z(x)$ *continuous and having continuous derivatives in a suitable interval* $|x-a|<h$ *which indentically the differential equations*

$$\frac{dy}{dx}=f_1(x,y,z) \quad and \quad \frac{dz}{dx}=f_2(x,y,z)$$

and have properties $y(a)=b$, $z(a)=c$, *where* a, b, c *are arbitrary numbers.*

According to the above theorem, there exist two cylinders $y=y(x)$ and $z=z(x)$ passing through the points (a,b,o) and (a,o,c) respectively such that $\frac{dy}{dx}=f_1$ and $\frac{dz}{dx}=f_2$.

The complete solution of these pair of equations, therefore, consists of the set of points which are common to both the cylinders, i.e. it consists of their curve of intersection Γ. This curve Γ passes throngh the point (a, b, c) and satisfes the pair of differential equations. Noting that a, b, c are arbitrary, it follows that the general sdution of these pair of equations consists of the curves formed by the intersection of a one-parameter system of cylinders of which $y = y(x)$ and z = $z(x)$ are two particular members. In other wordswe can say that general solution of a set of simultaneous equations of the form (1.18) will be a two-parameter family of curves in three-dimensional space.

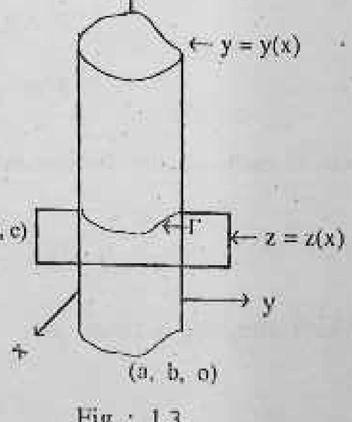


Fig : 1.3

Method of Solution

In solving the euqations of the form (1.18), we firstg note that if we can derive from these equations two relations of the type

$$u_1(x,y,z)=c_1,\ \ u_2(x,y,z)=c_2 \tag{1.19}$$

where c_1 and c_2 are arbitrary constants, then by varying these constants, we obtain a two-parameter family of curves satisfying the differential equations (1.19).

Methoed 1 : Let $u_1(x,y,z)=c_1$ be a suitable one-parameter system of surfaces. Since any tangential direction throngh the point (x,y,z) to the surface $u_1(x,y,z)=c_1$ satisfies the relation

$$\frac{\partial u_1}{\partial x}dx+\frac{\partial u_1}{\partial y}dy+\frac{\partial u_1}{\partial z}dz=0$$

this tangential direction to the integral curve must also be tangential direction to this surface. Hence we have

$$P\frac{\partial u_1}{\partial x}+Q\frac{\partial u_1}{\partial y}+R\frac{\partial u_1}{\partial z}=0$$

To find u_1, we try to look for three functions P', Q', R' such that

$$PP'+QQ'+RR'=0 \tag{1.20}$$

and
$$P'=\frac{\partial u_1}{\partial x},\ Q'=\frac{\partial u_1}{\partial y},\ R'=\frac{\partial u_1}{\partial z} \tag{1.21}$$

so that
$$P'dx+Q'dy+R'dz \tag{1.22}$$

is an exact differential du_1.

Similarly, we find u_2.

Example 1.2 . Find the integral curves of the equations

$$\frac{dx}{x\left(2y^4-z^4\right)}=\frac{dy}{y\left(z^4-2x^4\right)}=\frac{dz}{z\left(x^4-y^4\right)}$$

Solution. Here $P=x\left(2y^4-z^4\right)$, $Q=y\left(z^4-2x^4\right)$, $R=z\left(x^4-y^4\right)$.

Let us choose $P' = \frac{1}{x}, Q' = \frac{1}{y}, \ R' = \frac{2}{z}$. Then the condition (1.20) is satisfied and the expression $\frac{1}{x}dx + \frac{1}{y}dy + \frac{2}{z}dz = d\left\{\log\left(xyz^2\right)\right\}$, an exact differntial. Hence we have $u_1 = xyz^2$.

Again, if we choose $P' = x^3, \ Q' = y^3, \ R' = z^3$, then also the condition (1.20) is satisfied and the expression $x^3dx + y^3dy + z^3dz = d\left\{\frac{1}{4}\left(x^4 + y^4 + z^4\right)\right\}$ is an exact differential so that $u_2 = x^4 + y^4 + z^4$.

Hence the integral curves of the given differential equation are the members of the two-parameter family.

$$xyz^2 = c_1 \text{and} \ \ x^4 + y^4 + z^4 = c_2.$$

Method II. Suppose we can find three functions P', Q', R' such that

$$\frac{P' dx + Q' dy + R' dz}{PP' + QQ' + RR'}$$

is an exact differential dW', say. Also, let P'', Q'', R'' be three other functions such that

$$\frac{P'' dx + Q'' dy + R'' dz}{PP'' + QQ'' + RR''}$$

becomes an exact differential dW", say Noting that each of the above ratios is equal to dx/P, it follows that dW'=dW" so that there exists a relation $W' = W'' + C$ between x, y and z, C being an arbitrary constant.

Example 1.3 : Solve the equations $\quad \frac{dx}{mz - ny} = \frac{dy}{nx - lz} = \frac{dz}{ly - mx}$

Solution . Each of these ratios is equal to

$$\frac{\lambda dx + \mu dy + \upsilon dz}{\lambda(mz - ny) + \mu(nx - lz) + \upsilon(ly - mx)}$$

If λ, μ, υ are constant multipliers, then this expression will be exact differential provided it is of the form

$$\frac{1}{\rho}\frac{\lambda dx + \mu dy + \upsilon dz}{\lambda x + \mu y + \gamma z}$$

and this is possible if

$$-\rho\lambda + n\mu - m\upsilon = 0$$
$$-n\lambda + \rho\mu - l\upsilon = 0 \qquad (1.23)$$
$$m\lambda - l\mu - \rho\upsilon = 0$$

These equations possess a solution if ρ is a root of the equation

$$\begin{vmatrix} -\rho & n & -m \\ -n & -\rho & l \\ m & -l & -\rho \end{vmatrix} = 0, \text{ i.e. } \rho^3 + \rho\left(l^2 + m^2 + n^2\right) = 0 \qquad (1.24)$$

The roots are $\rho_1 = 0$, $(\rho_2, \rho_3) = \pm\sqrt{l^2 + m^2 + n^2}$. Substituting the value of ρ_1 in (1.23) and solving to find $\lambda_1 = l$, $\mu_1 - = m$, $\upsilon_1 = n$ we see that $lx + my + nz =$ constant $= c_1$, say,

Again, Substituting the values of ρ_2 and ρ_3 in (1.23) and solving for $\lambda_i, \mu_i, \upsilon_i (i = 1,2,3)$, we get

$$dW' = \frac{1}{\rho_2}\frac{\lambda_2 dx + \mu_2 dy + \upsilon_2 dz}{\lambda_2 x + \mu_2 y + \upsilon_2 z} \qquad dW' = \frac{1}{\rho_3}\frac{\lambda_3 dx + \mu_3 dy + \upsilon_3 dz}{\lambda_3 x + \mu_3 y + \upsilon_3 z}$$

We get $W' = \frac{1}{\rho_2} log(\lambda_2 x + \mu_2 y + \upsilon_2 z)$, $W' = \frac{1}{\rho_3} log(\lambda_3 x + \mu_3 y + \upsilon_3 z)$ so that

$$(\lambda_2 x + \mu_2 y + \upsilon_2 z)(\lambda_3 x + \mu_3 y + \upsilon_3 z) = const. = C_2,$$

where C is constant, Noting that $\rho_2 = \rho_3$ we find

$$(\lambda_2 x + \mu_2 y + \upsilon_2 z)(\lambda_3 x + \mu_3 y + \upsilon_3 z) = const. = C_2, \text{ say.}$$

Thus the required solutions are

$$lx + my + nz = C_1, \quad (\lambda_2 x + \mu_2 y + \upsilon_2 z)(\lambda_3 x + \mu_3 y + \upsilon_3 z) = const. = C_2.$$

Method III. Suppose one of the variables, say z is absent from one equation of the set (1.18). Then we can derive the integral curves in a simple way. Then if z is absent in P and Q, we have $\frac{dx}{P}=\frac{dy}{Q}$ i.e. $\frac{dy}{dx}=\frac{Q}{P}$ which has solution of the form $f(x,y,c_1)=0$. Elimination of either x or y from one of the other equations in (1.18), we obtain another relation between x and z or y ; y and z ; this will be the second equation of the solution.

Example 1.4 : Solve the equation

$$\frac{dx}{y^2+z^2-x^2}=\frac{dy}{-2xy}=\frac{dz}{-2xz}$$

Solution. From the last two equations we have $\frac{dy}{dz}=\frac{y}{z}\Rightarrow y=c_1z, c_1$ being constant. Also from the first and the last relations, we get

$$\frac{dx}{(C_1^2+1)z^2-x^2}=\frac{dz}{-2xz}\Rightarrow (c_1^2+1)dz=-\frac{2xzdx-x^2dz}{z^2}=-d\left(\frac{x^2}{z}\right)$$

$$\Rightarrow (c_1^2+1)z=-\frac{x^2}{z}+c_2\Rightarrow\left(\frac{y^2}{z^2}+1\right)z=-\frac{x^2}{z}+c_2,$$ c_2 being integration const.

so that $x^2+y^2+z^2=c_2z$

Hence the required solutions are

$$y=c_1z,\ x^2+y^2=z^2=c_2z.$$

§ 1.4. Orthogonal Trajectories of a System of Curves on a Surface.

Let $$F(x,y,z)=0 \tag{1.25}$$

be the equation of a surface and there is a system of curves on it. Then a system of curves of which lies on the surface (1.25) and cuts every curve of the system at right angles is called *system of orthogonal trajectories* on the surface of the given system of curves. We may, therefore, think of the original system of curves as the intersection of the surface (1.25) with the one-parameter family of surfaces

$$G(x,y,z)=C_1 \tag{1.26}$$

As an illustration, consider the system of circles (shown by full lines in Fig. 1.4) on the cone

$$x^2+y^2=z^2\tan^2\alpha$$

by the system of paralel planes $z=C_1$, where C_1 is a parameter. Obviously, the generators (shown by dotted lines Fig. 1.4) are orthogonal trajectories in this case.

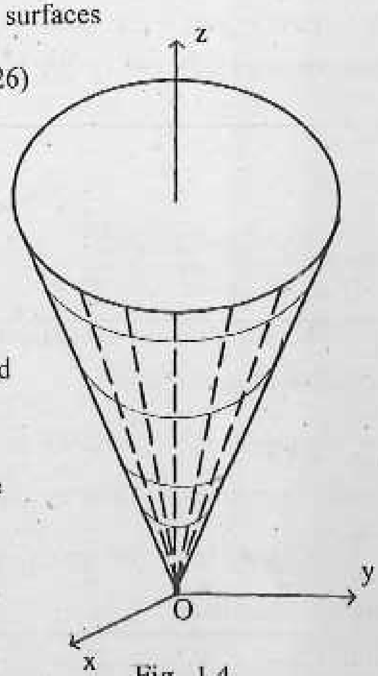


Fig. 1.4

In general the tragential direction (dx,dy,dz) to the given curve through the point (x,y,z) on the surface (1.25) satisfies the equations

$$\frac{\partial F}{\partial x}dx+\frac{\partial F}{\partial y}dy+\frac{\partial F}{\partial z}dz=0$$

and
$$\frac{\partial G}{\partial x}dx+\frac{\partial G}{\partial y}dy+\frac{\partial G}{\partial z}dz=0$$

so that
$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$$

(1.27)

where
$$P=\frac{\partial(F,G)}{\partial(y,z)},\quad Q=\frac{\partial(F,G)}{\partial(y,z)},\quad R=\frac{\partial(F,G)}{\partial(x,y)}$$

Since the curve through the point (x, y, z) of the orthogoal system has tangential direction (dx', dy', dz') (cf. Fig 1.5) lying on the surface (1.25) implying

$$\frac{\partial F}{\partial x}dx'+\frac{\partial F}{\partial y}dy'+\frac{\partial F}{\partial z}dz'=0 \tag{1.28}$$

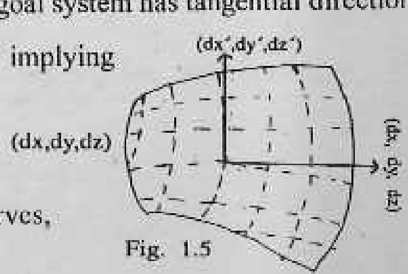


Fig. 1.5

and is perpendicular to the original system of curves, we have from (1.27)

$$Pdx' + Qdy' + Rdz' = 0 \tag{1.29}$$

Equations (1.28) and (1.29) together yield

$$\frac{dx'}{P'} = \frac{dy'}{Q'} = \frac{dz'}{R'} \tag{1.30}$$

where $P' = R\frac{\partial F}{\partial y} - Q\frac{\partial F}{\partial z}$, $Q' = P\frac{\partial F}{\partial z} - R\frac{\partial F}{\partial x}$, $R' = Q\frac{\partial F}{\partial x} - P\frac{\partial F}{\partial y}$ (1.31)

The solution of the equations (1.30) with the relation (1.25) gives the system of orthogonal trajectories.

Example 1.5 : Find the orthogonal trajectories on the conicoid $(x+y)z = 1$ of the conics in which it is cut by the system of planes $x - y + z = k$, where k is a parameter.

Solution. Here the given system of coincs on the conicoid is characterized by the pair of equations.

$$zdx + zdy + (x+y)dz = 0 \text{ and } dx - dy + dz = 0$$

which are equivalent to

$$\frac{dx}{x+y+z} = \frac{dy}{x+y-z} = \frac{dz}{-2z}$$

The system of orthogonal trajectories is, therefore, determined by the pair of equations

$$zdx + zdy + (x+y)dz = 0$$

and

$$(x+y+z)dx + (x+y-z)dy - 2zdz = 0$$

i.e. by

$$\frac{dx}{-2z^2 - (x+y)(x+y-z)} = \frac{dy}{(x+y)(x+y+z) + 2z^2} = \frac{dz}{-2z^2}$$

From the first and the last relations we have by putting $x + y = \frac{1}{z}$,

$$\frac{dx}{2z^2 + \frac{1}{z}\left(\frac{1}{z} - z\right)} = \frac{dz}{2z^2} \quad i.e. \quad dx = \left(1 + \frac{1}{2z^4} - \frac{1}{2z^2}\right)dz$$

Integrating, we get $x+c=z-\frac{1}{6z^3}+\frac{1}{2z}$ where c is a parameter. Hence the orthogonal trajectonies are given by the equations

$$x+c=z-\frac{1}{6z^3}+\frac{1}{2z},\ (x+y)z=1$$

1.5. Pfaffian Differential Equations

Let $F_i(i=1,2,......,n)$ be functions of some or all of the n independent variables x_1, x_2, ..., x_n. Then the expression of the form

$$\sum_{i=1}^{n} F_i(x_1,x_2,.....,x_n)dx_i \tag{1.32}$$

is called a *Pfaffian differential form* and the equation

$$\sum_{i=1}^{n} F_i(x_1,x_2,.....,x_n)dx_i=0 \tag{1.33}$$

is known as *Pfaffian differential equation.*

There is a fundamental difference between Pfaffian differential equations in two variables and those in higher number of variable.

For two variables x, y, we can write the equation (1.33) in the form

$$P(x,y).dx+Q(x,y)dy=0 \text{ i.e. } \frac{dy}{dx}=f(x,y) \tag{1.34}$$

where $F(x,y)=-P/Q$. If P and Q are defined single-valued in the xy-plane, $f(x,y)$ is also deficnd uniquely and is single-valued in the same plane. Thus the solution of (1.34) subject to the boundary condition $y=y_0$ at $x=x_0$ consists of the curve passing through this point and the tangent at each point of the curve is defined by (1.34), Hence the differential equation (1.34) defines a one-parameter family of curves in the xy-plane. I other words, there exists a function.

$$\Phi(x,y)=c \tag{1.35}$$

c being constant, which defines a function $y(x)$ satisfying the differential equation (1.34) identically at least in a certain region in the xy-plane.

The differential form $Pdx + Qdy$ is said to be exact or integrable if it can be written in the form $d\phi(x,y)$. Otherwise, we write the equation (1.35) in the differential form

$$\frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy = 0$$

Thus there exists a function $\phi(x,y)$ and a function $\mu(x,y)$ such that

$$\frac{1}{P}\frac{\partial\phi}{\partial x} = \frac{1}{Q}\frac{\partial\phi}{\partial y} = \mu.$$

By multiplying the equation (1.34) by μ, we see that it can be writte as

$$0 = \mu(Pdx + Qdy) = d\phi$$

The function $\mu(x,y)$ is called an *integrating factor* of the Pfaffian differential equation (1.34). Thus we have the following theorem :

Theorem 1.2 **:** *In the case of two variables, a Pfaffian differential equation always possesses an integrating factor.*

Now we consider Pfaffian differential equation in three variables x, y, z

$$P(x,y,z)dx + Q(x,y,z)dy + R(x,y,z)dz = 0 \tag{1.36}$$

Or, in vector notations

$$X.dr = 0 \tag{1.37}$$

where $X = (P,Q,R)$ and $dr = (dx,dy,dz)$. Before the discussions of the equation (1.37) we consider the following two lemmas.

Lemma 1.1 **:** *A necessary and sufficient condition that there exists a relation $F(u,v) = 0$ between two functions $u(x,y)$ and $V(x,y)$, not involving the variables x or y explicity, is that*

$$\frac{\partial(u,v)}{\partial(x,y)} = 0 \tag{1.38}$$

Proof. For necessity of the condition, we first note that

$$F(u,v) = 0 \tag{1.39}$$

is an identity in x and y. Differentiating this w.r.t x and y, we get respectively

$$\frac{\partial F}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial F}{\partial v}\frac{\partial v}{\partial x}=0$$

$$\frac{\partial F}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial F}{\partial v}\frac{\partial v}{\partial x}=0$$

Elimination of $\frac{\partial F}{\partial v}$ between these two equations yields

$$\frac{\partial F}{\partial u}\frac{\partial(u,v)}{\partial(x,y)}=0$$

Since (1.39) involves both u and v, $\frac{\partial F}{\partial u}\neq 0$ and hence $\frac{\partial(u,v)}{\partial(x,y)}=0$

To prove the sufficiency, we eliminate y from $u=(x,y)$ and $v=v(x,y)$ and obtain the relation

$$F(u,v,x)=0$$

Differentiating this w.r.t x and y, we obtain

$$\frac{\partial F}{\partial x}+\frac{\partial F}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial F}{\partial v}\frac{\partial v}{\partial x}=0$$

and $$\frac{\partial F}{\partial x}\frac{\partial v}{\partial y}+\frac{\partial F}{\partial u}\frac{\partial(u,v)}{\partial(x,y)}=0$$

which by *using* the condition (1.38) gives $\frac{\partial F}{\partial x}\frac{\partial v}{\partial y}=0$. Since $v=v(x,y)$, *so* $\frac{\partial v}{\partial y}\neq 0$ and, therefore, $\frac{\partial F}{\partial x}=0$. Thus the function F does not contain variable x explicitly. Similarly, F does not contain the variable y explicity.

Lemma 1.2 : *Let* **X** *be a vector function of x, y, z such that* **X**. *curl* **X** = 0 *and* μ *is also a function of x, y, z. The* $(\mu X)\cdot curl(\mu X) = 0$.

Proof : Let **X** =(P, Q, R). Then

$$(\mu X).curl(\mu X) = \mu P\left\{\frac{\partial}{\partial y}(\mu R) - \frac{\partial}{\partial z}(\mu Q)\right\} + \mu Q\left\{\frac{\partial}{\partial z}(\mu P) - \frac{\partial}{\partial x}(\mu R)\right\}$$

$$+\mu Q\left\{\frac{\partial}{\partial x}(\mu Q) - \frac{\partial}{\partial y}(\mu P)\right\}$$

$$= \mu^2\left\{P\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) + Q\left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) + R\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\right\}$$

$$+\mu\left\{PR\frac{\partial \mu}{\partial y} - PQ\frac{\partial \mu}{\partial z} + PQ\frac{\partial \mu}{\partial z} - QR\frac{\partial \mu}{\partial x} + QR\frac{\partial \mu}{dx} - PR\frac{\partial \mu}{dy}\right\}$$

$$= \mu^2 X.curlX$$

$$= 0$$

Which proves the lemma.

By the use of the factor $\frac{1}{\mu}$, the converse of Lemma 1.2 follows easily.

We now return to the Pfaffian differential equation (1.36). All equations of this type do not possess integrals. However, if we can find a function $\mu(x,y,z)$ such that the expression $\mu(Pdx + Qdy + Rdz)$ becomes an exact differential d, say, then the equation (1.36) is said to be *integrable*, $\mu(x + y + z)$ is termed as an *integrating factor* and the function $\phi(x,y,z)$ is known as the *primitive* of the differential equation.

The *criterion* for the Pfaffian differential equation (1.36) to be integrable is given by the following theorem.

Theorem 1.3 : *A necessary and sufficient condition for the Pfaffian differential equation X.dr=0 to be integrable is that X.curlX* = 0, *where X* = (*P, Q, R*) and dr = (*dx, dy, dz*).

Proof : For *necessity* of the condition, we first note that if the equation *X.dr* = 0 i.e. $Pdx + Qdy + Rdz = 0$ be integrable, then there exists a relation between the variables *x, y, z of* the type =C, C being constant, so that

$$\frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy + \frac{\partial F}{\partial z}dz = 0$$

and, therefore, there exists a function $\mu(x,y,z)$ such that

$$\mu P = \frac{\partial F}{\partial x}, \mu Q = \frac{\partial F}{\partial y}, \mu F = \frac{\partial F}{\partial z}, ie. \mu X = grad\ F$$

Hence, curl (μX) = curl grad $F = 0$, ie. (μX) curl $(\mu X) = 0$.

It, therefore, follows from Lemma 1.2 that X.curl $X = 0$.

To prove the *sufficiency,* we suppose the z is constant. Then the differential equation $X.\mathrm{dr} = 0$ reduces to $P(x,y,z)dx + Q(x,y,z)dy = 0$ which, by Theorem 1.2 possesses a solution of the from $\mathrm{U}(x,y,z) = \mathrm{C}_1$, where the constant C_1 may involve z. Also there exists a funciton $\mu(x,y,z)$ such that $\mu \mathrm{P} = \frac{\partial U}{\partial x}$ and $\mu \mathrm{Q} = \frac{\partial U}{\partial y}$. Substituting these in the equation $Pdx + Qdy + Rdz = 0$, we get

$$\frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}dy + \left(\mu R - \frac{\partial U}{\partial z}\right)dz = 0 \text{ ie, } dU + Kdz = 0 \tag{1.40}$$

where

$$K = \mu R - \frac{\partial u}{\partial z}$$

Since $\mu X = (\mu P, \mu Q, \mu R) = \left(\frac{\partial U}{\partial x}, \frac{\partial U}{\partial y}, \frac{\partial U}{\partial z} + k\right) = \text{grad } U + (o,o,K)$, we have

$$\mu X.curl(\mu X) = \left(\frac{\partial U}{\partial x}, \frac{\partial U}{\partial y}, \frac{\partial U}{\partial z} + K\right)\cdot\left(\frac{\partial K}{\partial y}, -\frac{\partial K}{\partial x} 0\right)$$

$$= \frac{\partial U}{\partial x}\frac{\partial K}{\partial y} - \frac{\partial U}{\partial y}\frac{\partial K}{\partial x} = \frac{\partial(U,K)}{\partial(x,y)}$$

Now by lemma 1.2, X.curlX = 0 implies $(\mu \mathrm{X})$. curl$(\mu \mathrm{X}) = 0$ and so $\frac{\partial(U,K)}{\partial(x,y)} = 0$. Hence using Lemma 1.2, it follows that there exists a relation between U and K which is independent of x and y but not necessarily of z. In other words, K can be expressed as a function of U and z alone, *ie.* $K = K(U,z)$ and the equation (1.40) gives $\frac{\partial U}{\partial x} + K(U,z) = 0$ which by Theorem 1.2, has a solution of the form $\Phi(U,z) = \text{constant} = C$, say, Replacement of U by its expression in terms of x, y, z, we obtain the solution in the form $F(x,y,z) = C$. Hence the equation $X.dr = 0$ is integrable.

Theorem 1.4 : *If the Pfaffian differential equation* $Pdx + Qdy + Rdz = 0$ *has an integrating factor, then we can find an infinity of them.*

Proof. Let $\mu(x,y,z)$ be an integrating factor of the Pfaffian differential equation $Pdx + Qdy + Rdz = 0$. Then there exists a function $\phi(x,y,z)$ such that

$$\mu P = \frac{\partial \phi}{\partial x}, \mu Q = \frac{\partial \phi}{\partial y}, \mu R = \frac{\partial \phi}{\partial z}$$

If $\Phi(\phi)$ be an arbitrary function of ϕ, then we can write the given equation as

$$\mu \frac{d\Phi}{d\phi}(Pdx + Qdy + Rdz) = 0, \text{ ie. } \frac{d\Phi}{d\phi}\left(\frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz\right) = 0$$

$\Rightarrow d\Phi = 0$ so that $\Phi(\phi) = \text{constant} = C$, say.

Thus if μ is an integrating factor yielding a solution $\phi = \text{constant}$, and if Φ is an arbitrary function of ϕ, then μ is also an integrating factor of the given equation. But Φ is arbitrary and, therefore, there exists an infinitely many integrating factors.

Example 1.6 : Verify that the differential equation $yzdx + 2xzdy - 3xydz = 0$ is integrable and find its primitive.

Solution. Here $X = (xyz, 2xz, -3xy)$ so that curl $X = (-5x, 4y, z)$. Hence X.Curl c = 0, and, therefore, the given differential equation is integrable (by Theorem 1.3).

Now if we treat z as a constant, the given equation reduces to $\frac{dx}{x} + \frac{2dy}{y} = 0$ which has the solution $U(x,y,z) = xy^2 = \text{const} = C_1$, say.

Thus $\mu \frac{1}{P}\frac{\partial U}{\partial x} = \frac{1}{yz} \cdot y^2 = \frac{y}{z}$ and in the notation of (1.41)

$$K = \mu R - \frac{\partial U}{\partial z} = \frac{y}{z} \cdot (-3xy) = -\frac{3xy^2}{z}$$

so that the equation (1.40) reduces to

$$y^2dx + 2xy \cdot dy - \frac{3xy^2}{z}dz = 0, \text{ ie. } d\left(\frac{xy^2}{z^3}\right) = 0$$

leding to the solution of the original equation as $xy^2 = Cz^3$, C is constant.

§ 1.6. Solution of Pfaffian Differential Equations is Three variables.

We now discuss the methods by which Pfaffian differential equations in three variables can be solved.

(a) *By inspection.* If the condition of integrability is satisfied, we can find the primitive of the equation by inspection. In particular, if the given equation is such that curl X = 0, then X = grad ϕ and the equation X.dr = 0 reduces to

$$\frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz = 0$$

which has the primitive $\phi(x, y, z) = const.$

Example 1.7 : Solve the equation $(yz + z^2)dx - zx.dy + zydz = 0$ by first showing that it is integrable.

Solution. We have $X = (yz + z^2, -xz, xy)$ so that curl X = $(2x, 2z, -2zy)$. Hence X.curlX =0. Thus the given equation is integrable.

Now the given equation can be written as

$$z(y + z)dx + x(ydz - zdy) = 0$$

$$\text{or, } (y + z)(zdx + xdz) - xzd(y + z) = 0$$

$$\text{or, } \frac{d(zx)}{zx} = \frac{d(y + z)}{y + z}$$

Thus the primitive of the given equation is $y + z = czx$, where c is constant.

(b) *Variable separable.* Sometimes Pfaffian differential equation can be written in the form

$$P(x)dx + Q(y)dy + R(z)dz = 0$$

for which the integral surfaces are given by

$$\int P(x)dx + \int Q(y)dy + \int R(z)dz = \text{constant}.$$

Example 1.8. Solve the equation $xy^2dx + zx^2dy - x^2y^2dz = 0$.

Solution. By dividing both sides of the given equation by x^2y^2z we get

$$\frac{dx}{x^2} + \frac{dy}{y^2} - \frac{dz}{z} = 0$$

Integrating the required integral surfaces are obtained as $ze^{\frac{x+y}{xy}} = c$, where c is constant.

(c) *One variable separable.* If one of the variables, say z, is separable, then the Pfaffian differential equation is of the form

$$P(x,y)dx + Q(x,y)dy + R(z)dz = 0$$

Noting the $X = \{P(x,y), Q(x,y), R(z)\}$, we have curl $X = \left(0,0\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$ so that the condition of integrability X. curl$X = 0$ implies $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$ and hence $Pdx + Qdy$ is an exact differential du, say. We may, therefore, write the given equation as $du + R(z)dz = 0$ so that the primitive is

$$u(x,y) + \int R(z)dz = \text{const.}$$

Example 1.9 : Verify that the equation $2yzdx - 2xzdy - (x^2 - y^2)(z^2 - 1)dz = 0$ is integrable and hence solve it.

Solution. The given equation can be written as

$$\frac{2y}{x^2 - y^2}dx - \frac{2x}{x^2 - y^2}dx - \frac{z^2 - 1}{z}dz = 0$$

so that $X = \left(\frac{2y}{x^2 - y^2}, -\frac{2x}{x^2 - y^2}, -\frac{z^2 - 1}{z}\right)$ and curl $X = (0, 0, 0)$ and hence the condition of integrability X.curlX = 0 is satisfied. We can rewrite the above equation in the form

$$2\frac{d(x/y)}{\left(\frac{x}{y}\right)^2-1}-\left(z-\frac{1}{z}\right)dz=0$$

Integrating, we get $\log\frac{\frac{x}{y}-1}{\frac{x}{y}+1}-\frac{1}{2}z^2+\log z=\log c$, where log c is constant.

Thus the required solution is $(x-y)=c(x+y)e^{\frac{1}{2}z^2}$

(d) *Homogeneous equation.* Let the functions P,Q,R in the Praffian different equation $P(x,y,z)dx+Q(x,y,z)dy+R(x,y,z)dz=0$ be homogeneous in x, y, z of the same degree n. Then this equation is transformed by the substitutions $y=ux, z=vx$, u and v are functions of x only, to the equation in the form

$$P(1,u,v)dx+Q(1,u,v)(udx+xdu)+R(1,u,v)(xdu+vdx)=0,$$ a factor x^n canceling out. This equation can be written as

$$\frac{dx}{x}+A(u,v)du+B(u,v)=0 \tag{1.42}$$

where

$$A(u,v)=\frac{Q(1,u,v)}{P(1,u,v)+uQ(1,u,v)+vR(1,u,v)}$$

$$B(u,v)=\frac{R(1,u,v)}{P(1,u,v)+uQ(1,u,v)+uR(1,u,v)}$$

The equation (1.42) can be solved by the method (c).

We may put the above result in another way. If the condition of integrabitity is satisfied and P, Q, R are homogeneous functions of x, y, z of the same degree and $xP + yQ + zR \neq 0$, then its reciprocal is an integrating factor of the given equation.

Example 1.10 : Verify that the equation $(y^2 + z^2 - x^2)dz - 2xydy - 2zxdz = 0$ is integrable and find its solution.

Solution. Here $X = y^2 + z^2 - x^2, -2xy, -2zx$ and curl$X$ = (0, 0, 0) so that the condition of integrabitity X. curl$X = 0$ is satisfied. Since the components of X are homegeneous functions of x, y, z of degree 2, we can reduce the given equation by putting $y = u(x).x$, $z = v(x).x$, to the form

$$(u^2 + v^2 - 1)dx - 2u(udx + xdu) - 2v(vdx + xdv) = 0$$

or,
$$\frac{dx}{x} + \frac{d(u^2 + v^2 + 1)}{u^2 + v^2 + 1} = 0$$

whose solution is $x(u^2 + v^2 + 1) = c$, where c is constant.

Returning to the original variables, the required solution is

$$x^2 + y^2 + z^2 = cx.$$

Exampl: 1.11 : Solve the equation $(y^2 + yz + z^2)dz + (z^2 + zx + x^2)dy + (x^2 + xy + y^2)dz = 0$

Solution. Here $X = (P, Q, R) = (y^2 + yz + z^2, z^2 + zx + x^2, x^2 + xy + y^2)$ and curl$X$ = $2(y - z, z - x, x - y)$ so that X.curl$X = 0$ and thus the condition of integrability is satisfied.

Also $F = Px + Qy + Rz = x(y^2 + yz + z^2) + y(z^2 + zx + x^2)$

$$+z(x^2 + xy + y^2) = (x + y + z)(yz + zx + xy) \neq 0$$

and $dF = (yz + yz + zy)d(x + y + z) + (x + y + z)(ydz + zdy + xdz + zdx + xdy + ydx)$

The integrating factor is $\mu(x, y, z) = \frac{1}{F} = \frac{1}{(x + y + z)(yz + zx + xy)}$

Multiplying both sides of the given equation by $\mu(x, y, z)$ we get

$$\frac{(y^2 + yz + z^2)dx + (z^2 + zx + x^2)dy + (x^2 + xy + y^2)dz}{(x + y + z)(yz + zx + xy)} = 0$$

$$\text{or, } \frac{dF}{F} - 2\frac{(yz + zx + xy)d(x + y + z)}{(x + y + z)(yx + zx + xy)} = 0$$

$$\text{or, } \frac{dF}{F} - 2\frac{d(x + y + z)}{x + y + z} = 0$$

leading to the solution $F = c(x + y + z)^2$ ie. $(yz + zx + xy) = c(x + y + z)$, where c is constant.

(e) *Method of reduction.* Suppose one of the variables, say z, is constant and then consider the equation $Pdx + Qdy = 0$ whose solution can be obtained in the form $\phi(x, y) = c$, say, where c is independent of x and y, but may depend on z. So taking differential of $P(x, y) = c$ and equating to $Pdx + Qdy + Rdz = 0$, we determine c. The integral of $\frac{dc}{dz}$ gives c as a function of z which when substituted in $\phi = c$ gives the required solution.

Example 1.12 : Solve $3x^2dx + 3y^2dy - \left(x^3 + y^3e^{2z}\right)dz = 0$

Solution. Here $X = \left\{3x^2, 3y^2, -\left(x^3 + y^3 + e^{2z}\right)\right\}$ and curlX $= \left(-3y^2, 3x^2, 0\right)$ so that X. curlX = 0 and, therefore, the condition of integrability in satisfied.

Taking the variable z to be constant, the given equation reduces to $3x^2dx + 3y^2dy = 0$ whose solution is $x^3 + y^3 = c$, where c is indepependent of x and y, but may depend on z. The differential of this solution leads to

$$3x^2dx + 3y^2dy - \frac{dc}{dz}dz = 0$$

Comparing this with the given equation, we get.

$$\frac{dc}{dz} = x^3 + y^3 + e^{2z} = c + e^{2z} \Rightarrow \frac{d}{dz}\left(ce^{-z}\right) = e^{z},$$

Integrating $c = c_1e^{z} + e^{2z}$, where c_1 is c[illegible] solution of the given equation is $x^3 + y^3 = c_1e^z + e^{2z}$.

(f) *Auxiliary equations.* For integrability of the equation $Pdx + Qdy + Rdz = 0$, the condition X.CurlX = 0 is to be satisfied, where X = (P, Q, R). This condition can be written as

$$P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = 0$$

Comparing this with the given equation, we get

$$\frac{dx}{\frac{\partial Q}{\partial z}-\frac{\partial R}{\partial y}}=\frac{dy}{\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}}=\frac{dz}{\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}}$$

These are called *auxiliary equations* and may be solved by the methods discussed earlier.

Let $f(x,y,z)=a$ and $g(x,y,z)=b$ be two integrals, We find A and B such that $\mathrm{A}df+\mathrm{B}dg=0$ becomes indentical with the given equation. Then using $f=a, g=b$, the values of A and B can be otained and the required solution is obtained by integration.

Example 1.13 : Solve the equation $(yz+xyz)dx+(zx+xyz)dy+(xy+xyz)dz=0$.

Solution. Here $\mathrm{X}=(yz+xyz, zx+xyz, xy+xyz)$, $\operatorname{curl}\mathrm{X}=(xz-xy, xy-yz, yz-xz)$ The anxiliary equations are

$$\frac{dx}{x(y-z)}=\frac{dy}{y(z-x)}=\frac{dz}{z(x-y)}$$

so that $dx+dy+dz=0$ and $\frac{dx}{x}+\frac{dy}{y}+\frac{dz}{z}=0$ leading to the solutions $f(x,y,z)=$ $x+y+z=$ const. and $g(x,y,z)=xyz=$ const. If the given equation is identically equal to $\mathrm{A}df+\mathrm{B}dg=0$, ie. to $\mathrm{A}(dx+dy+dz)+\mathrm{B}(yzdx+zxdy+xydz)=0$, ie, to $(\mathrm{A}+\mathrm{B}yz)dx+$ $(\mathrm{A}+\mathrm{B}zx)dy+(\mathrm{A}+\mathrm{B}xy)dx=0$, we have

$\mathrm{A}=xyz, \mathrm{B}=1$. Thus the equation $\mathrm{A}df+\mathrm{B}dg=0$ gives

$$xyz(dx+dy+dz)+(yzdx+zxdy+xydz)=0 \Rightarrow d(x+y+z)+\frac{d(xyz)}{xyz}=0$$

Integrating we get the required solution as $x+y+z+\log(xyz)=c$, where c is constant.

Exercises

1. Show that the direcdtion cosines of the tangent at the point (x,y,z) to the conic $ax^2+by^2+cz^2=1$, $x+y+z=1$ are proportional to $(by-cz, cz-ax, ax-by)$.

2. Find the integral curves of the following equations.

(i) $\frac{dx}{y(x+y)+az}=\frac{dy}{x(x+y)-az}=\frac{dz}{z(x+y)}$ [Ans. $x+y=c_1z$, $x^2-y^2-2xz=c_2$]

(ii) $\frac{dx}{y+\alpha z}=\frac{dy}{z+\beta x}=\frac{dz}{x-\upsilon y}$ [Ans. $(\lambda_1 x+\mu_1 y+\nu_1 z)=c_1(\lambda x+\mu_2 y+\nu_2 z$

$(\lambda_1 x+\mu_1 y+\nu_1 z)=c_2(\lambda_3 x+\mu_3 y+\nu_3 z)^{\rho_1/\rho_1}$

ρ_1, ρ_2, ρ_3 being roots of the equation $\rho^3+(\alpha+\beta+\gamma)\rho+(1+\alpha\beta\gamma)=0$ and λ_i, μ_i, ν_i are constant multipliers.]

(iii) $\frac{dx}{x+z}=\frac{dy}{y}=\frac{dz}{z+y^2}$ [Ans. $z=c_1 y+y^2, x=c_1 y \log y+c_1' y+y^2$]

(iv) $\frac{dx}{x(y-z)}=\frac{dy}{y(z-x)}=\frac{dz}{z(x-y)}$ [Ans. $x+y+z=c_1$, $xyz=c_2$]

(v) $\frac{dx}{xz-y}=\frac{dy}{yz-x)}=\frac{dz}{1-z^2}$ [Ans. $(x+y)(z+1)=c_1$, $(x-y)(z-1)=c_2$]

3. Find the orthogonal trajectories on the come $x^2+y^2=z^2\tan^2\alpha$ of its intersections, with the family of planes paralle to z = 0. [Ans. $x^2+y^2=z^2+\tan^2\alpha$, $z=c_1 y$]

4. Find the orthogonal trajectories on the surface $x^2+y^2+2fyz+d=0$ of its curves of intersection with planes parallel to the plane $x0y$. [Ans. $x^2+y^2+2fyz+d=0$] $fyz+d=cx$]

5. Find the equations of the system of curves on the cyliner $2y=x^2$ orthogonal to its intersections with the hyperboloids of the one-parameter system $xy=z+c$

[Ans. $2y=x^2, 3z+2(x-\frac{1}{2})=c_1$]

6. Verify that the following equations are integrable and find their primitives.

(i) $(y^2\_yz)dx+(z^2+zx)dy+(y^2-xy)dz=0$ [Ans. $y(x+z)=c(y+z)$]

(ii) $(1+yz)dx+x(z-x)dy-(1+xy)dz=0$ [Ans. $(yz+1)(cy+1)=xy+1$]

(iii) $yzdx+xzdy+xydz=0$ [Ans. $xyz=c$]

(iv) $yzdx+(z^2y-zx)dy+(x^2z-xy)dz=0$ [Ans. $2yz-x/y^2+z^2)=2cx$]

(v) $(6x+yz)dx+(xz-2y)dy+(xy+2z)dz=0$ [Ans. $3x^2-y^2+z^2+xyz=c$]

(vi) $ydx+xdy+2zdz=0$ [Ans. $zy+z^2=c$]

(vii) $yzdx + 2xzdy - 3xydz = 0$ [Ans. $\frac{xy^2}{z^3} = c$]

(viii) $(x^2z - y^3)dx + 3xy^2dy + x^3dz = 0$ [Ans. $x^2z + y^3 = cx$]

(ix) $a^2y^2z^2dx + b^2z^2x^2dy + c^2x^2y^2dz = 0$ [Ans. $\frac{a^2}{x} + \frac{b^2}{y} + \frac{c^2}{z} = C$]

(x) $x(y^2 - a^2)dx + y(x^2 - z^2)dy - z(y^2 - a^2)dz = 0$ [Ans $(x^2 - z^2)(y^2 - a^2) = c$]

(xi) $yz(y+z)dx + zx(x+z)dy + xy(x+y)dz = 0$ [Ans. $xyz = c(x+y+z)$]

(xii) $z(z+y^2)dx + z(z+x^2)dy - xy(x+y) = 0$ [Ans. $x(y^2 + z) = z(x+y)(1-cy)$]

(xiii) $2y(a-x)dx + [z - y^2 + (a-x)^2]dy - ydz = 0$ [Ans. $(a-x)^2 + z = y(c-y)$]

§ 1.7 Summary

The basic concepts fronm solid geometry are discussed in this unit as they are used most frequently in the study of the differential equations. Some properties of ordinary differential equations in more than two variables have also been incorporated because they play important roles in the theory of partial differential cqquations. Methods of solutions of the simultaneous differential equation $dx/p = dy/Q = dz/R$ and Pfaffian form $Pdx + Qdz = 0$ have also been discussed.

UNIT 2 ❑ PARTIAL DIFFERENTIAL EQUATIONS OF THE FIRST ORDER

§ 2.1 Partial Differential Equations

Most of the physical problems arising in science and technology involve two or more independent. Consequently, the dependent variable in such a case is a function of more than one variable and possesses partial derivatives with respect to several variables. As for example, consider the thermal effects in a solid body. Here the temperature θ may very from point to point in the solid as well as from time to time. Thus θ is a function of the space variables x, y, z and time t, i.e. $\theta=\theta(x, y, z, t)$. In such a phenomenon relating to temperature, we can obtain a relation between the derivaties of θ in the form

$$F\left(x, y, z, t, \theta, \frac{\partial\theta}{\partial x}, \frac{\partial\theta}{\partial y}, \frac{\partial\theta}{\partial t}, \frac{\partial^2\theta}{\partial z^2}, \cdots, \frac{\partial^2\theta}{\partial x\partial t}, \cdots\right)=0 \tag{2.1}$$

Such an equation relating partial derivaties is known as *partial differential equation*. A few well-known examples are

Heat or diffusion equation : $\dfrac{\partial\theta}{\partial t}=k\left(\dfrac{\partial^2\theta}{\partial x^2}+\dfrac{\partial^2\theta}{\partial y^2}+\dfrac{\partial^2\theta}{\partial z^2}\right)$

Laplace's equation : $\dfrac{\partial^2\phi}{\partial x^2}+\dfrac{\partial^2\phi}{\partial y^2}+\dfrac{\partial^2\phi}{\partial z^2}=0$ (2.2)

Nonlinear Burger equation : $\dfrac{\partial u}{\partial t}+u\dfrac{\partial u}{\partial x}=\mu\dfrac{\partial^2 u}{\partial x^2}$

In the case of two independent variable x and y, if z be the dependent variable, then usually we adopt the following notations :

$$p=\frac{\partial z}{\partial x},\ q=\frac{\partial z}{\partial y},\ r=\frac{\partial^2 z}{\partial x^2}, s=\frac{\partial^2 z}{\partial x\partial y}, t=\frac{\partial^2 z}{\partial y^2}, \tag{2.3}$$

The higher order derivative occuring in a partial differential equation is called its *order.* For example, the order of the questions in (2.2) is 2.

Classification of first-order partial differential equations.

1. *Linear equation.* A first-order partial differential equation is said to be *linear* if it is linear in p, q and z *i.e.* the equation is of the form

$$P(x,y)p+Q(x,y)=R(x,y)z+S(x,y)$$

For example, the equation $py+qx=x^3yz+y$ is a linear equations.

2. *Semi-linear equation.* A first-Order partial differential equation is said to be semi-linear if it is linear in p and q and the coefficients of p and q are functions of x and y only, i.e. the equation is of the form $P(x,y)p+Q(x,y)q=R(x,y,z)$

2. *Quasi-linear equation.* A first-order partial differential equation is said to be *quasilinear* if it is linear in p and q i.e. the equation is to the form

$P(x,y,z)p+Q(x,y,z)q=R(x,y,z)$

For example, the equation $x(z-2y^2)p+y(z-y^2-2x)q=z(z-y^3-2x^3)$ is equasi-linear equation.

4. *Nonlinear equation.* The partial differential equations which do not belong to the above three types are called *nonlinear equations.* For example, the equation $p^3+q^3=3pqz$ is nonlinear.

§ 2.2 Origin of first-Order Partial differential Equations.

Partial differential equations originate in many ways such as elimination of arbitrary constants or functions and in studying a physical or social phenomenon. Let us demonstrate how partial differential equations occur.

Case 1 : Elimination of arbitrary constants.

Let $f(x, y, z, a, b) = 0$ be a relation involving two independent variables x,y, *one dependant variable* z and two arbitrary constants a and b. Differentiating this equation with respect to x and y respectively, we get.

$$\frac{\partial f}{\partial x}+\frac{\partial f}{\partial z}\frac{\partial z}{dx}=0 \quad \text{and} \quad \frac{\partial f}{\partial y}+\frac{\partial f}{\partial z}\frac{\partial z}{dy}=0$$

Elimination of the constants a, b between these two relations and the given relation leads to the eliminant in the form (2.4)

$$F(x, \; y, \; z, \; p, \; q)=0$$

which is a first-order partial differntial equation.

Example 2.1. : Eliminate the constants a and b from the equation $2z=(ax+y)^2+b$

Solution, Differentiating the given equation with respect to x and y respectively, we get $p=a(ax+y)$, $P_{t_x}+Q_{t_y}+R_{t_z}=0$ so that $px+qy=(ax+y)^2=q^2$. Hence the required a $b$eliminant is $px+qy=q^2$ which is a nonlinear first-order differential equation.

Case II : Elimination of functions.

Let $u=u(x,y,z)$ and $v=v(x,y,z)$ be two given functions of x, y, z connected by the relation $\phi(u,v)=0$. Differentiation this relationpartially with respect to x and y we obtain respectively,

$$\frac{\partial\phi}{\partial u}\left(\frac{\partial u}{\partial x}+p\frac{\partial u}{\partial z}\right)+\frac{\partial\phi}{\partial v}\left(\frac{\partial v}{\partial x}+p\frac{\partial u}{\partial z}\right)=0$$

$$\frac{\partial\phi}{\partial u}\left(\frac{\partial u}{\partial y}+p\frac{\partial u}{\partial z}\right)+\frac{\partial\phi}{\partial v}\left(\frac{\partial v}{\partial y}+p\frac{\partial u}{\partial z}\right)=0$$

Eliminating $\frac{\partial\phi}{\partial u}$ and $\frac{\partial\phi}{\partial v}$ between these two equations, we get

$$\left(\frac{\partial u}{\partial y}\frac{\partial v}{\partial z}-\frac{\partial u}{\partial z}\frac{\partial v}{\partial y}\right)p+\left(\frac{\partial u}{\partial z}\frac{\partial v}{\partial x}-\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}\right)q=\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}$$

$$\text{or,} \quad \frac{\partial(u,v)}{\partial(y,z)}p+\frac{\partial(u,v)}{\partial(z,x)}q=\frac{\partial(u,v)}{\partial(x,y)}$$

$$\text{or} \quad Pp+Qq=R \qquad (2.5)$$

where

$$P=\frac{\partial(u,v)}{\partial(y,z)}, \; Q=\frac{\partial(u,v)}{\partial(z,x)}, \; R=\frac{\partial(u,v)}{\partial(x,y)}$$

The first-order linear equation (2.5) is called *Largangs equation* of the first-order. If the given relation between x, y and z contains two arbitrary functions, then (excepting some cases) the partial differential equations of higher order will be formed.

Example 2.2. : Form the partial differential equation by eliminating the arbitrary function f from the relation $f(x+y+z,\ x^2+y^2-z^2)=0$

Solution. The given equation can be written as $x+y+z=\phi(x^2+y^2-z^2)$ Differentiating both sides with respect to x and y we obtain respectively.

$$1+p=2\phi'(x^2+y^2-z^2)(x-zp), \quad 1+p=2\phi'(x^2+y^2-z^2)(x-zq)$$

so that $\dfrac{1+p}{x+zp}=\dfrac{1+q}{y-zq}$ leading to the partial differential equation $(y+z)p-(x+z)q=x-y$

§ 2.3 : Existence of Solutions of Partial Differential Equations.

This existence of solution of a partial differential equation is not guaranteed. However, its solution does exist provided the equation satisfies a set of conditions. Before the discussions of the existence of the solution, we first define a solution and its various types associated with a partial differential equation.

We have been is § 2.2. that a relation of the type $f(x,y,z,a,b)=0$ leads to a partial differential equation of first-order. Such a relation containing two arbitrary constants a and b is a *solution* of the first-order prtial differential equation and is called a complete *solution* or a *complete integral* of that equation. On the other hand, any relation of the type $f(u,v)=0$ involving an arbitrary function f connecting two known functions $u(x,\ y,\ z)$ *and* $v(x,\ y,\ z)$ and providing a solution of the first-order partial differential equation is called a *general solution or general integral* of the equation. The general solution can also be obtained as the locus of a parametric family of curve, called *characteristics* of a. The general solution of a first-order partial differential equation is a parametric family of surfaces, called *itergral surfaces.*

The *singular solution* or the *singular integral* is obtained from the complete integral by the elimination of arbitrary constants. Thus if, $f(x,y,z\,a,b)=0$ is the complete integral of the partial differential equation. $F(x,y,z,p,q)=0$, then the a b-eliminant from the equation $f=0$ $\dfrac{\partial f}{da}=0,\ \dfrac{\partial f}{\partial z}=0$ is the singular solution. A singular solution can also be obtained from the differential equation itself by eliminating p and q *from the equation* $F=0$ $\dfrac{\partial F}{dp}=0,\ \dfrac{\partial F}{\partial q}=0$.

Example 2.3 : The equation $z^2(p^2+q^2+1)=c^2$ has a complete integral in the form $(x-a)^2+(y-b)^2+z^2=c^3$ where a and b are arbitrary constants. Find the singular integral and a general integral assuming $b=a$.

Solution. Differentiating the relation $(x-a)^2+(y-b)^2+z^2=c^2$ partially w.r.t. a and b we get respectively $-2(x-a)=0, -2(y-b)=0$ i.e. $a=x, b=y$. Elimination of a and b gives $z^2=c^2$ i.e. $z=\pm c$ which is the singular integral.

When $z=\pm c$, $p=\frac{\partial z}{\partial x}=0, q=\frac{\partial z}{\partial y}=0$, and they satisfy the given differnetial equation.

Again making $b = a$, we get $(x-a)^2+(y-a)^2+z^2=c^2$. Differentiating w.r.f.a we have $-2(x-a)-2(y-a)=0$ i.e., $a=\frac{1}{2}(x+y)$. Eliminating a the general solution is $\frac{1}{2}(x-y)^2+z^2=c^2$ i.e. $(x-y)^2+2z^2=2c^2$

Existence theorem

For the existence of a solution of a first-order partial differential equation, the conditions to be satisfied are given in Cauchy problem which we state as follws :

Cauchy problem

Suppose

(a) the functions $x_0(\mu), y_0(\mu)$ and $z_0(\mu)$ and their first derivatives are continuous in the interval M : $\mu_1 < \mu < \mu_2$ and

(b) the function $F(x, y, z, p, q)$ in continuous in x, y, z, p, q in the region U of the $xyzpq$-space.

Then the problem is to establsh the existence of a function $\phi(x, y)$ having the following properties :

(i) $\phi(x, y)$ and its partial derivatives w.r.t x and y are continuous functions of x and y in a region R of the xy-space.

(ii) the point $\{x, y, \phi(x, y), \phi_x(x, y)\phi_y(x, y\} \in U$ and $F\{x, y\phi(x, y)\phi_x(x, y), \phi(x, y\}$ $=0, \forall x, y \in R$

(iii) the point $\{x_0(\mu), y_0(\mu)\} \in R$ and $\phi\{x_0(\mu), y_0(\mu)\}z_0, \forall \mu \in M$

Geometrically we can state Cauchy problem as follows : To prove the existence of a surface $z=\phi(x, y)$ which passes through the curve Γ with parametric equations

$$x=x_0(\mu),\ y=y_0(\mu),\ z=z_0(\mu) \tag{2.7}$$

and at every point of which the direction $(p, q, -1)$ of the normal is such that

To prove the existance of a solution of the partial differential equation (2.8 passing through a curve having equation (2.7), we hve to make some other assumptions regarding the function F and the curve Γ The existence theorem depends on the nature of these assumption. We now state the existence theorem without proof due to S. Kowalewski and is known as Cauchy-Kowaleski theorem.

Cauchy-Kowalewski theorem

Suppose a function g (y) and a i its derivatives are continuous for $|y-y_0|<\delta$ *and* x_0 *be a given number and* $z_0=g'(y_0)$, $q_0=g'(y_0)$. *Also we suppose that the function* $f(x,y,z,q)$ *and all its partial de vatives are continuous in the region* $S:|x-x_0|<\delta_1$, $|y-y_0|<\delta$, $|q-q_0|<\delta$. *Then there exists a unique function* $\phi(x,y)$ *such that*

(i) (x,y) *and all its partial erivatives are continuous in a region* $R:|x-x_0<\delta$, $|y-y_0|<\delta_2$;

(ii) $z=\phi(x,y)$ *is a solution of the equation*

$$\frac{\partial z}{\partial x}=f(x,y,\frac{\partial z}{\partial y}),\quad \forall\ x,\ y\in R;$$

and (iii) $\phi(x_0y)=g(y)$, $\forall y$ in he interval $|y-y_0|<\delta_1$

§ 2.4 : Linear Equations of the First Order

We have seen in § 2.2 that linear partial differential equations of the first order in two independent variables x, y and or dependent variable z are given by Lagrange's equation.

$$Pp+Qq=R \tag{2.5}$$

in which each of P, Q and R is a function of x, y and z and they do not involve p or $q\left(p=\frac{\partial z}{\partial x},q=\frac{\partial z}{\partial y}\right)$

The equation (2.5) can be generalised to n independent variables given by'

$$P_1p_1+P_2p_2+\cdots P_np_n=R \tag{2-9}$$

in which each of P_1, $P_2,\cdots P_n$, and R *is a function of the* n independent variables x_1, x_2, ... x_n, the dependent variable is z and $p_i=\frac{\partial z}{\partial x_i}(i=1,2,\cdots,n)$

The method of solving the linear equation (2.5) is given by the following theorem :

Theorem 2.1 : *The linear partial differential equation.*

$$Pp+Qq=R \tag{2.5}$$

has the general solution

$$\phi(u,v)=0 \tag{2.10}$$

where f is an arbitrary function u and v and $u(x,y,z)=c_1$ $v(x,y,z)=c_2$ *are solutions of the equations*

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R} \tag{2.11}$$

Proof Since $u(x,y,z)=c_1$ is a solution of the equation (2.11), the equations $u_x dx+u_y dy+u_z dz=0$ and $\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$ are compatible to each other so that we have

$$Pu_x+Qu_y+Ru_z=0$$

Similarly we get $\quad pv_x+Qv_y+Rv_z=0$

Solving these equations for P, Q and R, we obtain

$$\frac{P}{\frac{\partial(u,v)}{\partial(y,z)}}=\frac{Q}{\frac{\partial(u,v)}{\partial(z,x)}}=\frac{R}{\frac{\partial(u,v)}{\partial(x,y)}} \tag{2.12}$$

Now we have already seen in § 2.2. that the relation $\phi(u,v)=0$ leads to the partial differential equation

$$p\frac{\partial(u,v)}{\partial(y,z)}+q\frac{\partial(u,v)}{\partial(z,x)}=\frac{\partial(u,v)}{\partial(x,y)} \tag{2.13}$$

Substituting from equations (2.12), we find that (2.10) is a solution of the equation (2.5) when u and v are solutions of the equation (2.11).

The equations (2.11) are known as *Lagrante's auxilarly equations.*

Gemetrical interpretation of the equation Pp + Qq = R

The direction cosines of the normal of thesurface $z=f(x,y)$ at a point are proportional to $\frac{\partial z}{\partial x},\frac{\partial z}{\partial y},-1,$ i.e. to $p,q,-1$. We can write Lagrange's equations (2.5) in the form

$$Pp+Qq+(-1)\ R=0 \tag{2.14}$$

Thus the normal at a point to a given surface is perpendicular to a straight line whose direction cosines are in the ration $P:Q:R$. Also the equations $dx/_P=dy/_Q=dz/_R$ represent a family of curves, the tangent at any point of which has direction cosines in the ratio.

$P : Q : R$ Again the relation $\phi(u,v)=0$, where $u(x,y,z)$ = const. and $v(x,y,z)$= const, are two particular integrals of the equations (2.11) represents a surfaces through such curves. Now a curve of the family through any point on the surface lies entirely on the surface. Thus the normal to this surface at this point is at right angles to the tangent at the point to the curve. In other words, it is perpendicular to the straight line which has direction cosines proportional of $P : Q : R$.

Since the equations (2.11) and (2.14) define the same set of surfaces, they are equivalent and, therefore, the relation ϕ (u, v) = 0 is an integral of the equation (2.14) provided that u =const. and v = const, are two independent solutions of (2.11) and ϕ is an arbitrary function.

Example 2.4 : Solve $p\cos(x+y) + q\sin(x+y) = z$

Solution. Lagrange's auxiliary equations are

$$\frac{dr}{\cos(x+y)}=\frac{dy}{\sin(x+y)}=\frac{dz}{z} \tag{2.15}$$

The first two equations give

$$\frac{d(x+y)}{\cos(x+y)+\sin(x+y)}=\frac{d(x-y)}{\cos(x+y)-\sin(x+y)}$$

i.e
$$\frac{-\sin(x+y)+\cos(x+y)}{\cos(x+y)+\sin(x+y)}d(x+y)=d(x-y)$$

Integrating, we get $\log\{\cos(x+y)+\sin(x+y) = (x-y) + \log a$ being constant, so that $\{\cos(x+y)+\sin(x+y\}e^{y-x}=a$

Again from (2.15) we have

$$\frac{d(x+y)}{\cos(x+y)+\sin(x+y)}=\frac{dz}{z}$$

$$\Rightarrow \operatorname{cosec}\left(x+y+\frac{\pi}{4}\right)d\left(x+y+\frac{\pi}{8}\right)=\sqrt{2}\log z+\log b,$$ $\log b$ being constant, so that

$$\tan\left\{\frac{1}{2}(x+y)\frac{\pi}{8}\right\}z^{-\sqrt{2}}=b$$

Hence the solution of the given equation is

$$\phi\left[\{\cos(x+y)+\sin(x+y)\}e^{y-x},\tan\left\{\frac{1}{2}(x+y)+\frac{\pi}{8}\right\}z^{-\sqrt{2}}\right]=0$$

The method of solving the general linear equation (2.9) is given by the following theorem :

Theorem 2.2. Let $u_i(x_1, x_2 \cdots, x_n, z_i) = c_i$, $(i = 1, 2, \cdots, n)$, *be n independent solutions of the equations.*

$$\frac{dx_1}{P_1} = \frac{dx_2}{P_2} = \cdots = \frac{dx_n}{P_n} = \frac{dz}{R}$$

Then the relation $\Phi(u_1, u_2, \cdots, u_n) = 0$, *where* Φ *is arbiratry, is a general solution of the linear partial differential equation.*

$$P_1\frac{\partial z}{\partial x_1} + P_2\frac{\partial z}{\partial x_2} + \cdots + P_n\frac{\partial z}{\partial x_n} = R \qquad (2.16$$

Prof, since

$$u_1(x_1, x_2, \cdots, x_n\ z) = c_1, (i = 1, 2, \cdots, n)$$

are the solutions of the equations

$$J = \frac{\partial(F, G)}{\partial(p, q)} \neq 0.$$

the n equations $\sum_{j=1}^{n} \frac{\partial u}{\partial x_j} dx_j + \frac{\partial u_i}{\partial z} dz = 0, (i = 1, 2, \cdots, n)$

must be compatible with (2.17), i.e. we must have

$$\sum_{j=1}^{n} P_1 \frac{\partial u_i}{\partial x_j} + R\frac{\partial u_i}{\partial z} = 0,\ (i = 1, 2, \cdots, n)$$

Solving these equations for P_i we get

$$\frac{P_1}{\dfrac{\partial(u_1, u_2, \cdots, u_n)}{\partial(x_1, x_2, \cdots, x_{i-1}, z, x_{i+1}, \cdots x_n)}} = \frac{R}{\dfrac{\partial(u_1, u_2, \cdots, u_n)}{\partial(x_1, x_2, \cdots x_n)}} \qquad (2.18)$$

$(i = 1,\ 2, \ldots\ldots, n)$

where the Jacobian

$$\frac{\partial(u_1, u_2, \cdots, u_n)}{\partial(x_1, x_2, \cdots, x_n)} = \begin{vmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \cdots & \frac{\partial u_1}{\partial x_n} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \cdots & \frac{\partial u_2}{\partial x_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial u_n}{\partial x_1} & \frac{\partial u_n}{\partial x_2} & \cdots & \frac{\partial u_n}{\partial x_n} \end{vmatrix}$$

Now differnectiating the relation $\Phi(u_1, u_2, \cdots u_n) = 0$ with respect to $x_i (i = 1, 2, \cdots, n)$

we get $\sum_{j=1}^{n}\left(\frac{\partial\Phi}{\partial u_j}\frac{\partial u_j}{\partial x_1} + \frac{\partial u_j}{\partial z}\frac{\partial z}{\partial x_i}\right) = 0, \quad (i = 1, 2, \cdots, n)$

Eliminating the n quantities $\frac{\partial\Phi}{\partial u_1}, \frac{\partial\Phi}{\partial u_2}, \cdots, \frac{\partial\Phi}{\partial u_n}$ from the n equations, we have

$$\sum_{j=1}^{n}\frac{\partial(u_1, u_2, \cdots, u_n)}{\partial(x_1, x_2, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n)}\frac{dz}{dxj} + \frac{\partial(u_1, u_2, \cdots, u_n)}{\partial(x_1, x_2, \cdots, x_n)} = 0 \qquad (2.20)$$

Substitution from (2.18) into (2.19) leads to

$$P_1\frac{\partial z}{\partial x_1} + P_2\frac{\partial z}{\partial x_2} + \cdots + P_n\frac{\partial z}{\partial x_n} = R. \qquad (2.20)$$

Hence the function $\Phi(u_1, u_2, \cdots u_n)$ satisfies the equation (2.20) if $u_1, u_2, \cdots, u_n$ are n independent solutions of (2.17)

Example 2.5 : Solve $(t+y+z)\frac{\partial t}{\partial x} + (t+z+x)\frac{\partial t}{\partial y} + (t+x+y)\frac{\partial t}{\partial z} = x+y+z$

Solution : Here the auxiliary equations are

$$\frac{dx}{t+y+z} = \frac{dy}{t+z+c} = \frac{dz}{t+x+y} = \frac{dt}{x+y+z}$$

$$\Rightarrow \frac{d(x-y)}{-(x-y)} = \frac{d(y-z)}{-(y-z)} = \frac{d(z-t)}{-(z-t)} = \frac{d(x+y+z+t)}{3(x+y+z+t)}$$

Integration of the first two terms leads to $\frac{x-y}{y-z} =$ cons = tc$_1$ chile that of the second and third terms gives $\frac{y-z}{z-t} = c_2$ and the last two terms give $(x+y+z+t)^{1/3}(z-t) = c_3$

Hence the solution of the given equation is $\phi\left\{\frac{x-y}{y-z}, \frac{y-z}{z-t}, (x+y+z+t)^{1/3}(z-t)\right\} = 0$

§ 2.5 Integral Surfaces Passing Through a given Curve

Suppose the auxiliary equations.

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \qquad \text{ধ২.১১}(2.11)$$

has the two solution $u(x, y, z) = c_1$ and $v(x, y, z) = c_2$. Then according to the list section, any solution of the corresponding linear equation.

$$Pp+Qq=R \tag{2.5}$$

is of the form

$$\phi(u,v)=0 \tag{2.10}$$

arising from a relation

$$\phi(c_1,c_2)=0 \tag{2.21}$$

between the constants c_1 and c_2. Thus the problem is to consider the determination of the function ϕ in special circumstances.

To find the integral surface through the curve G having parametric equations $x=x(t)$, $y=y(t), z=z(t)$, t being a parameter, the particular solutions $u(x,y,z)=c_1$ and $v(x,y,z)=c_2$ must be such that

$$u\{x(t),y(t)\}=c_1,\ v[x(t),y(t),z(t)\}=c_2$$

from which the single variable t may be eliminated to obtain a relation of the type (2.21).

The desired solution is then given by (2.10).

Example 2.6 : Find the general integral of the equation $(x-y)p+(y-z-x)q=z$ and the equation of the integral surface of the differential equation which passes through the circle $z=1, x^2+y^2=1$

Solution. Here the auxiliary equations are

$$\frac{dx}{x-y}=\frac{dy}{y-z-x}=\frac{dz}{z}=\frac{dx+dy+dz}{0}=\frac{dx-dy+dz}{2(x-y+z)}$$

On integration, the fourth member gives $x+y+z=$ const = c_1 and the third and fifth members leas to $x-y+z=c_2z^2$. Thus the general integral is

$$\phi\left(x+y+z,\frac{x-y+z,}{z^2}\right)=0$$

Now the parametric equations of the given curve are $x=$ cost, y = sin, t, $z=1$ so that

cos t + sin t + 1 = c_1 and cost t – sin t + 1 = $c_1 \Rightarrow \cos t=\frac{1}{2}(c_1+c_2-2)$, $\sin t=\frac{1}{2}(c_1-c_2)$ and

hence $(c_1+c_2-2)^2+(c_1-c_2)^2=4$, i.e. $c_1^2+c_2^2-2(c_1+c_2)=0$

$$\Rightarrow (x+y+z)^2+\left(\frac{x-y+z}{z^2}\right)^2 2\left(x+y+z+\frac{x-y+z}{z^2}\right)=0$$

$$\Rightarrow z^4(x+y+z)(x+y+z-2)+(x-y+z)(x-y+z-2z^2)=0$$

which is the required equation of the integral surface.

Q 2.6 : Surfaces Orthogonal to a Given System of Surfaces

We now find a system of surfaces which cut orthogonally a one-parameter family of surfaces given by the .equation

$$F(x, y, z) = c, \tag{2.22}$$

c being a parameter.

First we note that the normal at any point (x,y,z) to the surface (2.22) and passing through this point has direction ratios (P, Q, R) = $\left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}\right)$

If the surface $z = f(x,y)$ cuts the system of surfaces (2.22) orthogonally, then its normal at the point (x,y,z) in the direction $\left(\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, -1\right)$ and is perpendicular to the direction (P,Q,R) of the normal to the surface (2.22) at the point.

Thus we have

$$\frac{\partial F}{\partial x}\frac{\partial z}{\partial x} + \frac{\partial F}{\partial y}\frac{\partial z}{\partial y} = \frac{\partial F}{\partial z}$$

i.e, $Pp + Qq = R$ (2.5)

Conversely, any solution of the linear partial differential equation (2.5) is orthogonal to every surface of the system given by (2.22), since (2.5) states that the normal to any solution of (2.5) is perpendicular to the normal to the system (2.22) passing through the same point.

Hence the equation (2.5) represents the general partial differential equation which determines the surfaces orthogonal to the system (2.22), i.e. the orthogonal surfaces of the system (2.22) are the surfaces generated by the integral curves of the equations

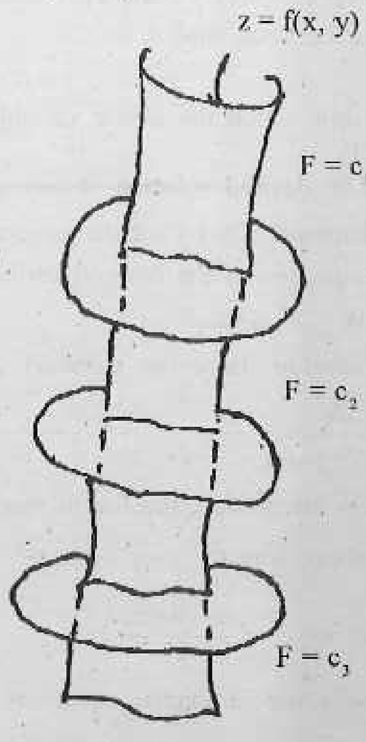


Fig. 2.1

$$\frac{dx}{\frac{\partial F}{dx}} = \frac{dy}{\frac{\partial F}{dy}} = \frac{dz}{\frac{\partial F}{dz}} \tag{2.23}$$

Example 2.7. : Find the surface which is orthogonal to the one-parameter system

$z = cxy(x^2+y^2)$ and which passes through the hyperbola $x^2 - y^2 = a^2, z = 0$

Solution. The given system of surfaces is

$F(x,y,z) = \frac{xy(x^2+y^2)}{z} = \frac{1}{c}$, c being a parameter.

The auxiliary equations are

$$\frac{dx}{\frac{\partial F}{dx}} = \frac{dy}{\frac{\partial F}{dy}} = \frac{dz}{\frac{\partial F}{dz}}$$

or, $$\frac{dx}{\frac{y(3x^2+y^2}{z}} = \frac{dy}{\frac{x(x^2+3y^2)}{z}} = \frac{dz}{\frac{xy(x^2+y^2)}{z^2}}$$

or $$\frac{dx}{yz(3x^2+y^2)} = \frac{dy}{zx(x^2+3y^2)} = \frac{dz}{-xy(x^2+y^2)} = \frac{xdx+ydy}{4xyz(x^2+y^2)} \qquad (2.24)$$

The third and fourth terms give $xdx + ydy + 4zdz = 0 \Rightarrow x^2 + y^2 + 4z^2 = c_1$

Also from (2.4) we get

$$\frac{x\,dx + y\,dy}{x^2+y^2} = z\frac{x\,dx + y\,dy}{x^2-y^2} \Rightarrow \frac{(x^2-y^2)}{x^2+y^2} = c_2$$

Now the given hyperbola has parametric equations $x = a\,sec,\,t$, y = a tan t, $z = 0$ and

therefore, $c_1 = a^2(\sec^2 t + \tan^2 t), c_2 = \frac{a^2}{\sec^2 t + \tan^2 t}$ so that $c_1c_2 = a^4$. Thus the required orthogonal surface is

$$(x^2+y^2+4z^2)\frac{(x^2-y^2)^2}{x^2+y^2} = c_1c_2 = a^4$$

i.e. $$(x^2+y^2+4z^2)(x^2-y^2)^2 = a^4(x^2+y^2)$$

§ Compatible Systems of first-Order Equations.

Two first-order partial differential equations

$F(x, y, z, p, q) = 0$ and $G(x, y, z, p, q) = 0$ (2.25)

are said to be *compatible* if a solution of the former satisfies the later and conversely.

We assume that $J=\frac{\partial(F,G)}{\partial(p,q)}\neq 0$. Then we can solve equations (2.25) for p and q in the form

$$p=p(x,y,z), q=q(x,y,z) \tag{2.26}$$

The conditon for the equations (2.25) to be compatible is that $dz+dpx+qdy$ must be integrable and this is possible provided (cf. Theorem § 1.3. 1.5, Unit I)

$$p\frac{\partial q}{\partial z}-p\frac{\partial p}{\partial z}+\frac{\partial q}{\partial x}-\frac{\partial q}{\partial y}=0$$

or, $$q_x+pq_z+p_y+ap_z \tag{2.27}$$

Differentiating the first equation of (2.25) with respect to x and z we get respectively

$$F_z+F_p p_x+F_q q_x=0 \tag{2.28a}$$

and $$F_z+F_p p_z+F_q q_z=0 \tag{2.28b}$$

Multiplying (2.28b) by p and adding the result to (2.28) we have

$$F_x+pF_z+F_p(p_x+pp_z)+F_q(q_x+pq_z)=0 \tag{2.29a}$$

Similarly, differentiating the second equation of (2.25) with respect to x and z and proceeding as above, we obtain

$$G_x+pG_z+G_p(p_x+pp)+G_q(q_x+pq_z)=0 \tag{2.29b}$$

Elimination of p_x+pp_z between (2.29a) and gives

$$\frac{\partial(F,G)}{\partial(x,p)}+p\frac{\partial(F,G)}{\partial(z,p)}-\frac{\partial(F,G)}{\partial(p,q)}(q_x+pq_z)=0$$

i.e. $$q_x+pq_z=-\frac{1}{J}\left[\frac{\partial(F,G)}{\partial(x,p)}+p\frac{\partial(F,G)}{\partial(z,p)}\right] \tag{2.30a}$$

Similarly, differentiation of (2.25) with respect to y and z and proceeding as above leads to

$$q_x+pq_z=-\frac{1}{J}\left[\frac{\partial(F,G)}{\partial(x,q)}+q\frac{\partial(F,G)}{\partial(z,q)}\right] \tag{2.30 b}$$

Substitutions of (2.30*a*) and (2.30*b*) into (2.27) give

$$\frac{\partial(F,G)}{\partial(x,p)}+p\frac{\partial(F,G)}{\partial(z,p)}+\frac{\partial(F,G)}{\partial(y,q)}+q\frac{\partial(F,G)}{\partial(z,q)}=0$$

which we write in short as

$$[F,G]=0$$

This is the required *compatibility condition.*

Example 2.8 : Show that the equations $xp - yq = x, x^2 p + q = xz$ are compatible and find their solution.

Solution. The given equations are

$$F(x,y,z,p,q) = xp - x = 0,\ G(x,y,z,p,q) = x^2 p + q - xz = 0$$

Then we have

$$\frac{\partial(F,G)}{\partial(x,p)} = xz - x^2(p+1),\ \frac{\partial(F,G)}{\partial(y,q)} = -q,\ \frac{\partial(F,G)}{\partial(z,p)} = x^2,\ \frac{\partial(F,G)}{\partial(z,q)} = -xy$$

so that $[F, G]$ $= xz - x^2(p+1) + px^2 - q - qxy = (xz - q) - x^2 - qxy$

$$= px^2 - x^2 - qxy = x(px - x - qy) = 0$$

Hence the given equations are compatible.

For the equation $F(x,y,z,p,q) = 0$, Lagrange's auxiliary equations are $\frac{dx}{x} = \frac{dy}{-y} = \frac{dz}{z}$ which lead to the solution $z - x = \phi(xy)$

2.8 : Nonlinear Partial Differential Equations of the First Order

We now proced to find the solutions of the partial differential equation of the form

$$F(x,y,z,p,q) = 0 \tag{2.4}$$

in which the function F is not necessarily linear. In 2.2, we have already seen that the partial differential equation of the two-parameter system of surfaces

$$f(x,y,z,a,b) = 0 \tag{2.32}$$

was of this form. We shall show later (see 2.9) that the converse is also true, i.e. any partial differential equation of the type (2.4) yields solutions of the type (2.32). Any envelope of the system (2.32) tonches at each of its points a member of the system. Thus it has the same set of values (x,y,z,p,q) as the particular surface and therefore, it must also be a solution of the differential equation. Hence we are led to the following three classes of integrals of a partial differential euqation of the form (2.4) :

(a) Two-parameter system of surfaces $f(x,y,z,a,b) = 0$

Such an integral is called a *complete integral.*

(b) Let the parameter b is connected to a by a relation of the from $b = \phi(a)$, where $\phi(a)$ is an arbitrary function of a. Then the one-parameter subsystem $f\{x,y,z,a,\phi(a)\} = 0$ of the system (2.32) forms its envelope and is called the *general integral of* (2.4).

(c) If the envelope of the two-parameter system of surfaces (2.4) exists, then it is also a solution of (2.4) and is called the *singular integral* of the euqation.

Example 2.9 : Verify that $z = ax + by + a + b - ab$ is a completet integral of the partial differential equation $z = px + qy + p + q - pq$ where a and b are arbitrary constants.

Show that the envelope of all planes corresponding to complete integrals provides a singular integral of the differential equation and determine a general integral by finding the envelope of those planes that pass through the origin.

solution. We have $f(x,y,z,a,b) = z - (ax + by + a + b - ab) = 0$ (2.33)

Then $p = \frac{\partial z}{\partial x} = a,\ q = \frac{\partial z}{\partial y} = b.$ Thus (2.38) is a complete integral of the given partial differential equation

$$z = px + qy + p + q - pq \tag{2.34}$$

The envelope of the two-parameter system (2.33) is obained by eliminating a and b from (2.34) and the equations.

$\frac{\partial f}{\partial a} = -(x + 1 - b) = 0$ and $\frac{\partial f}{\partial b} = -(y + 1 - a) = 0$, i.e. $a = y + 1,\ \ b = x + 1$

as $z = (x+1)(y+1)$ which is the required singular integral.

Now putting $b = \phi(a)$ we consider the one-paramater system

$$f\{x, y, z, a\phi(a)\} = z - ax - \phi(a)y - a - \phi(a) + a\phi(a) = 0 \tag{2.35}$$

so that $\frac{\partial f}{\partial a} = -x - \phi'(a)y - 1 - \phi'(a) + \phi(a) + a\phi'(a)$

Since the envelope of the planes passes through the origin, we have from (2.35)

$$-a - \phi(a) + a\phi(a) = 0 \ \text{ i.e. } \phi(a) = \frac{a}{a-1} \Rightarrow \phi'(a) = -\frac{1}{(a-1)^2}$$

Thus the relation $\frac{\partial f}{\partial a} = 0$ gives by putting the values of $\phi(a)$ and $\phi'(a)$, the value of a

as $a = \sqrt{\frac{y}{x}} + 1$. Substituting the values of a and $\phi(a)$ in (2.35) we get

$$z = x + y + 2\sqrt{xy} \Rightarrow (x + y - z)^2 = 4xy$$

which is the required general integral.

2.9 : Cauchy's Method of Characteristics

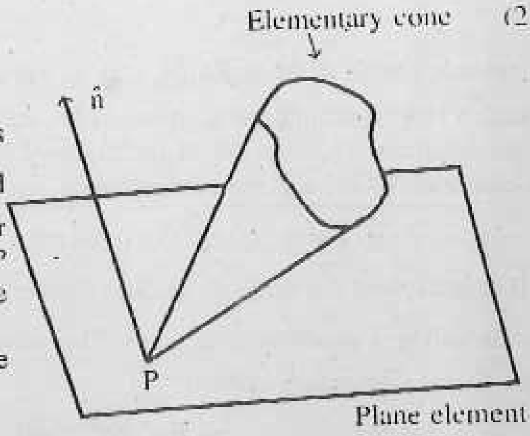
To Solve nonlinear partial differential equation of the form

$$F(x,y,z,p,q)=0 \qquad (2.4)$$

Cauchy introduced a method of solution based on geometrical ideas.

The plane through the point $P(x,y,z)$ with its normal parallel to the direction $\hat{n}$ with direction ratios $(p_o,q_o,-1)$ is uniquely specified by the set of numbers $A(x_o,y_o,z_o,p_o,q_o)$. Conversely, any set of five real numbers defines a plane in three-dimensional space. Such a set of five numbers (x,y,z,p,q) is called a plane element of the space. A plane element (x_o,y_o,z_o,p_o,q_o) satisfy the equation (2.4) is called an integral element of the equation at the point (x_o,y_o,z_o). We rewrite equation (2.4) in the form

$$q=G(x,y,z,p) \qquad (2.36)$$



Fin. 2.2

Let us keep x, y, z fixed and vary p. Then we obtain a set of plane elements $\{x_o,y_o,z_o,p,G(x_o,y_o,z_o,p)\}$ which depend on the single parameter P. The planer elements, therefore, envolpe a cone, with P as vertex, called the *elementary cone* of the equation (2.4) at the point P *(cf.* Fig 2.2).

Now let S be a surface given by the equation

$$z=g(x,y) \qquad (2.37)$$

in which the function $g(x,y)$ and its first partial derivatives with respect of a and y are assumed to be continuous in a region R of the xy-plane. Then the tangent plane at each point of S determines a plane element of the type $\{x_o,y_o,g(x_o,y_o),g_x(x_o,y_o),g_y(x_o,y_o)\}$ which is called the *tangent element* of the surface S at the point $\{x_o,y_o,g(x_o,y_o)\}$. Thus we have the result :

Theorem 2.3 : A *necessary and sufficient condition for a surface to be an integral surface of a partial differential equation is that at each point its tangent element should touch the elementary cone of the eqation.*

Let us now consider a curve Γ with parmetric equations $x=x(t), y=y(t), z=z(t)$. This curve Γ lies on the surface (2.37) if

$$z(t)=g\{x(t),y(t), \quad \forall t\in I,$$

where the interval I is given, If P_o is a point of Γ determined by the parameter t_o, then the direction ratios of the tangent line P_oP_1 (cf. Fig. 2.3) are $\{x'(t_o)y'(t_o),z'(t_o)\}$

where $x'(t_0)=\left(\frac{dx}{dt}\right)_{t=t_0}$ etc. This direction is perpendicular to the direction $(p_0,q_0,-1)$ if

$$z'(t_0)=p_0x'=p_0x'(t_0)+q_0y'(t_0)$$

Thus any set $\{x(t),y(t),z(t),p(t),q(t)\}$ (2.38)
of five real functions satisfying the condition
$z'(t)=p(t)x'(t)+q(t)y'(t)$ (2.39)

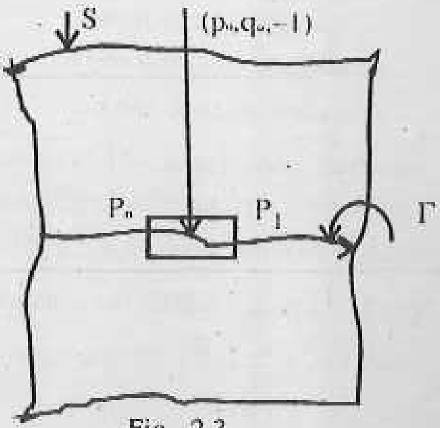


Fig. 2.3

defines a strip at the point (x,y,z) of the curve Γ.
If such a strip is an integral element of the equation (2.4), then it is called an *integral strip* of the equation. Thus the set of functions (2.38) is an integral strip of (2.4) if they satisfy the condition (2.39) and the further condition

$$F\{x(t),y(t),p(t),q(t)\}=0 \ \forall t\in I \tag{2.40}$$

If at each point the curve Γ touches a generator of the elementary cone, the corresponding strip is called a *characteristic strip.* The point $(x+dx,y+dy,z+dz)$ lies on the tangent plane to the elementary cone if

$$dz=pdx+qdy \tag{2.41}$$

where p, q, satisfy the equation (2.4). Now differentiating (2.4) with respect to p, we get

$$0=dx+\frac{dq}{dp}dy \tag{2.42}$$

and the equation (2.4), on differentiation with respect of p gives

$$\frac{\partial F}{\partial p}+\frac{\partial F}{\partial q}\frac{dq}{dp}=0 \tag{2.43}$$

Using (2.41) to (2.43) we have

$$\frac{dx}{F_p}=\frac{dy}{F_q}=\frac{dz}{pF_p+qF_q} \tag{2.44}$$

so that along a characteristis strip $x'(t),y'(t),z'(t)$ must be proportional to F_p,F_q,pF_p+qF_q
respectively. We choose the parameter t such that

$$x'(t)=F_p,\ y'(t)=F_q,\ z'(t)=pF_p+qF_q.$$

Now along a characteristic strip, p is a function of t so that

$$p'(t) = \frac{\partial p}{\partial x} x'(t) + \frac{\partial p}{\partial y} y'(t)$$

$$= \frac{\partial p}{\partial x}\frac{\partial F}{\partial p} + \frac{\partial p}{\partial y}\frac{\partial F}{\partial q}$$

$$= \frac{\partial p}{\partial x}\frac{\partial F}{\partial p} + \frac{\partial q}{\partial x}\frac{\partial F}{\partial q} \qquad \left[\because \frac{\partial p}{\partial y}\ \frac{\partial^2 z}{\partial y dx} = \frac{\partial q}{\partial x}\right]$$

Also differentiating (2.4) with respect to x, we get

$$= \frac{\partial F}{\partial x} + p\frac{\partial F}{\partial z} + \frac{\partial F}{\partial x} + \frac{\partial F}{\partial q}\frac{\partial q}{\partial x} = 0$$

or $$= \frac{\partial F}{\partial x} + p\frac{\partial F}{\partial z} p'(t) = 0$$

and, hence, on a characteristic strip

$$p'(t) = -(F_x + pF_z)$$

Similarly, we have

$$q'(t) = -(F_y + qF_z)$$

From the above discussions, we have the following system of five ordinary differential equations for the determination of the characteristic strip.

$$x'(t) = F_p,\ \ y'(t) = F_q,\ \ z'(t) = pF_p + qF_q,$$

$$p'(t) = -F_x - pF_z,\ \ q'(t) = -F_y - qF_z \qquad (2.45)$$

These equations are called *Cauchy's characteristic equations* of the partial differential equation $F(x,y,z,p,q)=0$

The main theorem about characteristic strip is given as follows :

Theorem 2.4 : *Along every characteristic strip of the equation* $F(x,y,z,p,q)=0$, *the fucntion* $F(x,y,z,p,q)$ *is constant.*

proof : Since along *a* characteristic strip

$$\frac{d}{dt}\{F\{x(t),y(t),z(t),p(t),q(t)\}]$$

$$= F_x x'(t) + F_y y'(t) + F_z z'(t) + F_p p'(t) + F_q q'(t)$$

$$= F_x F_p + F_y F_q + F_z(pF_p + qF_q) - F_p(F_x + pF_z) - F_q(F_y + qF_z) \qquad [By(2.45)]$$

$$= 0$$

We have $F(x,y,z,p,q) =$ constant along the strip.

As a corollary, we have the following result :

Theorem 2.5 : *If a characteristic strip contains at least one integral element of $F(x,y,z,p,q)=0$, them it is an integral strip of this equation.*

We are now in a position to *solve Cauchy's problem* stated earlier.

Let us find the solution of the equation $F(x,y,z,p,q)=0$ which passess through a curve Γ. Let the parametric equations of this curve be given by

$$x=\phi(u),\ y=\psi(u),\ z=x(u) \tag{2.46}$$

Then in the solution

$$x=x(x_o,y_o,z_o,p_o,q_o,t_o,t) \text{ etc.} \tag{2.47}$$

of the characteristic equations (2.45), we may take the initial values of x, y, z as

$$x=\phi(u),\ y_o=\psi(u),\ z_o=x(u)$$

The corresponding initial values of p_o, q_o are then obtained by the relations

$$x'(u)=p_o\phi'(u)+q_o\psi'(u) \text{ and } F\{\phi(u),\psi(u),x(u),p_o,q_o\}=0$$

Substituting these values of x_o, y_o, z_o, p_o, q_o and the appurpriate value of t_o in equation (2.47), we can express x, y, z in terms of two parameters t and in the form

$$x=X_1(u,t), y=Y_1(u,t), z=Z_1(u,t)$$

Elimination of u and t from these three equations leads to a relation of the form

$$\theta(x,y,z)=0 \tag{2.48}$$

Thisis the equation of the integral surface of the euqation $F(x,y,z,p,q)=0$ through a curve Γ.

Exmaple 2.10 : Determine the characteristics of the equation $z=p^2-q^2$ and findthe integral surface which passes through the parabola $4z+x^2=0,\ y=0$

Solution : The given equation is

$$F(x,y,x,p,q)=z-p^2+q^2=0 \tag{2.49}$$

Then the characteristics equations are

$$x'(t)=F_p=-2p,\ y'(t)=F_p=2q,\ z'(t)=pF_p+qF_q=2(q^2-p^2)$$

$$p'(t)=-(F_x+pF_z)=-p, q'(t)=-(F_y+qF_z)=-q$$

Now the given curve is $4z+x^2=0,\ y=0$, We choose the initial values as

$$x_o=2u,\ y_o=0,\ z_o=-u^2$$

Since $z_o'=p_ox_o'+q_oy_o'$ we have $p_o=-u$ and so from (2.49) $q_o=\sqrt{2u}$.

Now the equations $x'(t)=-2p$ and $p'(t)=-p$ give $dx=2dp \Rightarrow x=2p+c_1$, where c_1 is constant.

Also the equations $y'(t)=2q$ and $q'(t)=-q$ give $dy=-2dq \Rightarrow y+2q=c_2$ where c_2 is constant.

Using the initial conditions, we obtain $c_1 = 4u$, $c_2 = 2\sqrt{2u}$ Hence we have

$$x = 2p + 4u, \quad y = 2\sqrt{2u} - 2q$$

Again the equations $p'(t) = -p$ and $q'(t) = -q$ imply that $p = c_3 e^t$ and $q = q_4 e^{-t}$ which on using the initial conditions give $p = -ue^{-t}$, $q = \sqrt{2}ue^{-t}$ and hence

$$x = 2u\left(2 - e^{-t}\right), y = 2\sqrt{2u}\left(1 - e^{-t}\right)$$

Putting the values of p and q in the characteristic equation $z'(t) = 2\left(q^2 - p^2\right)$, we get $z'(t) = 2u^2e^{-2t} \Rightarrow z = -u^2e^{-2t} + c_5 \Rightarrow z = -u^2e^{-2t}$, since $c_5 = 0$ by initial conditions.
Thus the characteristics of the given equation are

$$x = 2u\left(2 - e^{-t}\right), y = 2\sqrt{2u}\left(1 - e^{-t}\right), z = -u^2e^{-2t} \tag{2.50}$$

The first two relations of (2.50) give $e^{-t} = \dfrac{\sqrt{2}x - 2y}{\sqrt{2}x - y}, u = \dfrac{\sqrt{2}x - y}{2\sqrt{2}}$ which when substituted into the third relation of (2.50) leads to $\left(x - \sqrt{2}y\right)^2 + 4z = 0$

This is the required equation of the integral surface.

§ 2.10 : Charpit's Method

A method of solving partial differential equations of the form

$$F(x, y, z, p, q) = 0 \tag{2.4}$$

based on the considerations § 2.7 has been given by Charpit. In this method, we first introduce another first-order differential equations

$$G(x, y, z, p, q, a) = 0 \tag{2.5}$$

where a is an arbitraty constant, so that
(i) equations (2.4) and (2.51) can be solved for p and q to give

$$p = p(x, y, x, a), \quad q = (x, y, z, a)$$

and (ii) the equation

$$dz = p(x, y, z, a)dx + q(x, y, z, a)dy \tag{2.52}$$

is integrable.

If we can find such a function $G(x, y, z, a)dx + q(x, y, z, a)dy$ then the equation (2.52) has the solution of the form

$$f(x, y, z, a, b) = 0 \tag{2.53}$$

containing two arbitrary constants a, b and this solution will also be a solution of the equation (2.4). It is obvious from the considerations of § 2.8 that the equation (2.35) is a complete integral of the equation (2.4)

Thus our problem is to find the second equation (2.51), compatible with the given equation (2.4), the conditions for which are given in § 2.7 as

$$J = \frac{\partial(F,G)}{\partial(p,q)} \neq 0 \text{ and } [F, G] = 0$$

Expansion of the last equation leads to the equivalent linear partial differential equation

$$F_p \frac{\partial G}{\partial x} + F_q \frac{\partial G}{\partial y} + \left(pF_p + qF_q\right)\frac{\partial G}{\partial_z} - \left(F_x + pF_z\right)\frac{\partial G}{\partial P} - \left(F_y + qF_z\right)\frac{\partial G}{\partial z} = 0$$

for the determination of G. A solution of this equation can be obtained by finding an integral of the subsidiary equation

$$\frac{dx}{F_p} = \frac{dy}{F_q} = \frac{dz}{pF_p + qF_q} = \frac{dp}{-(F_x + pF_z)} = \frac{dq}{-\left(F_y + qF_z\right)} \quad (2.54)$$

in accordance with Theorem 2.2. Th equations (2.54) are known as *Charpit's equations and they are equivalent to the characteristic equations (2.45).*

If we can find $G(x,y,z,p,q,a)$, then the problem reduces to that of solving for p and q and then to integrate the equation (2.52) by the methods gives in §. 1.6. of Unit I. It may be noted that not all of equations (2.54) are necessary, but that p or q must occur in the solution obtained.

Example 2.11 : Find the complete integral of the equation $2zx - px^2 - 2qxy + pq = 0$ by Charpit's method.

Solution. Let $F(x,y,z,p,q) = 2zx - px^2 - 2qxy + pq = 0$ Then Charpit's equations are

$$\frac{dz}{-x^2 + q} = \frac{dy}{-2xy} = \frac{dz}{-px^2 + pq - 2xyq} = \frac{dp}{-(2z - 2px - 2qy + 2px)} = \frac{dq}{-(-2qx + 2qx)}$$

so that $dq = 0 \Rightarrow q =$ const. $= a$ say. Then the given equation gives. $p = \dfrac{2xz - 2axy}{x^2 - a}$

Hence from the relation $dz = pdx + qdy$, we get

$$dz = \frac{2xz - 2axy}{x^2 - a}dx + ady \Rightarrow \frac{d(z - ay)}{z - ay} = \frac{d\left(x^2 - a\right)}{x^2 - a}$$

Integrating weget $z - ay = b\left(x^2 - a\right)$, i.e, $z = ay + b\left(x^2 - a\right)$, where b is constant.

Example 2.12 : Solve the equation $px + qy = z(1 + pq)^{1/2}$ by Charpit's method.

Solution. Let $F(x,y,z,p,q) = px + qy - z(1 + pq)^{1/2} = 0$. Then Charpit's equations are

$$\frac{dx}{x-\frac{1}{2}qz(1+pq)^{-\frac{1}{2}}}=\frac{dy}{y-\frac{1}{2}pz(1+pq)^{-\frac{1}{2}}}=\frac{dz}{px+qy-pqz(1+pq)^{-\frac{1}{2}}}=-\left\{\frac{dp}{p-p(1+pq)^{-\frac{1}{2}}}\right\}$$

$$=-\left\{\frac{dq}{q-q(1+pq)^{-\frac{1}{2}}}\right\}$$

The last two equations give $\frac{dp}{p}=\frac{dq}{q}\Rightarrow p=aq$, a being constant. The given euqation

then given $q=\frac{z}{\left\{(ax+y)^2-az^2\right\}^{\frac{1}{2}}}$. Thus the relation $dz=pdx+qdy$ leads to

$$dz=\frac{z(adx+dy)}{\left\{(ax+y)^2-az^2\right\}^{\frac{1}{2}}}\Rightarrow\sqrt{a}\left(t^2-z^2\right)^{\frac{1}{2}}dz=z\sqrt{a}dt$$ [Putting $\sqrt{a}t=ax+y$]

$$\Rightarrow\frac{dz}{z}=\frac{du}{\left(u^2-1\right)^{1/2}-u}$$ [Putting $t=uz$] $$\Rightarrow\frac{dz}{z}=-\left\{u+\left(u^2-1\right)^{\frac{1}{2}}\right\}du$$

Integrating $\log z+\frac{1}{2}u^2+\frac{1}{2}u\left(u^2-1\right)^{\frac{1}{2}}+\frac{1}{2}\log\left\{u+\left(u^2-1\right)^{\frac{1}{2}}\right\}=\text{cost.}=\text{b.say.}$

Putting $u=\frac{t}{z}=\frac{ax+y}{\sqrt{az}}$ we have the required solution

$$\log z+\frac{ax+y}{z^2}\left\{(ax+y)+\left(ax+y-\sqrt{az}\right)^{\frac{1}{2}}\right\}+\log\left\{(ax+y)+\left(ax+y-\sqrt{az}\right)^{\frac{1}{2}}\right\}=b$$

Some special types of first-order partial differential equations.

We now consider some special types of first-order partial differential equations which can be solved easily be Charpit's method.

(a) Equations involving only p and q : Let the equations are of the type

$$F(p,q)=0 \qquad (2.55)$$

Then Charpit's equations (2.54) give

$$\frac{dx}{F_p}=\frac{dy}{F_q}=\frac{dz}{pF_p+qF_q}=\frac{dp}{o}=\frac{dq}{0}$$

an obvious solution of which is $p=$ const. $=a$, say, so that the value of q can be obtained from (2.55) in the form $q=\phi(a)=$const. The solution of the equation $dz=adx+\phi(a)dy$ is then $z=ax+\phi(a)y+b$, were b is anothr constant.

We have chosen the equation $dp=0$ to provide our second equation. Sometimes it is more convenient to use the equation $dq=0$ i.e. $q=a$ constant.

Example 2.13 : Solve the equation $p+q=pq$

Solution : Let $F(p,q)=p+q-pq=0$. Then Charpit's equations are

$$\frac{dx}{1-q}=\frac{dy}{1-p}=\frac{dy}{p+q-pq}=\frac{dp}{o}=\frac{dq}{o}$$

Thus $dp=0\Rightarrow p=\text{const}=a$, say. From the given equation $q=\frac{a}{a-1}$. Hence the equation $dz=pdx+qdy$, i.e. $dz=adx+\frac{a}{a-1}dy$ leads to the required solution as $z=ax+\frac{a}{a-1}y+b$, where b is constant.

(b) Equations not involving independent variables x and y : Let the partial differential equation is of the form

$$f(z,p,q)=0 \tag{2.56}$$

Then Charpit's equations (2.54) are

$$\frac{dx}{F_p}=\frac{dy}{F_q}=\frac{dz}{pF_p+qF_q}=\frac{dp}{-pF_z}=\frac{dq}{-qF_z}$$

On integration, the last two retations give $p=aq$, a being constant. This equation along with (2.56) gives the expressions for p and q and the complete integral then follows by integrating the equation $dz=pdx+qdy$.

Example 2.14 : Find the complete integral of $zpq=p+q$.

Solution. Let $F(z,p,q)=zpq-p-q=0$. Then Charpit's equations are

$$\frac{dx}{zq-1}=\frac{dy}{zp-1}=\frac{dz}{2zpq-p-q}=\frac{dp}{-p^2q}=\frac{dq}{-pq^2}$$

The last two relations give $\frac{dp}{p}=\frac{dq}{q}\Rightarrow p=aq$, where a is constant. Then the given equation gives $q=\frac{a+1}{az}$ so that $p=\frac{a+1}{z}$. Hence from the equation $dz=pdx+qdy$, i.e. from

$dz = \frac{a+1}{z}dx + \frac{a+1}{az}dy,$ we get the required solution on integration as $az^2 = 2(a+1)(ax+y)+b$ where b is another constant.

(c) Separable equations : A first-order partial differential equation is said to be separable if it can be written in the form

$$\Phi(x,p) = \psi(y,q) \tag{2.57}$$

where $F(x,y,z,p,q) = \Phi(x,p) - \psi(y,q) = 0$ In this case Charpit's equations are

$$\frac{dx}{\Phi_p} = \frac{dy}{-\psi_q} = \frac{dz}{p\Phi_p - q\psi_q} = \frac{dp}{-\Phi_x} = \frac{dq}{-\psi_y}$$

The first and the fourth equations produce an ordinary differential equation $\frac{dp}{dx} + \frac{\Phi_x}{\Phi_p} = 0$ in x and p, i.e. an equation $\Phi_p dp + \Phi_x dx = 0$ whose solution can be obtanied in the form $\Phi(x,p) = \text{const} = a,$ say. Thus we have $\Phi(x,p) = a, \psi(y,q) = 0$ and then we proceed as in the general theory.

Example 2.15 : Find the complete integral of the equation $p^2q^2 + x^2y^2 = x^2q^2(x^2+y^2)$

Solution. We can write the given equation as $\frac{p^2}{x^2} - x^2 = y^2 - \frac{y^2}{q^2}$ and the equation is, therefore, separable. Let $\Phi(x,p) = \frac{p^2}{x^2} - x^2, \psi(y,q) = y^2 - \frac{y^2}{q^2}$ We then determine p and q from the relations $\frac{p^2}{x^2} - x^2 = a,\ y^2 - \frac{y^2}{q^2} = a,$ (a being arbitrary constant) as $p = x\sqrt{x^2+a},$ $q = \frac{y}{\sqrt{y^2-a}}$ Hence from the equation $dz = pdx + qdy,$ i.e. from $dz = x\sqrt{x^2+a}dx + \frac{y}{\sqrt{y^2-a}}dy,$ we get, on integration $z = \frac{1}{3}(x^2+a)^{3/2} + \sqrt{y^2-a},$ where b constant.

(d) *Clairaut's equations* : A first-order partial differential equation is said to be of Clairaut type if it can be written in the form

$$z = px + qy + f(p,q) \tag{2.58}$$

Then Charpit's equations are

$$\frac{dx}{x+f_p}=\frac{dy}{y+f_q}=\frac{dz}{px+qy+pf_p+qf_q}=\frac{dp}{0}=\frac{dq}{0}$$

so that $p=a$, $q=b$ where a and b are constants. Putting these values of p and q in the given equation, the complete integral is obtained of $z=ax+by+f(a,b)$,

Example 2.16 : Find the complete integral of the equation $pqz=p^2\left(xq+p^2\right)+q^2\left(yp+q^2\right)$

Solution. The given equation can be written as $z=px+qy+\dfrac{p^4+q^4}{pq}$ which is in Clairaut's form. Hence the required complete integral is $z=ax+by+\dfrac{a^2+b^4}{ab}$, where a, b are arbitrary constants.

§ 2.11 : Solutions Satisfying Given Conditions :

In this section we proceed to determine the surfaces which satisfy the partial differential equation

$$F(x,y,z,p,q)=0 \tag{2.4}$$

and some other conditions such as passing through a given curve or circumscribing a give surface. Also we consider now to derive one complete integral from the other.

First we determine the solution of the equation (2.4) which passes through a given curve Γ having parametric equations $x=x(t)$, $y=y(t)$, $z=z(t)$, t being parameter. For an integral surface of the equation (2.4) through the curve Γ it is either

(a) a particular case of the complete integral

obtained by giving a or b arbitrary values : or $f(x,y,z,a,b)=0$ (2.32)

(b) a particular case of the general integral corresponding to (2.32), that is the envelope of a one-parameter subsystem of (2.32) ; or

(c) the envelope of a two parameter system (2.32)

It is unlikely that the solution falls into categories (*a*) or (*c*). So we deal with the case (*b*) only. Suppose that a surface S passes through a curve Γ and is of type (*b*). Then at its every point, the envelope S is touched by a member of its subsystem. Let at a point P of Γ, it is touched by a member S_p, say S_p, of the subsystem and since s_p touches S at P, it also touches Γ at P. Hence S is the envolve of a one parameter subsystem of (2.32) each of whose members touches the curve, provided such a subsystem exists. To determine S, we consider the subsystem to be made up of those members of the family (2.32) which touch the curve Γ. The points of intersection of the surface (2.32) and the curve Γ are obtained by the equation

$$f\{x(t),y(t),z(t),a,b\}=0 \tag{2.59}$$

in terms of the parameter t. The curve Γ touches the surface (2.32) if the equation (2.59) has two equal roots, i.e., the equation (2.59) and the equation

$$\frac{\partial}{\partial t}\left[f\{x(t),y(t),z(t),a,b\}\right]=0 \tag{2.60}$$

have a common root, the condition for which is given by the t-eliminant of (2.59) and (2.60) as

$$\psi(a,b)=0 \tag{2.61}$$

which is a relation between a and b can be factorised to give

$$b=\psi_1(a),\ b=\psi_2(a),\ldots\ldots \tag{2.62}$$

each of which is a one-parameter subsystem. The envelope of these one-parameter subsystems is a solution of the problem.

Example 2.17 : Find a complete integral of the equation $p^2x+qy=z$ and hence derive the equation of an integral surface of which the line $y=1,\ x+z=0$ is a generator.

Solution : Let $F(x,y,z,p,q)=z-p^2x-qy=0$. Then Charpit's equations are

$$\frac{dx}{-2px}=\frac{dy}{-q}=\frac{dz}{-2p^2x-qy}=\frac{dp}{p^2+p}+\frac{dp}{0}$$

The last equation gives q = const. = a. The given equation then gives $p=\left(\frac{z-ay}{x}\right)^{\frac{1}{2}}$

so that the equation $dz=pdx+qdy=\left(\frac{z-ay}{x}\right)^{\frac{1}{2}}+ady$, i.e. $\frac{d(z-ay)}{(z-ay)^{1/2}}=\frac{dx}{x^{1/2}}$ yields on

integration $(z-ay)^{\frac{1}{2}}=x^{\frac{1}{2}}+b^{\frac{1}{2}}$ i.e. $(x+ay-z+b)^2=4bx$ (2.63)

which is the complete integral, b being constant.

Now the parametric equations of the given line are

$x=t,\ y=1,\ z=-t$

The intersection of (2.63) and (2.64) is determined by $(2t+a+b)^2=4bt$, i.e., by

$4t^2+4at+(a+b)^2=0$ which has equal roots if $a^2(a+b)^2$ i.e. if $b=-2a,0$.

The appropriate one-parameter subsystem is

$(x+ay-z-2a)^2=-8ax$, i.e. $a^2(y-2)^2+2a\{x(y+4)-z(y-2)\}+(x-z)^2=0$

and has for its envelope $\{x(y+4)-z(y-2)\}^2=(y-2)^2(x-z)^2$, i.e. $xy=z(y-2)$

The function z defined by this equation is the solution of the problem.

Next we consider the problem of finding one complete integral from another. Suppose

$$f(x,y,z,a,b)=0 \tag{2.32}$$

is a complete integral. We like to show that there exists another relation

$$g(x,y,z,c,d)=0 \tag{2.65}$$

involving two arbitrary constants c, d, which is also a complete integral. On the surface (2.65) we choose a curve Γ whose equations contain the constants as independent parameters. Then the envelope of the one-parameter subsystem of (2.32) touching the curve Γ is found out. Since this solution contains two arbitary constants, it is also a complete integral.

Example 2.18 : Show that the differential equation $2xz+q^2=x(xp+yq)$ has a complete integral $z+a^2x+axy+bx^2$ and deduce that $x(y+cx)^2=4(z-dx^2)$ is also a complete integral

Solution : Let $F(x,y,z,p,q)=2xz+q^2-x(xp+yq)=0$. Then Charpit's equations are

$$\frac{dx}{-x^2}=\frac{dy}{2q-xy}=\frac{dz}{-px^2+eq^2-2xy}=\frac{dp}{-2z+yq}=\frac{dq}{-qx}$$

From the first and last relations, we get on integration $q=ax$, whose a is an arbitrary constant. Then from the given equation we have $p=\dfrac{2z+a^2x-axy}{x}$ so that the equation $dz=pdx+qdy$ gives

$$dz=\frac{2z+a^2x-axy}{x}dx+axdy \Rightarrow \frac{dz}{x^2}-\frac{2z}{x^3}dx-\frac{a^2}{x^2}dx=a\frac{xdy-ydx}{x^2}$$

$$\Rightarrow d\left(\frac{z}{x^2}\right)+d\left(\frac{a^2}{x}\right)=ad\left(\frac{y}{x}\right)\Rightarrow \frac{z}{x^2}+\frac{a^2}{x}=a\frac{y}{x}+b, \text{ on integration}$$

i.e. $z+a^2x=axy+bx^2$ which is a complete integral, b being constant.

To show that

$$x(y+cx)^2=4(z-dx^2) \tag{2.66}$$

is also a complete integral, we consider the curve $\Gamma: y=0,\ z=\dfrac{c^3x^3+4dx^2}{4}$

on the surface (2.66). At the intersections of $z+a^2x=axy+bx^2$ and the curve Γ we have $c^3x^2+4(d-b)x+4a^2=0$

which has equal roots if $b=d\pm ac$, Taking $b=d+ac$ the subsystem has the equation

$z+a^2x=axy+(d+ac)x^2$ i.e. $a^2x-x(cx+y)a+z-dx^2)=0$

which has the envelope $x^2(cx+y)^2=4x(z-dx^2)$, i.e. $x(y+cx)^2=4(z-dx^2)$

This is, therefore, a complete integral.

Lastly, we outline the procedure in determining an integral surface which circumscribes a given surface. Two surfaces circumscribe each other if they touch along a given curve, e.g., a conicoid and its enveloping cylinder, and this curve of contact may not be a plane curve. Let the partial differential equation (2.4) : $F(x,y,z,p,q)=0$. has a complete integral (2.32) . $f(x,y,z,a,b)=0$. Our object is to find, with the use of (2.32), an integral surface of (2.4) which circumscribes the surface $\sum$ whose equation is

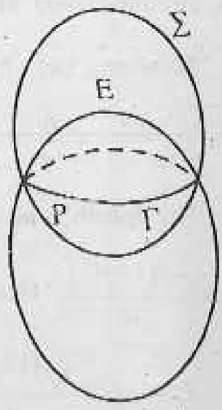


Fig. 2.4

$$\psi(x,y,z)=0 \tag{2.67}$$

Let us consider the surface

$$E:u(x,y,z)=0 \tag{2.68}$$

of the required kind. Then it is one of the three kinds

(*a*), (*b*) or (*c*) listed above. Owing to frequent occurence, we consider the possibility (*b*). Since E is the envelope of a one-parameter subsystem S of a two-parameter system (2.32), it is touched at each of its points, and in particular, at each point P of Γ by a member S_p of the subsystem S. Since S_p touches E at P $\sum$ at P. Hence the equation (2.68) is the equation of the envelope of a set of surfaces (2.32) touching the surface (2.67). We now find the surfaces (2.32) which touch E and see whether they provide a solution of the problem.

Now the surface (2.32) touches the surface (2.68) if the equations (2.32), (2.67) and

$$\frac{fx}{\psi_x}=\frac{f_y}{\psi_y}=\frac{f_z}{\psi_z} \tag{2.69}$$

are consistent and the conditions for which is the elimination of x,y,z from these four equations yielding a relation of the type

$$\lambda(a,b)=0 \tag{2.70}$$

between a and b. This equationcan be factorised into a set of relations of the form

$$b=x_1,(a),b=x_2(a),..... \tag{2.71}$$

each of which defines a subsystem of (2.32) whose members touch (2.67). The points of contact lie on the surface whose equation is obtained by the elimination of a and b from the equations (2.69) and (2.71). The intersection of this surface with $\sum$ is the curve

Γ. Each of the relations (2.71) defines a subsysem whose envelope E touches $\sum$ along Γ.

Example 2.19 : Show that the integral surface of the equation $2y(1+p^2)=pq$ which is circumscribed about the cone $x^2+z^2=y^2$ has the equation $z^2=y^2(4y^2+4x+1)$

Solution. Let $F(x,y,z,p,q)=2y(1+p^2)-pq=0$ The Charpit's equations are

$$\frac{dx}{4yp-q}=\frac{dy}{-p}=\frac{dz}{4yp^2-2pq}=\frac{dp}{0}=\frac{dp}{-(2+2p^2)}$$

The fourth relation gives p=const. = a, say, so that from the given equation we have $q=\frac{2y(1+a^2)}{a}$. Hence the equation $dz=pdx+qdy$ leads to

$dz=adx+\frac{2y(1+a^2)}{a}dy \Rightarrow z=ax+\frac{y^2(1+a^2)}{a}+b$, on integration, b being a constant. This is a complete integral.

Now let $f(x,y,z,a,b)=z-ax-\frac{y^2(1+a^2)}{a}-b=0$ and $\psi(x,y,z)=x^2+y^2-z^2=0$ Then the relation $\frac{f_x}{\psi_x}=\frac{f_y}{\psi_y}=\frac{f_z}{\psi_z}$ gives $\frac{-a}{2x}=\frac{-2y(1+a^2)}{-2ay}=\frac{1}{2z}$ so that

$x=-\frac{a^2}{2(1+a^2)}, z=\frac{a}{2(1+a^2)}$. Hence from $f=0$, we get $y^2=\frac{a(a-2b)}{2(1+a^2)}$

Using the above values of x,y,z we find from the equation $\psi=0$ that the relation $b=a/4$ defines a subsystem whose envelope is a surface of the required kind. The envelope of the subsystem $(4x+4y^2+1)a^2-4az+4y^2=0$ is obviously $z^2=y^2(4x+4y^2+1)$.

§ 2.12 : Jacobi's Method

Another method of solving partial differential equation of the form

$$F(x,y,z,p,q)=0 \tag{2.4}$$

is due to Jacobi. The method lies on the fact that if there is a relation

$$u(x,y,z)=0 \tag{2.72}$$

between x,y,z then

$$p=\frac{u_1}{u_3}, q=\frac{u_2}{u_3} \tag{2.73}$$

where $u_i = \dfrac{\partial u}{\partial x_i} (i = 1,2,3)$. Substituting (2.73) into (2.4) we obtain a partial differential equation of the type

$$f(x, y, z, u_1, u_2, u_3) = 0 \tag{2.74}$$

in which the new dependent variable u does not appear.

The fundamental idea of Jacobi's method is to introduce another two first-order partial differential equations involving two arbitrary constants a and b of the type

$$g(x, y, z, u, u_2, u_3, a) = 0 \text{ , } \frac{\partial x}{f_{u_1}} = f_{u_2}\frac{\partial g}{\partial y} + f_{u_3}\frac{\partial g}{\partial z} - f_x\frac{\partial g}{\partial u_1} - f_y\frac{\partial g}{\partial u_2} - f_z\frac{\partial g}{\partial u_3} = 0 \tag{2.75}$$

such that

(a) equations (2.74) and (2.75) are solvable for u_1, u_2, u_3

and (b) the equation $du = u_1 dx + u_2 dy + u_3 dz$ (2.76)

is integerable.

Now the equation (2.74) and (2.75) must be mutually compatible so that

$$[f, g] = 0, \ [g, h] = 0, \ [h, f] = 0$$

The equation $[f, g] = 0$ implies

$$\frac{\partial(f, g)}{\partial(x, u_1)} + \frac{\partial(f, g)}{\partial(x, u_2)} + \frac{\partial(f, g)}{\partial(x, u_3)} = 0$$

i.e. $f_{u_1} \dfrac{\partial g}{\partial x} + f_{u_2}\dfrac{\partial g}{\partial y} + f_{u_3}\dfrac{\partial g}{\partial z} - f_x\dfrac{\partial g}{\partial u_1} - f_y\dfrac{\partial g}{\partial u_2} - f_z\dfrac{\partial g}{\partial u_3} = 0$

which has subsidiary equations

$$\frac{dx}{f_{u_1}} = \frac{dy}{f_{u_2}} = \frac{dz}{f_{u_3}} = \frac{du_1}{f_x} = \frac{du_2}{f_y} = \frac{du_3}{f_z} \tag{2.77}$$

Any two solutions of (2.77) involving u_1, u_2 or u_3 serve the purpose of equations (2.75), ; provided that the conditions of compatibility are satisfied. We solve (2.74) and (2.75) for u_1, u_2, or u_3 and putting these values in (2.76) the required solution is obtained after integration.

The advantage of Jacobi's method is that it can be generalised to any number of variables. If we are to solve the partial differential equation

$$f_i(x_1, x_2, \ldots x_n, u_1, u_2 \ldots, u_n) = 0 \tag{2.78}$$

where $u_i = \partial u / \partial x_i$, (i=1,3,...., n), then the auxiliary equation is

$$\frac{dx_1}{f_{u_1}} = \frac{dx_2}{f_{u_2}} = ... = \frac{dx_n}{f_{u_n}} = \frac{du_1}{-f_{x_1}} = ... = \frac{du_n}{-f_{x_n}}$$

involving (n−1) arbitrary constants. We then solve these equations for $u_1, u_2 ... u_n$ and determine u by integrating Pfaffian equation . The solution is then obtained containing n arbitrary constants. On the other hand, Charpit's method cannot be generalised directly.

Example 2.20 : Solve the equation $p^2x + q^2y = z$ by Jacobi's method.

Solution : The given equation is $F(x, y, z, p, q) = p^2x + q^2y - z = 0$

Putting $p = -\frac{u_1}{u_3}$, $q = -\frac{u_2}{u_3}$ in this equation we get

$$f(x, y, z, u_1, u_2, u_3) = xu_1^2 + xu_2^2 - zu_3^2 = 0$$

(2.79)

Hence Jacobi's equations (2.77) are

$$\frac{dx}{2u_1x} = \frac{dy}{2u_2y} = \frac{dz}{-2u_2z} = \frac{du_1}{-u_1^2} = \frac{du_2}{-u_2^2} = \frac{du_3}{u_3^2}$$

From the first and fourth equations, we have $\frac{dx}{x_1} + \frac{2du_1}{u_1} = 0 \Rightarrow xu_1^2 = a$

i.e. $u_1 = \sqrt{\frac{a}{x}}$. Also, from the second and fifth equations we get $\frac{dy}{y} + \frac{2du_2}{u_2} = 0 \Rightarrow yu_2^2 = b$

i.e. $u_2 = \sqrt{\frac{b}{y}}$. Hence from (2.79) we get $u_3 = \sqrt{\frac{a+b}{z}}$

Thus the equation $du = u_1dx + u_2dy + u_3dz$ gives

$$du = \sqrt{\frac{a}{x}}dx + \sqrt{\frac{b}{y}}dy + \sqrt{\frac{a+b}{z}}dz$$

Integration leads to

$$u = 2\sqrt{ax} + 2\sqrt{by} + 2\sqrt{(a+b)z} + c$$

Example 2.21 : Solve the equation $px + qy = pq$ by Jacobi's method.

Solution. The given equation is $F(x, y, z, p, q) = px + qy - pq = 0$. Putting

$p=-\frac{u_1}{u_3}$, $q=-\frac{u_2}{u_3}$ in this equation we get.

$$f(x,y,z,u_1,u_2,u_3)=u_1u_3x+u_2u_3y+u_1u_2=0 \tag{2.80}$$

Hence Jacobi's equations (2.77) are

$$\frac{dx}{u_3x+u_2}=\frac{dy}{u_3y+u_1}=\frac{dz}{u_3x+u_2y}=\frac{du_1}{-u_1u_3}=\frac{du_2}{-u_2u_3}=\frac{du_3}{0}$$

The last equation gives $du_3=0 \Rightarrow u_3=$ const. = c (say).

From the fourth and fifth equations we derive $\frac{du_2}{u_2}=\frac{du_1}{u_1} \Rightarrow u_2=bu_1$

where b is constant. Hence from (2.80) we get

$u_1cx+bu_1cy+bu_1^2=0 \Rightarrow u_1=-\frac{c(x+by)}{b}\ (\because u_1\neq 0)$ so that $u_2=-c(x+by)$

Thus the equation $du=u_1dx+u_2dy+u_3dz$ gives

$$du=-\frac{c}{b}(x+by)(dx+bdy)+c\,dz$$

leading to the solution

$$u=-\frac{c}{2}(x+by)^2+cz+a$$

§ 213. Application of First-order Equation : Hamilton-Jacobi Equation :

One of the most important first order partial differential equation which occurs in mathematical physics is the Hamilton-Jacobi equation given by

$$H\left(q_1,q_2 \dots q_n ; \frac{\partial S}{\partial q_1},\frac{\partial S}{\partial q_2},\dots,\frac{\partial S}{\partial q_n}\right)=0 \tag{2.81}$$

In this equations $H(q_1,q_2 \dots q_n ; p_1,p_2 \dots p_n)$ devotes the Hamiltonian of a dynamical system of n generalised coordinates $q_1,q_2,\dots,q_n$ and conjugate momenta $p_1,p_2,\dots,p_n$. Here the dependent variable is absent and, therefore, it is of the type (2.78) of § 2.12. The characteristic equations are given by

$$\frac{dt}{1}=\frac{dq_1}{\partial H/\partial p_1}=...=\frac{dq_n}{\partial H/\partial p_n}=\frac{dp_1}{-\partial H/\partial q_1}=...=\frac{dp_n}{-\partial H/\partial q_n} \tag{2.82}$$

which are equivalent to the Hamiltonian equations of motion

$$\frac{dq_i}{dt}=\frac{\partial H}{\partial p_i},\ \frac{dp_i}{dt}=-\frac{\partial H}{\partial q_i} \quad (i = 1,2,...,n) \tag{2.83}$$

If we write the equation (2.81) as

$$S=-Wt+S_1$$

then we have

$$H\left(q_1,q_2,...q_n;\frac{\partial S_1}{\partial q_1},...\frac{\partial S_1}{\partial q_n}\right)=W \tag{2.84}$$

As an example, let us consider a dynamical system with two degrees of freedom with Hamiltonian

$$H=\frac{Pp_x^2+Qp_y^2}{2(X+Y)}+\frac{\xi+n}{X+Y} \tag{2.85}$$

Where P, X, ξ are functions of x alone and Q,n,η are functions of y alone. Then the equation (2.84) gives

$$\frac{1}{2}(Pp_x+QP_y)+(\xi+\eta)-W(X+Y)=0$$

Then one of the characteristic equations is

$$\frac{dx}{Pp_x}+\frac{dp_x}{\frac{1}{2}P'p_x+\xi'-WX'}=0$$

which has the solution $P_x=\{2(WX-\xi+a)\}^{\frac{1}{2}}$

where a is an arbitrary constant. Similarly, we have $p_y=\{2(WY-\eta+b)\}^{\frac{1}{2}}$

where b is an arbitrary constant, Noting that $p_x=p_x(x)$ and $p_y=p_y(y)$, we have

$$S=-Wt+\sqrt{2}\int(WX-\xi+a)^{\frac{1}{2}}dx+\sqrt{2}\int(WY-\eta+b)^{\frac{1}{2}}dy$$

Thus a solution of the Hamilton-Jacobi equation can always be obtained for a Hamiltonion of the form (2.84).

Exercises

1. Formulate the partial differential equations by eliminating arbitrary constants or functions from the following :

(i) $z=(x+a)(y+b)$ [Ans. $pq=z$]

(ii) $ax^2+by^2+z^2=1$ [Ans. $z(px+qy)=z^2-1$]

(iii) $x^2+y^2+(z-c)^2=a^2$ [Ans. $py-qx=0$]

(iv) $(x-a)^2+(y-b)^2+z^2=1$ [Ans. $z^2(1+p^2+q^2)=1$]

(v) $z=xy+f(x^2+y^2)$ [Ans. $x^2-y^2=qx-py$]

(vi) $z=x+y+f(xy)$ [Ans. $px-qy=x-y$]

(vii) $z=f(xy/z)$ [Ans. $xp(z-yq)=qy(z-px)$]

(viii) $z=f(x-y)$ [Ans. $p+q=0$]

2. Find the general integrals of the linear partial differential equations.

(i) $p(y+zx)-q(x+yz)=x^2-y^2$ [Ans. $f(x^2+y^2-z^2,xy+z)=0$]

(ii) $z(px-qy)=y^2-x^2$ [Ans. $f(x^2+y^2+z^2,xy)=0$]

(iii) $px(x+y)=qy(x+y)-(x-y)(2x+2y+z)$ [Ans. $f\{(x+y)(x+y+z),xy\}=0$]

(iv) $x(x^2+3y^2)p-y(3x^2+y^2)q=2z(y^2-x^2)$ [Ans. $f\left\{\frac{xy}{z},z(x^2+y^2)\right\}=0$]

(v) $px(z-2y^2)=(z-qy)(z-y^2-2x^2)$ [Ans. $f\left\{\frac{y}{z},\frac{z}{x}-\frac{a^2z^2}{x}+x^2\right\}=0$]

(vi) $y^2p-xyq=x(z-2y)$ [Ans. $f(x^2+yz,x^2+y^2)=0$]

3. Find the integral surface of the linear partial differential equation

$$x(y^2+z)p-y(x^2+z)q=(x^2-y^2)z$$

with contains the straight line $x+y=0,z=1$ [Ans. $f(x^2+yz,x^2+y^2)=0$]

4. Find the equation of the intetral surface of the differential equation

$$2y(z-3)p+(2x-z)q=y(2x-3)$$

which passes through the circle $z=0,x^2+y^2=2x$ [Ans. $x^2+y^2-2x=z^2-4z$]

5. Find the general integral of the partial differential equation

$$(2xy-1)p+(z-2x^2)q=2(x-yz)$$

and also the particular integral which passes through the line $x=1,\ y=0$

[Ans. $x^2+y^2-xz-y+z-1=0$]

6. Find the integral surface of the equation

$$(x-y)y^2p+(y-x)x^2q=(x^2+y^2)z$$

through the curve $xz=a^3, y=0$ [Ans. $z^3(x^3+y^3)^2=a^9(x-y)^3$]

7. Find te surface which interest the surface of the system

$$z(x+y)=c(3z+1)$$

orthogonally and which passes through the circle $x^2+y^2=1, z=1$

[Ans. $x^2+y^2=2z^3+z^2-2$]

8. Find the equation of the system of surface which cut orthogonally the cones of the system $x^2+y^2+z^2=cxy$. [Ans. $x^2+y^2+z^2=f(x^2-y^2/z^2)$]

9. Show that the equations $xp=yq, z(xp+yq)=2xy$ are compatible and solve them.

[Ans. $z^2=c_1+2xy$]

10. Show that the equation $z=px+qy$ is compatible with any equation $f(x,y,z,p,q)=0$ that is homogeneous in x,y and z.

Solve completely the simultaneous equations

$$z=px+qy, 2xy(p^2+q^2)=z(py+qx)$$ [Ans. $z^2=c(x^2+y^2)$ or $z^2=cxy$]

11. If $u_1=\partial u/\partial x$, $u_2=\partial u/\partial y$, $u_3=\partial u/\partial z$, show that the equations

$f(x,y,z,u_1,u_2,u_3)=0$, $g(x,y,z,u_1,u_2,u_3)=0$

are compatible if $\dfrac{\partial(f,g)}{\partial(x,u_1)}+\dfrac{\partial(f,g)}{\partial(y,u_2)}+\dfrac{\partial(f,g)}{\partial(z,u_3)}=0$

12. Verify that the equations

(a) $z=\sqrt{2x+a}+\sqrt{2y+b}$

(b) $z^2+\mu=2(1+\lambda^{-1})(x+\lambda y)$

are both complete intergrals of the partial differential equation

$$z = \frac{1}{p} + \frac{1}{q}$$

Show, further, that the complete integral (b) is the envelope of the one-parameter subsystem by taking

$$b = -\frac{a}{\lambda} - \frac{\mu}{1+\lambda}$$

in the solution (a)

13. Find the characteristics of the equation $pq = z$ and determine the integral surface which passes through the parabola $x=0,\ y^2 = z$.

[Ans. Characteristics : $x = 2v(e^t - 1), y = \frac{1}{2}v(e^t + 1), z = v^2 e^{2t}; 16z = (x+4y)^2$]

14. Find the solution of the equation $z = \frac{1}{2}(p^2 + q^2)(p - x)(q - y)$ which passes through the x-axis. [Ans. $z = \frac{1}{2}y(4x - 3y)$]

15. Solve the following equations by Charpit's or Jacobi's method :

(i) $px + qy = pq$ [Ans. $az = \frac{1}{2}(y + ax)^2 + b$]

(ii) $(p^2 + q^2)y = qz$ [Ans. $(z^2 - a^2y^2)^{\frac{1}{2}} = an + b$]

(iii) $z - px - qy = p^2 + q^2$ [Ans. $z - ax + yb + a^2 + b^2$]

(iv) $p^2 + q^2 - 2px - 2q + 1 = 0$ [Ans. $(a^2 + 1)z = b + \frac{1}{2}u^2 \pm \sqrt{\left[\frac{1}{2}u \quad (u^2 - a^2 - 1)\right]}$

$-\frac{1}{2}(a^2 + 1)\log\{u + \sqrt{(u^2 - a^2 - 1)}\}$ where $u = ax + y$]

(v) $2xz - px^2 - 2qxy + pq = 0$ [Ans. $z - ay = b(x^2 - a)$]

(vi) $q + px = p^2$ [Ans. $z = \frac{1}{4}\left[x^2 \pm \left\{x\sqrt{x^2 + 4a} + a\log\left(x + \sqrt{x^2 + 4a}\right)\right\} + ay + b\right]$

(vii) $z = pq$ [Ans. $2\sqrt{az} = ax + y + b$]

(viii) $px + qy = z(1 + pq)^{\frac{1}{2}}$ [Ans. $\log z + \frac{1}{2}u^2 + \frac{1}{2}u\sqrt{u^2 - 1} + \frac{1}{2}\log\left(u + \sqrt{u^2 - 1}\right) = b$

where $u = (ax + y)/z\sqrt{a}$]

(ix) $(p + q)(px + qy) - 1 = 0$ [Ans. $z = \frac{2}{\sqrt{(1+a)}}\sqrt{(ax + y)} + b$]

(x) $p=(qy+z)^2$ [Ans. $yz=ax+z\sqrt{ay}+b$]

(xi) $pxy+pq=qy-yz=0$ [Ans. $(z-ax)(y+a)^a=be^y$]

(xii) $(p^2+q^2)x=pz$ [Ans. $z=bx^a y^{\frac{1}{a}}$]

(xiii) $(p^2+q^2)y=qz$ [Ans. $(x+b)^2+y^2=az^2$]

(xiv) $z^2=pqxy$ [Ans. $z=bx^a y^{\frac{1}{a}}$]

(xv) $2(z+xp+yq)=yp^2$ [Ans. $z=\frac{ax}{y^2}+\frac{b}{y}-\frac{a^2}{4y^2}$]

(xvi) $pq=1$ [Ans. $z=\frac{ax}{y^2}+\frac{b}{y}-\frac{a^2}{4y^2}$]

(xvii) $p^2z^2+q^2=1$ [Ans. $az(1+a^2z)^{\frac{1}{2}}-log\left\{az+(1+a^2z^2)^{\frac{1}{2}}\right\}=2a(ax+y+b)$]

(xviii) $p^2y(1+x^2)=qx^2$ [Ans. $z=a\sqrt{1+x^2}+\frac{1}{2}a^2y^2+b$]

(xix) $(p+q)(z-xp-yq)=1$ [Ans. $z=ax+by+\frac{1}{a+b}$]

(xx) $p+q=pq$ [Ans. $z=ax+\frac{ay}{a-1}+b$]

(xxi) $zpq=p+q$ [Ans. $z^2=2(a+1)\left(x+\frac{y}{a}\right)+b$]

(xxii) $p^2q^2+x^2y^2=x^2q^2(x^2+y^2)$ [Ans. $z=\frac{1}{3}(x^2+y^2)^{\frac{3}{2}}+(y^2-a^2)^{\frac{1}{2}}+b$]

16. Find a complete integeral of the partial differential equation $(p^2+q^2)x=qz$ and deduce the solution which passes through the curve $x=0,\ z^2=4y$.

[Ans.Complete integral : $z^2=a^2x^2+(ay+b)^2$, Solution : $(2y-z^2)^2=4(x^2+y^2)$]

17. Show that the equation $xpq+yq^2=1$ has complete integrals $(a)(z+b)^2=4(ax+y)$, $(b)kx(z+h)=k^2y+x^2$ and deduce (b) from (a).

18. Show that the only integral surface of the equation $2q(z-px-qy)=1+q^2$ which is circumscribed about the paraboloid $2x=y^2+z^2$ is the enveloping cylinder which touches it along its section by the plane $y+1=0$

19. Show that the integral surface of the equation $z(1-q^2)=2(px+qy)$ which passes through the line $x=1$, $y=hz+k$ has equation $(y-kx)^2=z^2\{(1+h^2)x-1\}$

20. Find the complete integral of the differential equation $(y+zq)^2=z^2(1+p^2+q^2)$ circumscribed about the surface $x^2-y^2=2y$

[Ans. $(x-a)^2+y^2+z^2-2by=0, (y^2+4y+2z^2)^2=8x^2y^2$]

21. Show how to solve, by Jacobi's method, a partial differential equation of the type

$$f\left(x,\frac{\partial u}{\partial x},\frac{\partial u}{\partial Z}\right)=f\left(y,\frac{\partial u}{\partial y},\frac{\partial u}{\partial z}\right)$$

and illustrate the method by finding a complete integral of the equation

$$2x^2y\left(\frac{\partial u}{\partial x}\right)^2\frac{\partial u}{\partial z}=x^2\frac{\partial u}{\partial y}+2y\left(\frac{\partial u}{\partial x}\right)^2$$ [Ans. $u(ax^2-b)^{\frac{1}{2}}+ay^2+\frac{z}{b}+c$]

§ 2.14 Summary

This unit is dealt with partial differential equations of first order, their origin, formulation and solutions by different methods like Lagrange, Charpit and Jacobi. Existence of the solution has also been indicated. Compatibility systems of first order equations are also considered and Cuachy's method of characteristics for solving nonlinear partial differential equations has been discussed.

UNIT 3 ☐ SECOND ORDER PARTIAL DIFFERENTIAL EQUATIONS

§ 3.1 INTRODUCTION

In this unit we confine ourselves to a preliminary discussion of second order partial differential equations, and then in the following three chapters we shall consider in more detail the three main types of second order linear partial differential equations. Though we are mainly concerned with second order equations, we shall however, deal with some partial differential equations of order higher than the second, viz. higher order linear partial differential equations with constant coefficients.

§ 3.2 The Origin of Second-order Equations

Let a function z be defined by an expression of the form

$$z = f(u) + g(v) + w \tag{3.1}$$

in which f and g are arbitrary functions of u and v respectively and u, v, w are known functions of x and y. Let us write

$$p = \frac{\partial z}{\partial x},\quad q = \frac{\partial z}{\partial y},\quad r = \frac{\partial^2 z}{\partial x^2},\quad s = \frac{\partial^2 z}{\partial x \partial y},\quad t = \frac{\partial^2 z}{\partial y^2} \tag{3.2}$$

Differentiating both sides of (3.1) with respect to x and y respectively. we get

$$p = f'(u)u_x + g'(v)v_x + w_x$$

$$q = f'(u)u_y + g'(v)v_y + w_y$$

so that

$$r = f''(u)u_x^2 + g''(v)v_x^2 + f'(u)u_{xx} + g'(v)v_{xx} + w_{xx},$$

$$s = f''(u)u_x u_y + g''(v)v_x v_y + f'(u)u_{xy} + g'(v)v_{xy} + w_{xy},$$

$$t = f''(u)u_y^2 + g''(v)v_y^2 + f'(u)u_{yy} + g'(v)v_{yy} + w_{yy}.$$

The above five equations contain four arbitrary quantities . Eliminating these four quantities from the five equations we obtain the relation

$$\begin{vmatrix} p-w_x & u_x & v_x & 0 & 0 \\ q-w_y & u_y & v_y & 0 & 0 \\ r-w_{xx} & u_{xx} & v_{xx} & u_x^2 & v_x^2 \\ s-w_{xy} & u_{xy} & v_{xy} & u_xu_y & v_xv_y \\ t-w_{yy} & u_{yy} & v_{yy} & u_y^2 & v_y^2 \end{vmatrix} = 0 \tag{3.3}$$

Equation (3.3) involves only derivatives p,q,r,s,t and know functions of x and y and is, therefore, a second-order partial differential equation. Expanding the determinant on the left hand side of (3.3) in terms of elements in the first column, we obtain an equation of the form.

$$Rr + Ss + Tt + Pp = Qq = W \tag{3.4}$$

where each of R, S, T, R, Q and W is a function of x and y. Hence the relation (3.1) is a solution of the second-order partial equation (3.4). It may be noted that the equation (3.4) is a particular type in which the dependent variable does not occur.

Example 3.1 : Show that if f and g are arbitrary functions of thier respective arguments, then $u = f(x - vt + i\alpha y) + g(x - vt - i\alpha y)$ is a solution of

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$

where $\alpha^2 = 1 - \frac{v^2}{c^2}$.

Solution. We have $u = f(x - vt + i\alpha y) + g(x - vt - i\alpha y)$ (3.5)

Differenting (3.5) twice partially with respect to x, y and t we get respectively

$$\frac{\partial^2 u}{\partial x^2} = f''(x - vt + i\alpha y) + g''(x - vt - i\alpha y)$$

$$\frac{\partial^2 u}{\partial y^2} = -\alpha^2 \left[f''(x - vt + i\alpha y) + g''(x - vt - i\alpha y) \right]$$

$$\frac{\partial^2 u}{\partial t^2} = v^2 \left[f''(x - vt + i\alpha y) + g''(x - vt - i\alpha y) \right]$$

It follows that

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\left(1-\alpha^2\right)\left[f''(x-vt+i\alpha y)+g''(x-vt-i\alpha y)\right],$$

i.e. $$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$

3.3. Linear partial Differential Equations with Constant Coefficients (Two Indepent Variables.)

An equation of the form $F(D,D')z=f(x,y)$ (3.6)

where $F(D,D')$ denotes a differential operator of the tupe

$$F(D,D')=\sum_r\sum_s C_{rs}D^r D'^s \qquad (3.7)$$

C_{rs} being constsnts and $D\equiv \partial/\partial y$, $D'\equiv \partial/\partial y$, is called a *linear partial differential equation with constant coefficients in two independent variable x and y.*

The most general solution, i.e. the solution containing the exact number of arbitrary functions of the corresponding linear homogeneous equation

$$F(D,D')z=0 \qquad (3.8)$$

is called te *complementary function* of the equation (3.6) and any solution of (3.6) is called a *particular integral* of (3.6).

Theorem 3.1: *Let u be the complementary function and z_1 be a particular a of the linear partial differential equation $F(D,D')z=f(x,y)$. Then $u+z_1$ is a general solution of this equation.*

Proof : First we note that the solution $u+z_1$ contains the correct number of arbitrary functions for a general solution of the equation $F(D,D')z=f(x,y)$. Also, $F(D,D')u=0$ and $F(D,D')z_1=f(x,y)$. So we have

$$F(D,D')(u+z_1)=F(D,D')u+F(D,D')z_1=f(x,y).$$

Hence $u+z_1$ is a general solution of the given equation.

Theorem 3.2 : *Let $u_1, u_2, \ldots, u_n$ be n solutions of te linear homogeneous partial differential equation $F(D,D')z=0$. Then $\sum_{r=1}^{n} C_r u_r$, where C_r's are arbitraru constants, is also a solution of the equation.*

Proof : We have $F(D,D')(C_r u_r) = C_r F(D,D')u_r$

$$\text{and } F(D,D')\sum_{r=1}^{n} u_r = \sum_{r=1}^{n} F(D,D')u_r$$

for any set of functions u_r, Therefore,

$$F(D.D')\sum_{r=1}^{n} C_r u_r = \sum_{r=1}^{n} c_r F(D,D')u_r = 0$$

This prove the theorem.

The linear differential operators can be classified into *two* main types :

(a) $F(D.D')$ is *reducible* if it can be written as the product of linear factors of the form $\alpha D + \beta D' + \gamma$ where α, β and γ are constants and

(b) $F(D.D')$ is *irreducible* if it cannot be so written.

For example, $F(D,D') = D^2 + DD' - 2D'^2 - D - 2D'$ can e written as the product $(D+2D')(D-D'-1)$ and, therefore, it is reducible. But $F(D,D') = D^2 - 2D'$ cannot be documposed into two linear factors and hence it is irreducible.

Rules for finding complementary functions.

(a) Reducible equations.

Theorem 3.3 : *If the operator $F(D,D')$ is reducible, the order of the linear factors is unimportant.*

Proof : We have $(\alpha_r D + \beta_r D' + \gamma_r)(\alpha_s D + \beta_s D' + \gamma_s)$

$$= \alpha_r \alpha_s D^2 + (\alpha_s \beta_r + \alpha_r \beta_s)DD' + \beta_r \beta_s D'^2 + (\alpha_s \alpha_r + \gamma_r \alpha_s)D$$
$$+ (\gamma_s \beta_r + \gamma_r \beta_s)D' + \gamma_r \gamma_s$$
$$= (\alpha_s D + \beta_s D' + \gamma_s)(\alpha_r D + \beta_r D' + \gamma_r)$$

for any reducible operator and, therefore, we can write

$$F(D,D') = \prod_{r=1}^{n}(\alpha_r D + \beta_r D' + \gamma_r)$$

Theorem 3.4 *: Let $\alpha_r D + \beta_r D' + \gamma_r$ be a factor of $F(D,D')$ and $\phi_r(\xi)$ is an arbitrary function of the single variable ξ. Then if $\alpha_r \neq 0$*

$$u_r = exp\left(-\frac{\gamma_r x}{\alpha_r}\right)\phi_r(\beta_r x - \alpha_r y)$$

is a solution of the equation $F(D,D')z = 0$

Proof : Differentiating $u_r = exp\left(-\frac{\gamma_r x}{\alpha_r}\right)\phi_r(\beta_r x - \alpha_r y)$ with respect to x and y we get respectively.

$$Du_r = -\frac{\gamma_r}{\alpha_r}u_r + \beta_r\, exp\left(-\frac{\gamma_r x}{\alpha_r}\right)\phi'(\beta_r x - \alpha_r y)$$

and
$$D'u_r = -\alpha_r\, exp\left(-\frac{\gamma_r x}{\alpha_r}\right)\phi'(\beta_r x - \alpha_r y)$$

so that $(\alpha_r D + \beta_r D' + \gamma_r)u_r = 0$. Now by Theorem 3.3

$$F(D,D')u_r = \left\{\prod_{\substack{s=1\\ s\neq r}}^{n}(\alpha_s D + \beta_s D' + \gamma_s)\right\}(\alpha_r D + \beta_r D' + \gamma_r)u_r = 0$$

Similarly we can prove the following theorem.

Theorem 3.5 *: Let $\beta_r D' + \gamma_r, (\beta_r \neq 0)$. be a factor of $F(D,D')$ and $\phi_r(\xi)$ is an arbitrary fimction of the single variable ξ. Then $u_r = exp\left(-\frac{\gamma_r \gamma}{\beta_r}\right)\phi_r(\beta_r x)$ is a solution of the equation $F(D,D')z = 0$.*

Example 3.2 : Solve the equation $(D^2 - DD' - 2D'^2 + 2D + 2D')z = 0$

Solution : The given equation is $(D+D')(D-2D'+2)z=0$. Hence by Theorem 3.4, the solution is $z=\phi_1(x-y)+e^{-2x}\phi_2(2x-y)$

Case of repeated factors :

If $F(D,D')$ had repeated factors of the form $(\alpha_r D+\beta_r D'+\gamma_r)^n$, the solution of the equation $F(D,D')z=0$ corresponding to a factor of this type can be obtained by applying Theorems 3.4 and 3.5. For example, if $n=2$, then for the solution of the equation

$$(\alpha_r D+\beta_r D'+\gamma_r)^2 z=0 \tag{3.9}$$

we put $Z=(\alpha_r D+\beta_r D'+\gamma_r)^2 z$ so that $(\alpha_r D+\beta_r D'+\gamma_r)Z=0$ and by Theorem 3.4, it has the solution

$$Z=exp\left(-\frac{\gamma_r x}{\alpha_r}\right)\phi_r(\beta_r x-\alpha_r y), \text{ provided } \alpha_r\neq 0.$$

To find z, we solve the first-order linear partial differential equation

$$\alpha_r\frac{\partial z}{\partial x}+\beta_r\frac{\partial z}{\partial y}+\gamma_r z=exp\left(-\frac{\gamma_r x}{\alpha_r}\right)\phi_r(\beta_r x-\alpha_r y) \tag{3.10}$$

Using the method of § 2.4 of Unit 2, we find that Lagrange's auxiliary equations are

$$\frac{dx}{\alpha_r}=\frac{dy}{\beta_r}=\frac{dz}{-\gamma_r z+exp\left(-\frac{\gamma_r x}{\alpha_r}\right)\phi_r(\beta_r x-\alpha_r y)}$$

From the first two equations, we get $\beta_r x-\alpha_r y=const.=c_1\,(say)$, and, therefore,

$$\frac{dx}{\alpha_r}=\frac{dz}{-\gamma_r z+exp\left(-\frac{\gamma_r x}{\alpha_r}\right)\phi_r(c_1)}$$

This is a first-order linear ordinary differential equation having the solution.

$$z=\frac{1}{\alpha_r}\{\phi_r(c_1)x+c_2\}exp\left(-\frac{\gamma_r x}{\alpha_r}\right)$$

Hence the solution of the equation (3.10) and, therefore, of (3.9) is given by

$$z=\{x\phi_r(\beta_r x-\alpha_r y)+\psi_r(\beta_r x-\alpha_r y)\}exp\left(-\frac{\gamma_r x}{\alpha_r}\right)$$

where the functions ϕ_r and ψ_r arbitarary.

By the method of induction, this result can readily be generalised to give

Theorem 3.6 : *Let* $(\alpha_r D+\beta_r D'+\gamma_r)^n$, $(\alpha_r \neq 0)$ *be a factor of* $F(D,D')$ *and the functins* $\phi_{r1}, \phi_{r2}, \ldots\ldots\ldots \phi_{rn}$ *are arbitrary. Then a solution of* $F(D,D')z=0$ *is* exp

$$\left(-\frac{\gamma_r x}{\alpha_r}\right)\sum_{s=1}^{n}x^{s-1}\phi_{rs}(\beta_r x-\alpha_r y)$$

Similarly, we have the generalisation of Theorem 3.5 as

Theorem 3.7 : *Let* $(\beta_r D'+\gamma_r)^m, (\beta_r \neq 0)$ *be a factor of* $F(D,D')$ *and the functions* $\phi_{r1}, \phi_{r2}, \ldots\ldots\ldots \phi_{rm}$ *are arvitrary. Then a solution of* $F(D,D')z=0$ *is* exp $\left(-\frac{\gamma_r y}{\beta_r}\right)\sum_{s=1}^{m}x^{s-1}\phi_{rs}(\beta_r x)$.

We are now in a position to state the complementary function of the equation (3.6) if the operator $F(D,D')$ is reducible. Theorems 3.4 and 3.6 show that if $F(D,D')$ is reducible and is of the type

$$F(D,D')=\prod_{r=1}^{n}(\alpha_r D+\beta_r D'+\gamma_r)^{m_r} \tag{3.11}$$

and none of the α_r's is zero, then the corresponding complementary function is

$$u=\sum_{r=1}^{n}exp\left(-\frac{\gamma_r x}{\alpha_r}\right)\sum_{s=1}^{m_r}x^{s-1}\phi_{rs}(\beta_r x-\alpha_r y) \tag{3.12}$$

where functions $\phi_{rs}(s=1,2,\ldots\ldots,n_r;r=1,2,\ldots\ldots,n)$ are arbitrary. If some of the α_r's are zero, the necessary modifications in (3.12) can be made by means of Theorems 3.5 and 3.7. From (3.11), it follows that the order of the euqation. (3.8) is $m_1 + m_2 + ...+ m_n$. Since the solution (3.12) contains the same number of arbitrary functions, this is the complete complementary function.

Example 3.3 : Solve $\left(D^3 - 4D^2D' + 4DD'^2\right)z = 0$

Solution. We have $D(D-2D')^2 z = 0$. Thus the solution is

$$z = \phi_0(-y) + \sum_{s=1}^{2} x^{s-1}\phi_{rs}(-2x - y)$$

$$\text{i.e. } z = f_0(y) + f_1(2x + y) + xf_2(2x + y).$$

(b) Irreducible equations.

For irreducible operator it may not always be possible to find a solution containing full number of arbitrary functions. However, it is possible to find solutions containing as many arbitrary constants as we desire. The method of finding such solutions follows from the following theorem.

Theoremn 3.8. : $F(D,D')e^{ax+by} = F(a,b)e^{ax+by}$

Proff : Since the operator $F(D,D')$ is made up of terms of the type $c_{rs}D^rD'^s$ and

$$D^r\left(e^{ax+by}\right) = a^r e^{ax+by},\ D'^s\left(e^{ax+by}\right)b^s e^{ax+by}$$

we have $\left(c_{rs}D^rD'^s\right)\left(e^{ax+by}\right) = c_{rs}a^rb^se^{ax+by}$.

The theorem follows by recombining the terms of the operator $F(D,D')$.

Note : Although the theorem is true for both reducible and irreducible operators, we make use of it only for irreducible case.

To find the complementary function of the equations (3.8), we first split the operator $F(D,D')$ into factors. The reducible factors are treated by the method (a). We treat the irreducible factors as follows. By Theorem 3.8, we see that e^{ax+by} is a solution of the equation $F(D,D')z = 0$ provided $F(a,b) = 0$. Hence

$$z = \sum c_r \exp(a_r x + b_r y) \tag{3.13}$$

where a_r, b_r, c_r are constants, is also a solution of $F(D, D')z = 0$, provided the constants a_r, b_r are connected by the relation

$$F(a_r, b_r) = 0 \tag{3.14}$$

The series (3.13) may not be finite. If it is infinite, then it should be uniformly convergent for a solution of the equation.

Example 3.4 : Solve $(D - 2D' + 5)(D^2 + D' + 3)z = 0$.

Solution : Here the second factor on the left hand side of the given equation is irreducible. Hence the solution is

$$z = e^{-5x}\phi(y + 2x) + \sum_r c_r e^{a_r x + b_r y}$$

where $a_r^2 + b_r + 3 = 0$.

Rules for finding particular integrals.

The particular integral (P.I.) of the equation

$$F(D, D')z = f(x, y) \tag{3.6}$$

is given by

$$\text{P.I.} = \frac{1}{F(D, D')} f(x, y)$$

I. Let $f(x,y)$ be a polynomial in x and y. *i.e.*, $f(x,y) = \sum a_{rs} x^r y^s$, where r, s are positive integers or zero and a_{rs} are constants so that

$$\text{P.I.} = \frac{1}{D^n f(D'/D)} f(x, y)$$

Here $\{f(D'/D)\}^{-1}$ is expanded by binomial theorem and if the highest power of y in $f(x,y)$ be m, the we retain terms upto $(D'/D)^m$.

Example 3.5 : Solve the equation $\frac{\partial^2 z}{\partial x^2} + 2\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = x^2 + xy + y^2$.

Solution : The given equation is $(D + D')^2 z = x^2 + xy + y^2$.

The complementary function is $z=\sum_{r=1}^{2} x^{r-1}\phi_{r}(x-y)=\phi_1(x-y)+x\phi_2(x-y)$

The particular integral P. I = $\dfrac{1}{(D+D')^2}(x^2+xy+y^2)=\dfrac{1}{D^2}\left(1+\dfrac{D'}{D}\right)^{-2}(x^2+xy+y^2)$

$$=\frac{1}{D^2}\left(1-\frac{2D'}{D}+\frac{3D'^2}{D^2}\ldots\ldots\right)(x^2+xy+y^2)$$

$$=\frac{1}{D^2}\left[(x^2+xy+y^2)-\frac{2}{D}(x+2y)+\frac{3}{D^2}2\right]$$

$$=\left(\frac{x^4}{12}+\frac{x^3y}{6}+\frac{x^2y^2}{2}\right)-2\left(\frac{x^4}{24}+\frac{x^3y}{3}\right)+6\cdot\frac{x^4}{24}$$

$$=\frac{1}{4}(x^4-2x^3y+2x^2y^2).$$

Hence the complete solution of the given equation is

$$z=\phi_1(x-y)+x\phi_2(x-y)+\frac{1}{4}(x^4-2x^3y+2x^2y^2)$$

II. *Let* $f(x,y)=e^{ax+by}$. By Theorem 3.8, we have

$$F(D,D')e^{ax+by}=F(a,b)e^{ax+by}$$

so that, $\dfrac{1}{F(D,D')}e^{ax+by}=F(a,b)e^{ax+by}$, provided $F(a,b)\neq 0$.

Example 3.6. : Solve $(D^2-DD'-2D'^2+2D+2D')z=e^{3x+4y}$

Solution. The given equation can be written as $(D+D')(D-2D'+2)z=e^{3x+4y}$

The complementary function is $z=\phi_1(x-y)+e^{-2x}\phi_2(2x+y)$.

The P.I. $=\dfrac{1}{3^2-3.4-2.4^2+2.3+2.4}e^{3x+4y}=-\dfrac{1}{21}e^{3x+4y}$

Hence the general solution of the given equation is $z=\varphi(x-y)+e^{-2x}\varphi_2(2x+y)-\dfrac{1}{21}e^{3x+4y}$

III. *Let* $f(x,y)=\sin(ax+by)$. If $F(D,D')=\phi(D^2,DD',D'^2)$, then we have $\phi(D^2,DD',D'^2)\sin(ax+by)=\phi(-a^2,-ab,-b^2)\sin(ax+by)$, so that the

$$\text{P.I.}=\frac{1}{F(D,D')}\sin(ax+by)=\frac{1}{\phi(D^2,DD',D'^2)}\sin(ax+by)$$

$$=\frac{1}{\phi(-a^2,-ab,-b^2)}\sin(ax+by),\text{ provided }\phi(-a^2,-ab,-b^2)\neq 0$$

Similar result holds for $f(x,y)=\cos(ax+by)$.

Example 3.7. Solve the equation $\frac{\partial^2 z}{\partial x^2}-\frac{\partial^2 z}{\partial x\partial y}=\sin x\cos 2y$

Solution. The given equation is $D(D-D')z=\frac{1}{2}[\sin(x+2y)-\sin(x-2y)]$

The complementary function is $z=\phi_1(y)+\phi_2(x+y)$

$$\text{The P.I.}=\frac{1}{2}\left[\frac{1}{-1^2-(-1.2)}\sin(x+2y)-\frac{1}{-1^2-(-1.-2)}\sin(x-2y)\right]$$

$$=\frac{1}{2}\sin(x+2y)+\frac{1}{6}\sin(x-2y)$$

Hence the general solution is $z=\phi_1(y)+\phi_2(x+y)+\frac{1}{2}\sin(x+2y)+\frac{1}{6}\sin(x-2y)$

IV. *If* $F(a,b)=0$, *the above methods fail.* In such cases $(bD-aD')$ is a factor of $F(D,D')$. So we can write $F(D,D')=(bD-aD')G(D,D')$, where $G(D,D')\neq 0$.

Consider now $(bD-aD')z=f(ax+by)$. The subsidiary equations are then

$$\frac{dx}{b}=\frac{dy}{-a}=\frac{dz}{f(ax+by)}.$$

The first two relations give $ax+by=const.=c(say)$. Also the relation $\frac{dx}{b}=\frac{dz}{f(ax+by)}$

i.e. the relation $\frac{dx}{b}=\frac{dz}{f(c)}$ gives $z=\frac{x}{b}f(c)=\frac{x}{b}f(ax+by)$.

Hence the P.I. is now given by

$$\frac{1}{F(D,D')}f(ax+by)=\frac{x}{b}\frac{1}{G(D,D')}f(ax+by)=\frac{x}{bG(a,b)}\phi(ax+by)$$

where $\phi(ax+by)$ is obtained after integration of $f(ax+by)$ and it is being supposed that $G(a,b)\neq 0$.

Next consider the relation $F(D,D')=(bD-aD')G(D,D')$

Differentiating with respect to D, we get

$$F'(D,D')=bG(D,D')+(bD-aD')G'(D,D')$$

so that $F'(a,b)=bG(a,b)$. Thus the P.I. can be written as $\frac{x}{F'(a,b)}\phi(ax+by)$.

Generally if $F(a,b)=0$, we write $F(D,D')=\left(D-\frac{a}{b}\right)^r G(D,D')$ where $G(D,D')\neq 0$.

Then

$$\text{P.I.} =\frac{1}{F(D,D')}f(ax+by)=\frac{1}{G(a,b)}\frac{1}{\left(D-\frac{a}{b}D'\right)^r}f(ax+by)$$

Example 3.8. : Solve the equation $\left(D^2-5DD'+4D'^2\right)z=\sin(4x+y)$

Solution. The complementary function is $z=\phi_1(x+y)+\phi_2(4x+y)$

Since $F'(D,D')=2D-5D'$, we have $F'(4,1)=-3$. Hence, the

$$\text{P.I.} =x\frac{1}{F'(4,1)}\cos(4x+y)=-\frac{x}{3}\cos(4x+y)$$

Thus the general solution is $z=\phi_1(x+y)+\phi_2(4x+y)-\frac{x}{3}\cos(4x+y)$

Example 3.9 : Solve $r-2as+a^2t=e^{ax+y}$

Solution. The given equation is $\left(D^2-2aDD'+a^2D'^2\right)z=e^{ax+y}$

The complementary function is $z=\phi_1(ax+y)+x\phi_2(ax+y)$ and since $F(D,D')=(D-a)^2$, we have the P.I. $= \frac{x^2}{2!} e^{ax+y}$.

Hence the general solution is $z = \phi_1(ax+y) + x\phi_2(ax+y) + \frac{x^2}{2} e^{ax+y}$

V. *Let* $f(x,y) = e^{ax+by} V(x,y)$. Here it is easy to see that

$$\frac{1}{F(D,D')}\left\{e^{ax+by}V(x,y)\right\} = e^{ax+by}\frac{1}{F(D+a,D'+b)}V(x,y).$$

Example 3.10 : Solve $D(D-2D')(D+D')z = e^{x+2y}(x^2+4y^2)$

Solution. The complementary function is $z = \varphi_1(y) + \phi_2(2x+y) + \varphi_3(x-y)$.

The P.I. $= \frac{1}{D(D-2D')(D+D')} e^{x+2y}(x^2+4y^2)$

$$= e^{x+2y}\frac{1}{(D+1)(D+1-2D'-4)(D+1+D'+2)}(x^2+4y^2)$$

$$= -\frac{1}{3}e^{x+2y}\frac{1}{(1+D)(3-D+2D')}\left(1+\frac{D+D'}{3}\right)^{-1}(x^2+4y^2)$$

$$= -\frac{1}{3}e^{x+2y}\frac{1}{(1+D)(3-D+2D')}\left(1+\frac{D+D'}{3}+\frac{D^2+2DD'+D'^2}{9}-\ldots\right)(x^2+4y^2)$$

$$= -\frac{1}{9}e^{x+2y}\frac{1}{(1+D)}\left(1-\frac{D-2D'}{3}\right)^{-1}\left(x^2+4y^2-\frac{2x}{3}-\frac{8y}{3}+\frac{10}{9}\right)$$

$$= -\frac{1}{9}e^{x+2y}\frac{1}{(1+D)}\left(1+\frac{D-2D'}{3}+\frac{D^2-4DD'+4D'^2}{9}+\ldots\right)\left(x^2+4y^2-\frac{2x}{3}-\frac{8y}{3}+\frac{10}{9}\right)$$

$$= -\frac{1}{9}e^{x+2y}(1-D+D^2\ldots)\left(x^2+4y^2-8y+\frac{58}{9}\right)$$

$$= -\frac{1}{9}e^{x+2y}\left(x^2+4y^2-8y+\frac{58}{9}-2x+2\right)$$

Thus the general solution of the given equation is

$$z=\phi_1(y)+\phi_2(2x+y)+\phi_3(x-y)-\frac{1}{81}e^{x+2y}\left(9x^2+36y^2-18x-72y+76\right)$$

Example 3.11 : Solve $\left(D^2+D'\right)\left(D-D'-D'^2\right)z=e^{2x+3y}\cos(x+2y)$

Solution : Since each of D^2+D' and $D-D'-D'^2$ is irreducible, so the the complementary function is

$$z=\sum_{r=1}^{\infty}c_r e^{a_r x+b_r y}+\sum_{s=1}^{\infty}c_s e^{A_s x+B_s y}$$

where $a_r^2+b_r=0$ and $A_s-B_s-B_s^2=0$

$$\text{The P.I.}=\frac{1}{\left(D^2+D'\right)\left(D-D'-D'^2\right)}e^{2x+3y}\cos(x+2y)$$

$$=e^{2x+3y}\frac{1}{\left[(D+2)^2+D'+3\right]\left[D+2-(D'+3)-(D'+3)^2\right]}\cos(x+2y)$$

$$=e^{2x+3y}\frac{1}{\left(D^2+4D+D'+7\right)\left(D-7D'-D'^2-10\right)}\cos(x+2y)$$

$$=e^{2x+3y}\frac{1}{\left(-1^2+4D+D'+7\right)\left(D-7D'+2^2-10\right)}\cos(x+2y)$$

$$=e^{2x+3y}\frac{1}{4D^2-27DD'-7D'^2-18D-48D'-36}\cos(x+2y)$$

$$=e^{2x+3y}\frac{1}{4\left(-1^2\right)-27(-1.2)-7\left(-2^2\right)-18D-48D'-36}\cos(x+2y)$$

$$=-\frac{1}{6}e^{2x+3y}\frac{1}{3D+8D'-7}\cos(x+2y).$$

$$=-\frac{1}{6}e^{2x+3y}\frac{3D+8D'+7}{9D^2+48DD'+64D'^2-49}\cos(x+2y)$$

$$= -\frac{1}{6}e^{2x+3y}\frac{3D+8D'+7}{9(-1^2)+48(-1.2)+64(-2^2)-49}\cos(x+2y)$$

$$= \frac{1}{2460}e^{2x+3y}[-3\sin(x+2y)-16\sin(x+2y)+7\cos(x+2y)]$$

$$= \frac{1}{2460}e^{2x+3y}[7\cos(x+2y)-19\sin(x+2y)]$$

Hence the general solution of the give equation is

$$z = \sum_{r=1}^{\infty} c_r e^{a_r x + b_r y} + \sum_{s=1}^{\infty} c_s e^{A_s x + B_s y} + \frac{1}{2460}e^{2x+3y}[7\cos(x+2y)-19\sin(x+2y)]$$

Example 3.12 : Solve $(D^2 + DD' - 6D'^2)z = x^2\sin(x+y)$.

Solution. The given equation is $(D-2D')(D+3D')z = x^2\sin(x+y)$. The complementary function is $z = \phi_1(2x+y) + \phi_2(3x-y)$.

To find the P.I. we note that $\sin(x+y) = \text{Im}\, e^{i(x+y)}$

$$\text{P.I.} = \text{Im}\left[\frac{1}{(D-2D')(D+3D')}x^2 e^{i(x+y)}\right]$$

$$= \text{Im}\left[e^{i(x+y)}\frac{1}{(D+i-2D'-2i)(D+i+3D'+3i)}x^2\right]$$

$$= \text{Im}\left[\frac{1}{4i}e^{i(x+y)}\frac{1}{D-2D'-i}\left(1+\frac{D+3D'}{4i}\right)^{-1}x^2\right]$$

$$= \text{Im}\left[\frac{1}{4i}e^{i(x+y)}\frac{1}{D-2D'-i}\left(1-\frac{D+3D'}{4i}-\frac{D^2+6DD'+D'^2}{16}+\dots\right)x^2\right]$$

$$= \text{Im}\left[\frac{1}{4i}e^{i(x+y)}\frac{1}{D-2D'-i}\left(x^2-\frac{2x}{4i}-\frac{2}{16}\right)\right]$$

$$= \text{Im}\left[\frac{1}{4i}e^{i(x+y)}\{1+i(D-2D')\}^{-1}\left(x^2+\frac{ix}{2}-\frac{1}{8}\right)\right]$$

$$= \text{Im}\left[\frac{1}{4i}e^{i(x+y)}\left\{1-i(D-2D')-\left(D^2-4DD'+4D'^2\right)+...\right\}\left(x^2+\frac{1}{2}ix-\frac{1}{8}\right)\right]$$

$$= \text{Im}\left[\frac{1}{4}e^{i(x+y)}\left\{x^2+\frac{1}{2}ix-\frac{1}{8}-i\left(2x+\frac{i}{2}\right)-2\right\}\right]$$

$$= \frac{1}{4}\text{Im}\left[\cos(x+y)+i\sin(x+y)\right]\left[x^2-\frac{13}{8}-i\frac{3}{2}x\right]$$

$$= \frac{1}{4}\left[\left(x^2-\frac{13}{8}\right)\sin(x+y)-\frac{2}{3}x\cos(x+y)\right]$$

Hence the general solution of the given equation is

$$z=\phi_1(2x+y)+\phi_2(3x-y)+\frac{1}{4}\left(x^2-\frac{13}{8}\right)\sin(x+y)-\frac{3}{8}x\cos(x+y)$$

VI. *General method of finding particular integral.*

The method is applicable to all cases where $f(x,y)$ is not of the form given earlier or when the above methods fail.

$$\text{The P.I.} = \frac{1}{F(D,D')}f(x,y).$$

To evaluate this, we consider the equation $(D-mD')z=f(x,y)$. This can be written as $p-mq=f(x,y)$ so that Lagrange's subsidiary equations are

$$\frac{dx}{1}=\frac{dy}{-m}=\frac{dz}{f(x,y)}$$

From the first two relations we get $y+mx=c$, a constant. Also the equations

$$\frac{dz}{f(x,y)}=\frac{dx}{1}$$

give $z=\int f(x,y)dx=\int f(x,c-mx)dx$.

Thus $z=\dfrac{1}{D-mD'}f(x,y)=\int f(x,c-mx)dx,$

in which the constant c is to be replaced by $y+mx$ after integration.

Now, if the given equation is of the form $F(D,D')z = f(x,y)$, where $F(D,D') = (D - m_1D')$ $(D - m_2D')...(D - m_nD')$, then the

$$\text{P.I.} = \frac{1}{(D - m_1D')(D - m_2D')...(D - m_nD')} f(x,y)$$

This can be evaluated by the repeated application of the above method.

Example 3.13 : Solve $r - t = \tan^3 x \tan y - \tan x \tan^3 y$

Solution. The given equation is

$$\left(D^2 - D'^2\right)z = \tan^3 x \tan y - \tan x \tan^3 y$$

i.e. $(D + D')(D - D')z = \tan^3 x \tan y - \tan x \tan^3 y$

The complementary function is $z = \phi_1(x + y) + \phi_2(x - y)$

$$\text{The P.I.} = \frac{1}{D + D'} \cdot \frac{1}{D - D'} \tan x \tan y\left(\tan^2 x - \tan^2 y\right)$$

$$= \frac{1}{D + D'} \int \tan x \tan y\left(\sec^2 x - \sec^2 y\right) dx$$

$$= \frac{1}{D + D'} \int \left[\tan x \sec^2 x \tan(c - x) - \tan x \tan(c - x) \sec^2(c - x)\right] dx$$

$$\left[\because \text{ corresponding to } (D - D')z = 0,\ y + x = c\right]$$

$$= \frac{1}{D + D'}\left[\frac{1}{2}\tan^2 x \tan(c - x) + \frac{1}{2}\int \tan^2 x \sec^2(c - x)\right.$$

$$\left. + \frac{1}{2}\tan x \tan^2(c - x) - \frac{1}{2}\int \tan^2(c - x)\sec^2(c - x) dx\right]$$

$$= \frac{1}{2(D + D')}\left[\tan^2 x \tan(c - x) + \tan x \tan^2(c - x) + \int\left\{\sec^2 x - \sec^2(c - x)\right\} dx\right]$$

$$= \frac{1}{2(D + D')}\left[\tan^2 x \tan y + \tan x \tan^2 y + \tan x + \tan y\right]$$

[replacing c by $x + y$]

$$= \frac{1}{2(D+D')}\left[\tan y \sec^2 x + \tan x \sec^2 y\right]$$

$$= \frac{1}{2}\int\left[\tan(c'+x)\sec^2 x + \tan x \sec^2(c'+x)\right]dx$$

[$\because$ corresponding to $(D+D')z = 0,\ y - x = c'$]

$$= \frac{1}{2}\tan x \tan(c'+x)$$

$$= \frac{1}{2}\tan x \tan y.$$

Hence the general solution of the given equation is $z = \phi_1(x+y) + \phi_2(x-y) + \frac{1}{2}\tan x \tan y$.

§ 3.4. Homogeneous Equations with Variable Coefficients : Cauchy-Euler Equations.

A partial differential equation with variable coefficients can sometimes be reduced to an equation with constant coefficients by suitable substitutions. One such form is a homeogeneous equation of the form $F(xD, yD') = f(x,y)$. To find the solution of such an equation we put $x = e^u, y = e^v$, i.e. $u = \log x, v = \log y$.

Let $D \equiv \frac{\partial}{\partial x},\ D' \equiv \frac{\partial}{\partial y},\ \vartheta \equiv \frac{\partial}{\partial u},\ \vartheta' \equiv \frac{\partial}{\partial v}$

Now, $Dz = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = \frac{1}{x}\frac{\partial z}{\partial u}$, i.e. $xD \equiv \vartheta$,

$$D^2 z = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{1}{x}\frac{\partial z}{\partial u}\right) = -\frac{1}{x^2}\frac{\partial z}{\partial u} + \frac{1}{x^2}\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right)$$

$$= -\frac{1}{x^2}\frac{\partial z}{\partial u} + \frac{1}{x^2}\frac{\partial^2 z}{\partial u}, \text{ i.e. } (xD)^2 \equiv \vartheta(\vartheta - 1)$$

and, in general, $(xD)^m = \vartheta(\vartheta-1)(\vartheta-2)......(\vartheta-m+1)$

Similarly, $(yD')^n = \vartheta'(\vartheta'-1)(\vartheta'-2)......(\vartheta'-n+1)$

These substitutions reduce the given equation into an equation with constant coefficients and the equation can be solved by the methods discussed in Section 3.3.

Example. 3.14. Solve the equation $\left(x^2D^2 - 4y^2D'^2 - 4yD' - 1\right)z = x^2y^3\log y$.

Solution. Let $x = e^u, y = e^v, \vartheta \equiv \frac{\partial}{\partial u}, \vartheta' \equiv \frac{\partial}{\partial v}$. Then the given equation reduces to

$$\left[\vartheta(\vartheta-1) - 4\vartheta'(\vartheta'-1) - 4\vartheta - 1\right]z = ve^{2u+3v}$$

i.e. $\left(\vartheta^2 - 4\vartheta'^2 - \vartheta - 1\right)z = \vartheta e^{2u+3v}$

The complementary function is $z = \sum_{r=1}^{\infty} c_r e^{a_r u + b_r v}$ where $a_r^2 - 4b_r^2 - a_r - 1 = 0$

$$\text{The P.I.} = \frac{1}{\vartheta^2 - 4\vartheta'^2 - \vartheta - 1}\vartheta e^{2u-3v}$$

$$= e^{2u-3v}\frac{1}{(\vartheta+2)^2 - 4(\vartheta'+3)^2 - (\vartheta+2) - 1}v$$

$$= e^{2u-3v}\frac{1}{\vartheta^2 - 4\vartheta'^2 + 3\vartheta - 24\vartheta' - 35}v$$

$$= -\frac{1}{35}e^{2u+3v}\left[1 - \left\{\frac{1}{35}\left(3\vartheta - 24\vartheta' + \vartheta^2 - 4\vartheta'^2\right)\right\}\right]^{-1}v$$

$$= -\frac{1}{35}e^{2u+3v}\left(1 - \frac{24}{35}\vartheta'\right)v$$

$$= -\frac{1}{35}e^{2u+3v}\left(v - \frac{24}{35}\right)$$

$$= -\frac{1}{1225}e^{24+3v}(35v - 24)$$

Returning to the orginal variables x and y, the general solution of the given equation is

$$z = \sum_{r=1}^{\infty} c_r x^{a_r} y^{b_r} - \frac{1}{1225}x^2y^3(25\log y - 24)$$

§ 3.5. Classification of Second-order Partial Differential Equations

Let us consider second-order linear partial differential equations of the type

$$Rr + Ss + Tt + f(x,y,z,p,q) = 0 \qquad (3,15)$$

$$\text{or, } Lz + f(x,y,z,p,q) = 0 \tag{3.16}$$

where $L = R\frac{\partial^2}{\partial x^2} + S\frac{\partial^2}{\partial x \partial y} + T\frac{\partial^2}{\partial y^2}$ and R, S, T are continuous functions of x and y possessing continuous partial derivatives of all orders with respect ot x, y. A second order partial differential equation which is linear with respect to the second order derivatives, i.e. r, s and t is said to be a *quasi-linear partial differential equation of second order*.

The equation (3.15) is said to be *hyperbolic, parabolic or elliptic* according as $S^2 - 4RT > 0$, $S^2 - 4Rt = 0$ or $S^2 - 4Rt < 0$ at a point (x_0, y_0). If this is true for alll points in a domain Ω, then (3.15) is said to be hyperbolic, parabolic or elliptic in that domain.

It the number of independent variables is two or three, then we can always find a transformation which reduces the given partial differential equation to a caronical (or normal) form and the transformed equation takes a simple form which is easily solvable. However, if the number of independent variables is more than three, the it is not always possible to find such a transformation.

Canonical (or normal) form

In order to reduce the equation (3.15) to a cononical form, we apply the transformation

$$\xi = \xi(x,y), \eta = \eta(x,y) \tag{3.17}$$

where the functions ξ and η are assumed to be continuously differentiable and the Jacobian

$$J = \frac{\partial(\xi,\eta)}{\partial(x,y)} = \begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix} = \xi_x\eta_y - \xi_y\eta_x \neq 0 \tag{3.18}$$

in the domain where the equation (3.15) holds.

Now, we have

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial z}{\partial \eta}\frac{\partial \eta}{\partial x} = z_\xi \xi_x + z_\eta \eta_x \text{ and similarly } q = z_\xi \xi_y + z_\eta \eta_y$$

Also

$$r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(z_\xi \xi_x + z_\eta \eta_x\right) = z_{\xi\xi}\xi_x^2 + 2z_{\xi\eta}\xi_x\eta_x + z_{\eta\eta}\eta_x^2 + z_\xi \xi_{xx} + z_\eta \eta_{xx}$$

$$\text{Similarly, } s = \frac{\partial^2 z}{\partial x \partial y} = z_{\xi\xi}\xi_x\xi_y + z_{\xi\eta}\left(\xi_x\eta_y + \xi_y\eta_x\right) + z_{\eta\eta}\eta_x\eta_y + z_\xi \xi_{xy} + z_\eta \eta_{xy}$$

$$t = \frac{\partial^2 z}{\partial y^2} = z_{\xi\xi}\xi_y^2 + 2z_{\xi\eta}\xi_y\eta_y + z_{\eta\eta}\eta_y^2 + z_\xi \xi_{yy} + z_\eta \eta_{yy}.$$

Substituting the above values of p, q, r, s and t in (3.15), the equation reduces to the form

$$A(\xi_x,\xi_y)\tilde{z}_{\xi\xi} + 2B(\xi_x,\xi_y,\eta_x,\eta_y)\tilde{z}_{\xi\eta} + A(\eta_x,\eta_y)\tilde{z}_{\eta\eta} = F(\xi,\eta,\tilde{z},\tilde{z}_\xi,\tilde{z}_\eta) \tag{3.19}$$

where $A(u,v) = Ru^2 + Suv + Tv^2$

$$B(u_1,v_1,u_2,v_2) = Ru_1u_2 + \frac{1}{2}S(u_1v_2 + u_2v_1) + Tu_1v_2$$

$$F(\xi,\eta,z,\tilde{z}_\xi,\tilde{z}_\eta)$$

$$= -\left[\tilde{z}_\xi\left(R\xi_{xx} + S\xi_{xy} + T\xi_{yy}\right) + \tilde{z}_\eta\left(R\eta_{xx} + S\eta_{xy} + T\eta_{yy}\right) + f\left(\xi,\eta,z,\tilde{z}_\xi,\tilde{z}_\eta\right)\right] \quad (3.20)$$

We can easily verify that, in general

$$B^2(\xi_x,\xi_y,\eta_x,\eta_y) - A(\xi_x,\xi_y)A(\eta_x,\eta_y) = (S^2 - 4RT)J^2$$

where J is given by (3.18).

Here the following *three* cases arise :

Case I : $S^2 - 4RT > 0$

In this case the equation $R\lambda^2 + S\lambda + T = 0$ has two real and distinct roots λ_1 and λ_2 (say). We choose ξ and η such that

$$\xi_x = \lambda_1\xi_y, \eta_x = \lambda_2\eta_y$$

Now the equation $\xi_x - \lambda_1\xi_y = 0$ gives $\dfrac{dx}{1} = \dfrac{dy}{-\lambda_1} = \dfrac{d\xi}{0}$

so that $d\xi = 0$, i.e. $\xi(x,y) =$constant and $\dfrac{dy}{dx} + \lambda_1(x,y) = 0$. (3.22)

Similarly, we have $\eta(x,y) =$constant and $\dfrac{dy}{dx} + \lambda_2(x,y) = 0$ (3.23)

Let the solutions of the equations (3.22) and (3.23) be given by

$$\xi = f_1(x,y) \text{ and } \eta = f_2(x,y) \quad (3.24)$$

Now, $A(\xi_x,\xi_y) = R\xi_x^2 + S\xi_x\xi_y + T\xi_y^2 = \xi_x^2(R\lambda_1^2 + S\lambda_1 + T) = 0$ and similarly $A(\eta_x,\eta_y) = 0$. Hence the equation (3.21) gives

$$B^2 = (S^2 - 4RT)J^2 > 0 \quad (3.25)$$

Thus the equation (3.19) reduces to an equation of the form

$$\tilde{z}_{\xi\eta} = \phi(\xi,\eta,\tilde{z},\tilde{z}_\xi,\tilde{z}_\eta) \quad (3.26)$$

which is the required *connonical form for the hyperbolic partial differential equation.*

Case II : $S^2 - 4RT = 0$

Here the equation $R\lambda^2 + S\lambda + T = 0$ has equal roots λ,λ (say).

We choose $\xi = f_1(x,y)$ where $f_1(x,y)$ =constant is a solution of the equation

$$\frac{dy}{dx} + \lambda(x,y) = 0.$$

Since $A(\xi_x, \xi_y) = 0$, $S^2 - 4RT = 0$ and, therefore, from (3.21) we have B = 0. However, $A(\eta_x, \eta_y) \neq 0$, otherwise η will depend on ξ. Then the equation (3.19) reduces to the form

$$\bar{z}_{\eta\eta} = \psi(\xi, \eta, \bar{z}, \bar{z}_\xi, \bar{z}_\eta) \tag{3.27}$$

This is the required *cononical form of parabolic partial differential equation*.

Case III : $S^2 - 4RT < 0$

The roots λ_1, λ_2 in this case of the equation $R\lambda^2 + S\lambda + T = 0$ are complex and therefore, ξ and η are comple conjugate we put $\xi = \alpha + i\beta$ and $\eta = \alpha - i\beta$ so that $\alpha = \frac{1}{2}(\xi + \eta)$ and $\beta = \frac{i}{2}(\eta - \xi)$.

Then

$$\frac{\partial \bar{z}}{\partial \xi} = \frac{1}{2}\left(\frac{\partial \bar{z}}{\partial \alpha} - i\frac{\partial \bar{z}}{\partial \beta}\right), \frac{\partial \bar{z}}{\partial \eta} = \frac{1}{2}\left(\frac{\partial \bar{z}}{\partial \alpha} + i\frac{\partial \bar{z}}{\partial \beta}\right), \frac{\partial^2 \bar{z}}{\partial \xi \partial \eta} = \frac{1}{4}\left(\frac{\partial^2 \bar{z}}{\partial \alpha^2} + \frac{\partial^2 \bar{z}}{\partial \beta^2}\right)$$

Now $A(\xi_x, \xi_y) = \xi_x^2 (R\lambda_1^2 + S\lambda_1 + T) = 0$. Similarly, $A(\eta_x, \eta_y) = 0$.

Hence the equation (3.21) gives $B^2 < 0$ and the equation (3.19) reduces to

$$\frac{\partial^2 \bar{z}}{\partial \alpha^2} + \frac{\partial^2 \bar{z}}{\partial \beta^2} = \kappa(\alpha, \beta, \bar{z}, \bar{z}_\alpha, \bar{z}_\beta) \tag{3.28}$$

Equation (3.28) is the required *cononical form of elliptic partial differential equation*.

Example 3.15 : Reduce the equation

$$z_{xx} - 2\sin x z_{xy} - \cos^2 x z_{yy} - \cos x z_y = 0$$

into its cononical form and hence solve this.

Solution. Here $R = 1$, $S = -2\sin x$, $T = -\cos^2 x$ so that $s^2 - 4RT = 4 > 0$.

Hence the given equation is of hyperbolic type.

Let $\xi = \xi(x,y)$, $\eta = \eta(x,y)$. Transforming the given equation in the new variables, we have

$$A(\xi_x, \xi_y)\bar{z}_{\xi\xi} + 2B(\xi_x, \xi_y, \eta_x, \eta_y)\bar{z}_{\xi\eta} + A(\eta_x, \eta_y)\bar{z}_{\eta\eta} = F(\xi, \eta, \bar{z}, \bar{z}_\xi, \bar{z}_\eta)$$

where A, B and F are obtained from (3.20) as

$$A(u,v) = u^2 - 2\sin x . uv - \cos^2 x v^2$$

$$B(\xi_x,\xi_y,\eta_x,\eta_y)=\xi_x\eta_x-\sin x(\xi_x\eta_y+\xi_y\eta_x)-\cos^2 x\xi_y\eta_y$$

$$F(\xi,\eta,\tilde{z},\tilde{z}_\xi,\tilde{z}_\eta)=-\left[\tilde{z}_\xi(\xi_{xx}-2\sin x.\xi_{xy}-\cos^2 x\xi_{yy})+\tilde{z}_\eta(\eta_{xx}-2\sin x\eta_{xy}-\cos^2 x\eta_{xy})\right]$$

Now consider the equation $R\lambda^2+S\lambda+T=0$ i.e. $\lambda^2-2\sin x\lambda-\cos^2 x=0$ whose roots are $\lambda=\sin x\pm 1$. Let $\lambda_1=\sin x+1$, $\lambda_2\sin x-1$. We determine ξ and η as the solutions of the equations $\frac{dy}{dx}+\sin x+1=0$ and $\frac{dy}{dx}+\sin x-1=0$ given by $\xi=y-\cos x+x$ and $\eta=y-\cos x-x$. Then

$$\xi_x=\sin x+1,\ \xi_y=1,\ \xi_{xx}=\cos x,\ \xi_{xy}=0,\ \xi_{yy}=0$$

$$\eta_x=\sin x-1,\ \eta_y=1,\ \eta_{xx}=\cos x,\ \eta_{xy}=0,\ \eta_{yy}=0$$

It follows that $A(\xi_x,\xi_y)=\xi_x^2-2\sin x.\xi_x\xi_y-\cos^2 x.\xi_y^2$

$$=(\sin x+1)^2-2\sin x(\sin x+1)-\cos^2 x.1$$
$$=0$$

$$A(\eta_x,\eta_y)=\eta_x^2-2\sin x\eta_x\eta_y-\cos^2 x.\eta_y^2$$
$$=(\sin x-1)^2-2\sin x(\sin x-1)-\cos^2 x.1$$
$$=0$$

$$B(\xi_x,\xi_y,\eta_x,\eta_y)=\xi_x\eta_x-\sin x(\xi_x\eta_y+\xi_y\eta_x)-\cos^2 x\xi_y\eta_y$$
$$=(\sin x+1)(\sin x-1)-\sin x(\sin x+1+\sin x-1)-\cos^2 x.1$$
$$=\sin^2 x-1-2\sin^2 x-\cos^2 x$$
$$=-2\neq 0$$

and $F(\xi,\eta,\tilde{z},\tilde{z}_\xi,\tilde{z}_\eta)=-\left[\tilde{z}_\xi(1.\cos x-2\sin x.0-\cos^2 x.0-\cos x.1)\right.$

$$\left.+\tilde{z}_\eta(1.\cos x-2\sin x.0-\cos^2 x.0-\cos x.1)\right]$$
$$=0.$$

Thus the cononical form of the given equation is

$$\tilde{z}_{\xi\eta}=-\frac{1}{4}F(\xi,\eta,\tilde{z},\tilde{z}_\xi,\tilde{z}_\eta)=0$$

Intergating this equation with respect to ξ, we obtain $\tilde{z}_\eta=f(\eta)$, where f is arbitrary. Integrating again with respect to η, we have

$$\bar{z}=\int f(\eta)d\eta+f_1(\xi)=f_2(\eta)+f_1(\xi)$$

i.e. $$z=f_1(y-\cos x)+f_2(y-\cos x-x)$$

which is the required solution.

Example 3.16 : Reduce the equation

$$y^2\frac{\partial^2 z}{\partial x^2}-2xy\frac{\partial^2 z}{\partial x\partial y}+x^2\frac{\partial^2 z}{\partial y^2}=\frac{y^2\partial z}{x\,\partial x}+\frac{x^2}{y}\frac{\partial z}{\partial y}$$

to its canonical form and hence slove it

Solution. Here. $R=y^2, S=-2xy, T=x^2$ so that $S^2-4RT=0$. Hence the given equation is parabolic type.

We now consider the equation $R\lambda^2-S\lambda+T=0$ i.e. the equation $y^2\lambda^2-2xy\lambda+x^2=0\Rightarrow\lambda=\frac{x}{y},\frac{x}{y}$, Therefore, we have the equation $\frac{dy}{dx}+\frac{x}{y}=0\Rightarrow x^2-y^2$ const.

We choose $\xi(x,y)=x^2-y^2$. The function can be chosen aribtrarily provided that the Jacobian of the transformation is not zero. Thus we take. $\xi=x^2+y^2, \eta=x^2-y^2$. Then

$$\xi_x=2x,\ \xi_y=2y,\ \xi_{xx}=2,\ \xi_{xy}=0,\ \xi_{yy}=2$$

$$\eta_x=2x,\ \eta_y=2y,\ \eta_{xx}=2,\ \eta_{xy}=0,\ \eta_{yy}=2$$

Now, $$A(\xi_x,\xi_y)=R\xi_x^2+S\xi_x\xi_y+T\xi_y^2=4y^2x^2-2xy.2x.2y+x^2 4y^2=0$$

$$A(\eta_x,\eta_y)=R\eta_x^2+S\eta_x\eta_y+T\eta_y^2=4y^2x^2-2xy(-4xy)+x^2.4y^2=16x^2y^2$$

$$B(\xi_x,\xi_y,\eta_x,\eta_y)=R\xi_x\eta_x+\frac{1}{2}S(\xi_x\eta_y+\xi_x\eta_y)+T\xi_y\eta_y$$

$$=y^2.4x^2-xy(-4xy+4xy)+x^2(-4y^2)$$

$$=0$$

$$F(\xi,\eta,\bar{z},\bar{z}_\xi,z_\eta)=-\left[z_\xi\left(R\xi_{xx}+S\xi_{xy}+T\xi_{yy}\right)+z_\eta(R\eta_{xx}+S\eta_{yy}+T\eta_{yy})\right]$$

$$+f(\xi,\eta,\bar{z},\bar{z}_\xi,\bar{z}_\eta)]$$

$$=-\left[\bar{z}_\xi(2y^2-2xy.0+x^2.2)+\bar{z}_\eta(2y^2-2xy.0+x^2.2)\right]$$

$$-\frac{y^2}{x}(z_\xi.2x+z_\eta.2x)-\frac{x^2}{y}(z_\xi.2y-z_\eta.2y)]$$

$$=-\left[2\bar{z}_\xi(y^2+x^2)+2\bar{z}_\eta(y^2-x^2)-2y^2(z_\xi+z_\xi)-2x^2(\bar{z}_\xi-\bar{z}_\eta)\right]$$

$$=0$$

Thus the required canonical form of the given equation is $\bar{z}_{\eta\eta}=0$.

Integrating this twice with respect to η, we get $\bar{z}=\eta f_1(\xi)+f_2(\xi)$, where $f_1(\xi)$ and $f_2(\xi)$ are arbitrary functions of x. Thus the solution of the given equation is

$$z=(x^2-y^2)f_1(x^2+y^2)+f_2(x^2-y^2)$$

Example 3.17. : Reduce the following equation to the canonical form :

$$(1+x^2)z_{xx}+(1+y^2)z_{yy}+x\,z_x+yz_y=0$$

Solution. for the given equation, $R=1+x^2$, $S=0$, $T=1+y^2$ so that $S^2-4RT=-4(1+x^2)(1+y)<0$. Hence the given equation is elliptic type.

Now consider the equation $(1+x^2)\lambda^2+(1+y^2)=0$. The roots of this equation are $\lambda=\pm\sqrt{\frac{1+y^2}{1+x^2}}$. The characteristic equations are

$$\frac{dy}{dx}-i\sqrt{\frac{1+y^2}{1+x^2}}=0,\ \frac{dy}{dx}+i\sqrt{\frac{1+y^2}{1+x^2}}=0$$

whose solutions are in $\left(x+\sqrt{1+y^2}\right)-i\ln\left(y+\sqrt{1+y^2}\right)=c_1$ and

$\ln\left(x+\sqrt{1+x^2}\right)+i\ln\left(y+\sqrt{1+y^2}\right)=c_2$. We choose

$$\xi(x,y)=\ln\left(x+\sqrt{1+x^2}\right)-i\ln\left(y+\sqrt{1+y^2}\right)$$

$$\eta(x,y)=\ln\left(x+\sqrt{1+x^2}\right)+i\ln\left(y+\sqrt{1+y^2}\right)$$

Introducing the second transformation $\alpha=\frac{1}{2}(\xi+\eta),\beta=\frac{i}{2}(\eta-\xi)$,

we get $\alpha=\ln\left(x+\sqrt{1+x^2}\right)$, $\beta=-\ln\left(y+\sqrt{1+y^2}\right)$

Proceeding exactly along the same lines as in the Example 3.15 and 3.16, we obtain $A(\xi_x,\xi_y)=A(\eta_x,\eta_y)=0, B(\xi_x,\xi_y,\eta_x,\eta_y)=2$ and $F(\alpha,\beta,\bar{z},\bar{z}_\alpha,\bar{z}_\beta)=0$.

Hence the required canonical form of the given equation is $\bar{z}_{\alpha\alpha}+\bar{z}_{\beta\beta}=0$

§ 3.6 Riemann's Method of Solution of Linear Hyperbolic Equations.

Let us consider the linear partial differential equation of the hyperbolic type

$$\frac{\partial^2 z}{\partial x\partial y}+a\frac{\partial z}{\partial x}+b\frac{\partial z}{\partial y}+cz=f(x,y)$$

$$\text{i.e. } L(z)=f(x,y) \tag{3.29}$$

Where L denotes the linear differential operator $\frac{\partial^2}{\partial x\partial y}+a\frac{\partial}{dx}+b\frac{\partial}{\partial y}+c$ and a, b, c are functions of x, y having continuous first order partial derivatives with respect to x and y. Let w be another function of x and y having continouous first order partial derivatives. Then we have

$$w\frac{\partial^2 z}{\partial x\partial y}-z\frac{\partial^2 w}{\partial x\partial y}=\frac{\partial}{\partial y}\left(w\frac{\partial z}{\partial x}\right)-\frac{\partial}{\partial x}\left(z\frac{\partial w}{\partial y}\right)$$

$$wa\frac{\partial z}{\partial x}+z\frac{\partial}{\partial x}(aw)=\frac{\partial}{\partial x}(awz)$$

$$wb\frac{\partial z}{\partial y}+z\frac{\partial}{\partial y}(bw)=\frac{\partial}{\partial y}(bwz)$$

It is evident from these relations that

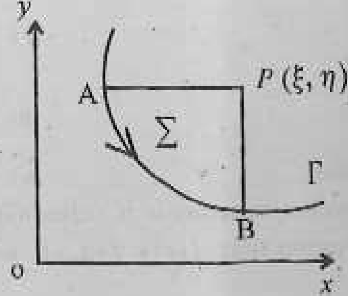
$$wL(z)-zL^*(w)=\frac{\partial U}{\partial x}+\frac{\partial V}{\partial y} \tag{3.30}$$

where the operator L* is defined in such a way that

$$L^*(w)=\frac{\partial^2 w}{\partial x\partial y}-\frac{\partial}{\partial x}(aw)-\frac{\partial}{\partial y}(bw)+cw \tag{3.31}$$

and $$U=awz-z\frac{\partial w}{\partial y},\ V=bwz+w\frac{\partial z}{\partial x} \tag{3.32}$$

The operator L* defined by (3.31) is called the adjoint operator of L and the equation (3.30) is known is *Lagranve identity.* If L*=L, then the operator L is called *self-adjoint operator.*



Now consider an arc AB of a curve Γ and $P(\xi,\eta)$ be any point on the xy-plane. Let PA and PB be parallel to the x and y axes respectively and $\sum$ be the area enclosed by the contour $ABPA$. Then by Green's theorem

$$\iint_{\Sigma}[wL(z)-zl^*(w)]dx\,dy$$

$$= \iint_{\Sigma} \left(\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} \right) dxdy$$

$= \int_{c} (U\,dy - Vdx)$, where c is the closed contour $APBA$

$$\int_{A}^{B} (Udy - Vdx) + \int_{B}^{P} Udy - \int_{P}^{A} Vdx \tag{3.33}$$

Now $$\int_{P}^{A} Vdx = \int_{P}^{A} \left(bwx + w\frac{\partial z}{\partial x} \right) dx = \int_{P}^{A} z\left(bw - \frac{\partial w}{\partial x} \right) dx + \int_{P}^{A} \frac{\partial}{\partial x}(zw)dx$$

$$\int_{P}^{A} z\left(bw - \frac{\partial w}{\partial x} \right) dx + [zw]_A - [zw]_P$$

so that $$[zw]_P = [zw]_A = \int_{P}^{A} z\left(bw - \frac{\partial w}{\partial x} \right) dx - \int_{P}^{A} Vdx$$

$$= [zw]_A = \int_{P}^{A} z\left(bw - \frac{\partial w}{\partial x} \right) dx - \int_{B}^{P} Udy - \int_{A}^{B} (Udy - Vdx)$$

$$+ \iint_{\Sigma} [wL(z) - zL^*(w)] dxdy$$

(by using (3.333)

$$= [zw]_A = \int_{P}^{A} z\left(bw - \frac{\partial w}{\partial x} \right) dx - \int_{B}^{P} z\left(aw - \frac{\partial w}{\partial y} \right) dy - \int_{A}^{B} wz(ady - abz)$$

$$+ \int_{A}^{B} \left(z\frac{\partial w}{\partial y}\partial y + w\frac{\partial z}{\partial x}dx \right) + \iint_{\Sigma} [wL(z) - zL^*(w)] dxdy \tag{3.340}$$

(By using (3.32)

Since the function w has been assumed to be arbitrary, we can choose this function in such way that

i) $L^*(w) = 0$ throughout the xy-plane

ii) $\frac{\partial w}{\partial x} = b(x, y)$ when $y = \eta$

iii) $\frac{\partial w}{\partial y} = a(x, y)$ when $x = \xi$

and iv) $w = 1$ when $x = \xi,\ y = \eta$

such a function w is called *Green's function* or Riemann-Green's function for the problem. Noting that $L(z) = f(x, y)$, we have from (3.34)

$$[z]_P=[zw]_A-\int_A^B wz(ady-bdx)+\int_A^B\left(z\frac{\partial w}{\partial y}dy+w\frac{\partial z}{\partial x}dx\right)$$

$$+\iint_\Sigma wf(x,y)\,dx\,dy \tag{3.35}$$

This gives the value of z at any point P when the values of z and $\frac{\partial z}{\partial x}$ are given on the curve AB. On the other hand, if the values of z and $\frac{\partial z}{\partial y}$ are prescribed on the curve, then using the result.

$$[zw]_B-[zw]_A=\int_A^B\left\{\frac{\partial}{\partial x}(zw)dx+\frac{\partial}{\partial y}(zw)dw\right\}$$

we rewrite, the question (3.35) in the form

$$[z]_p=[zw]_B-\int_A^B wz(ady-bdx)+\int_A^B\left(z\frac{\partial w}{\partial x}dx+w\frac{\partial z}{\partial y}dy\right)$$

$$+\iint_\Sigma wf(x,y)\,dxdy \tag{3.36}$$

Additing (3.35) and (3.36) we obtain

$$[z]_P=\frac{1}{2}\{[zw]_A+[zw]_B\}-\int_A^B wz(ady-bdz)+\frac{1}{2}\int_A^B w\left(\frac{\partial z}{\partial y}dy-\frac{\partial z}{\partial x}dx\right)$$

$$-\frac{1}{2}\int_A^B z\left(\frac{\partial w}{\partial x}dx-\frac{\partial w}{\partial y}dy\right)+\iint_\Sigma wf(x,y)dxdy \tag{3.37}$$

Thus the solution of the equation (3.29) at the point $\sum$ is obtained in terms of the prescribed values of $z, \frac{\partial z}{\partial x}$ or $\frac{\partial z}{\partial y}$ along a curve by meant of either of the formulas (3.35), (3.36) or (3.37). *This method is of immense importance to find the solution of one-dimensional wve equation by Rieman's method.*

Example 3.18. Prove that for the equation

$$\frac{\partial^2 z}{\partial x\partial y}+\frac{2}{x+y}\left(\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}\right)=0$$

the Riemann-Green's function is given by

$$w(x,y,\xi,\eta)=\frac{(x+y\{2xy+(x-y)+2\xi\eta\}}{(\xi+\eta)^3}$$

Hence find the solution of the differential equation which satisfies the conditions $z=0$

$$\frac{\partial z}{\partial x}=3x^2 \text{ on } y=x$$

Solution. The given equation is $L(z)=\frac{\partial^2 z}{\partial x\partial y}+\frac{2}{x+y}\frac{\partial z}{\partial x}+\frac{2}{x+y}\frac{\partial z}{dy}=0$ and its adjoint equation is (*c.f.* equation (3.31)

$$L^*(w)=\frac{\partial^2 w}{\partial x\partial y}-\frac{\partial}{\partial x}\left(\frac{2w}{x+y}\right)-\frac{\partial}{\partial y}\left(\frac{2w}{x+y}\right)=0$$

Such that

i) $L^*(w)=0$ throughout the *xy*-plane

ii) $\frac{\partial w}{\partial x}=\frac{2}{x+y}w$ on $y=\eta$

iii) $\frac{\partial w}{\partial y}=\frac{2}{x+y}w$ on $x=\xi$

and iv) $w=1$ at (ξ,η)

Now if $w=\frac{(x+y)\{2xy+(\xi-\eta)(x-y)+2\xi\eta\}}{(\xi+\eta)^3}$

then $$\frac{\partial w}{\partial x}=\frac{1}{(\xi+\eta)^3}[4xy+2y^2+2x(\xi-\eta)++2\xi\eta]$$

$$\frac{\partial^2 w}{\partial x\partial x}=\frac{4(x+y)}{(\xi+\eta)^3}$$

It is easy to verfy that all the four conditions (*i*) to (*iv*) are satisfied with these values. Hence the function w is the Riemann-Green function for the given equation.

Now $wL(z)-zL^*(w)=\frac{\partial U}{\partial x}+\frac{\partial V}{\partial y}$

where $U=\frac{2zw}{x+y}-z\frac{\partial w}{\partial y}, V=\frac{2zw}{x+y}+z\frac{\partial w}{\partial y}$. Hence we have

$$\iint_{\Sigma} wL(z)-zL^*(w)dxdy=\int_{C_p}(Udy-Vdx)$$

$$= \int_B^A (Udy - Vdx) + \int_A (Udy - Vdx) + \int_P^B (Udy - Vdx)$$

which on using the conditions (i) to (iv) and the fact that $y = \eta$ on AP and $x = \xi$ on BP, gives

$$[Z]_p = [zw]_A - \int_B^A w \frac{\partial z}{\partial x} dx$$

$$= [zw]_A - 3\int_\xi^\eta x^2 . \frac{2x(2x^2 + 2\xi\eta)}{(\xi + \eta)^3} dx \left[\because \frac{\partial z}{\partial x} = 3x^2 \; on \; AB \right]$$

$$= -\frac{12}{(\xi + \eta)^3} \int_\xi^\eta \left(x^5 + x^3 \xi\eta \right) dx$$

i.e. $\quad z(\xi, \eta) = (\xi - \eta)(2\xi^2 - \xi\eta + 2\eta^2)$

Hence $z(x, y) = (x - y)\left(2x^2 - xy + 2y^2\right)$ is the required solution of the given equation.

Exercises

1. If $u = f(x + iy) + (x - iy)$, where f and g are aarbitrary function,s show that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

2. If Verfy that the partial differential equation $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = \frac{2z}{x}$ is satisfied by $z = \frac{1}{x}\phi(y - x) + \phi'(y - x)$, where ϕ is an arbitrary function.

3. if $z = f\left(x^2 - y\right) + g(x^2 + y)$, where f and g are arbitrary functions. prove that

$$\frac{\partial^2 z}{\partial x^2} - \frac{1}{x}\frac{\partial z}{\partial x} = 4x^2 \frac{\partial^2 z}{\partial y^2}$$

4. Solve the following equations :

   i) $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = x - y$ [Ans $z = \phi_1(x + y) + \phi_2(x - y) + \frac{1}{4}x(x - y)^2$]

ii) $(D^2-D')z=2y-x^2$ [Ans. $z=\sum_r c_r c^{a_r x+b_r y}+x^2 y$ where a_r^2 - b_r=0]

iii) $r+s-2t=e^{x+y}$ [Ans. $z=\phi_1(x+y)+\phi_2(x-2y)+\frac{1}{3}xe^{x+y}$]

iv) $r-4s+4t=e^{2x+y}$ [Ans. $z=\phi_1(y+2x)+x\phi_2(y+2x)+\frac{1}{3}x^2e^{y+2x}$]

v) $2r-5s+2t=5\sin(2x+y)$ [Ans. $z=\phi_1(2y+x)+\phi_2(y+2x)-\frac{5}{3}x\cos(y+2x)$]

vi) $\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial x\partial y}-6\frac{\partial^2 z}{\partial y^2}=y\cos x$

[Ans. $z=\phi_1(y-3x)+\phi_2(y+2x)+\sin x-x\cos x+\frac{10}{9}\sin(y+2x)$]

vii) $\frac{\partial^2 z}{\partial x^2}+3\frac{\partial^2 z}{\partial x\partial y}+2\frac{\partial^2 z}{\partial y^2}=x+y$ [Ans. $z=\phi_1(x-y)+\phi_2(y-2x)+\frac{x^2 y}{2}-\frac{x^3}{3}$]

viii) $(D^3-7DD'^2-6D'^3)z=\sin(x+2y)+e^{2x+y}$

Ans. $z=\phi_1(y-x)+\phi_2(y+3x)+\phi_3(y-2x)-\frac{1}{75}\cos(x+2y)-\frac{1}{12}e^{2x+y}$]

ix) $\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial x\partial y}=\sin x\cos 2y$ [Ans. $z=\phi_1(y)+\phi_2(y+x)+\frac{1}{2}\sin(x+2y)-\frac{1}{6}\sin(x-2y)$]

x) $\frac{\partial^3 z}{\partial x^3}-3\frac{\partial^3 z}{\partial x^2\partial y}+4\frac{\partial^3 z}{\partial y^3}=e^{x+2y}$ [Ans. $z=\phi_1(x+y)+\phi_2(x-2y)+\frac{1}{3}xe^{x+y}$]

xi) $(D^2+5DD'+5D'^3)z=\cos(x-y)+x^2+xy+y^2$

[Ans. $z=\phi_1(y-x)+\phi_2(y-2x)+\phi_3(y+3x)+\frac{1}{4}x\cos(x-y)+\frac{5}{72}x^6$

$$+\frac{1}{60}x^5(1+21y)+\frac{1}{24}x^4y^2+\frac{1}{6}x^3y^3]$$

xiii) $(D^2+D+D')z=x^2y$ [Ans. $z=\sum_r c_r e^{\alpha_r x+b_r y}+\frac{1}{360}(x^6-6x^5+15x^4+180x^2y)$

where $a_r^2+a_r+b_r=0$]

xiv) $(D-2D'+5)(D^2+D'+3)z=\sin(2x+3y)$

[Ans. $z=e^{-5x}\phi(y+2x)+\sum_r c_r e^{a_r x+b_r y}+\frac{1}{410}[7\sin(2x+3y)-19\cos(2x+3y)]$ where

$a_r^2+b_r+3=0$]

xv) $D(D-2D')(D+D')z=e^{x+2y}(x^2+2y^2)$

[Ans. $z=\phi_1(y)+\phi_2(y+2x)+\phi_3(y-x)-\frac{1}{81}e^{x+2y}(9x^2+36y^2-18x-72y+76)$]

xvi) $(D^2D'+D'^2-2)z=e^{2y}\cos 3x+e^x\sin 2y$

[Ans. $z=\sum_r c_r e^{a_r x+b_r y}-\frac{1}{16}e^{2y}\cos 3x-\frac{1}{20}e^x(\cos 2y+3\sin 2y)$, where $a_r^2 b_r+b_r^2-2=0$]

xviii) $(D+D'-1)(D+D'-3)(D+D')z=e^{x+y+2}\cos(2x-y)$ [Ans. [$z=e^x\phi_1(y-x)$]

$+e^{3x}\phi_2(y-x)+\phi_3(y-x)-\frac{1}{10}e^{x+y+2}\{\sin(2x-y)+2\cos(2x-y)\}$]

5. Solve the following equations :

i) $(x^3y^2D^3D'^2-x^2y^3D^2D'^3)z=0$ [Ans. $z=\phi_1(y)+\phi_3(x)+x\phi_3(y)+y\phi_4(x)+\phi_5(xy)$]

ii) $(xyDD'-y^2D'^2-3xD+2yD')z=0$ [Ans. $z=\phi_1(xy)+y^3\phi_2(x)$]

iii) $(x^2D^2+xyDD'-2y^2D'^2-xD-6yD']z=0$ [Ans. $z=\phi_1(y/x^2)+\phi_2(xy)$]

iv) $\{x^2D^2-2xyDD'+y^2D'^2-n(xD+yD')+n\}z=x^2+xy+x^3$

[Ans. $z=x\phi_1(x/y)+x^n\phi_2(y/x)-\frac{x^2+y^2}{n-2}-\frac{1}{2}\frac{x^3}{n-3}$]

v) $(x^2D^2-2xyDD'-3y^2D'^2+xD-3yD')z=-x^2y\sin(\log x^2)$

[Ans. $z=\phi_1(x^3y)+\phi_2(y/x)-\frac{1}{56}x^2y\{4\cos(\log x^2)+7\sin(\log x^2)\}$]

vi) $(x^2D^2-xyDD'-2y^2D'^2+xD-2yD')z=\log(y/x)-\frac{1}{2}$

[Ans. $z=\phi_1(x^2y)+\phi_2(y/x)+\frac{1}{2}(\log x)^2\log y+\frac{1}{2}\log x\log y$]

viii) $\left(x^2yD^2D'-xy^2DD'^2-x^2D^2+y^2D'^2\right)z=\frac{x^3+y^3}{xy}$

[Ans. $z = x\phi_1(y) + y\phi_2(x) + \phi_3(xy) - \frac{1}{6}\frac{x^3 - y^3}{xy}$]

viii) $(x^2D^2 - 4xyDD' + 4y^2D'^2)z = x^3y^4$ [Ans. $[z = \phi_1(x^2y) + x\phi_2(x^2y) + \frac{1}{30}x^3y^4]$

ix) $\frac{1}{x^2}D^2 - \frac{1}{x^3}D = \frac{1}{y^2}D'^2 - \frac{1}{y^3}D'$ (Hints : Put $u = \frac{1}{2}x^2, v = \frac{1}{2}y^2$)

[Ans. $z = \phi_1(x^2 + y^2) + \phi_2(x^2 - y^2)$]

6. Solve the following equations by reducing into canonical form :

i) $yz_{xx} + (x+y)z_{xy} + xZ_{yy} = 0$ [Ans. $z = f_1(y - 3x) + f_2\left(y - \frac{x}{3}\right)$]

ii) $3z_{xx} + 10z_{xy} + 3z_{yy} = 0$ [Ans: $z = f_1(y - 3x) + f_2\left(y - \frac{x}{3}\right)$]

iii) $\frac{\partial^2 z}{\partial x^2} + 2\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = 0$ [Ans. $z = (x+y)f_1(x-y) + f_2(x-y)$]

7. Reduce the following equations to canonical form :

i) $z_{xx} + xz_{yy} = 0$ [Ans. $z_{\xi\eta} = \frac{1}{6(\xi - \eta)}(z_\xi - z_\eta)$, hyperbolic if $x < 0$, $z_{\alpha\alpha} + z_{\beta\beta} + \frac{1}{3\beta}z_\beta = 0$ elliptic if $x > 0$.

ii) $\sin^2 x.z_{xx} + \sin 2x.z_{xy} + \cos^2 x.z_{yy} = x$ [Ans. $\{1 - e^{2(\eta - \xi)}\}z_{\eta\eta} = \sin^{-1} e^{\eta - \xi} - z_\xi$, parabolic]

iii) $z_{xxx} + \frac{2N}{x}zx = \frac{1}{a^2}z_{tt}$ [Ans. $z_{\xi\eta} + \frac{N}{\xi + \eta}(z_\xi + z_\xi) = 0$ hyperbolic]

iv) $z_{xx} + 2z_{xy} + 4z_{yy} + 2z_x + 3z_y = 0$ [Ans. $z_{\alpha\alpha} + z_{\beta\beta} = -\frac{1}{3}(z_\alpha + 2\sqrt{3}z_\beta)$, elliptic]

v) $y^2z_{xx} - x^2z_{yy} = 0$ $x > 0$, $y > 0$ [Ans. $2(\xi^2 - \eta^2)z_{\xi\eta} - \eta z_\xi + \xi z_\eta = 0$, hyperbolic]

vi) $e^x z_{xx} + e^y z_{xx} = z$ [Ans. $z_{\alpha\alpha} + z_{\beta\beta} = z - \frac{1}{\alpha}z_\alpha - \frac{1}{\beta}z_\beta$, elliptic]

vii) $4z_{xx} + 5z_{xy} + z_{yy} + z_x + z_y = 2$ [Ans. $z_{\xi\eta} = \frac{1}{3}z_\eta - \frac{8}{9}$ hyperbolic]

8. Construct the adjoints of the folowing differential operators

i) $L(z) = z_{xx} + z_{yy}$ [Ans. $L^*(w) = w_{xx} + w_{yy}$,, L is self-adjoint]

ii) $L(z) = z_{xx} - z_t$ [Ans. $L^*(w) = w_{xx} + w_t$]

iii) $L^*z) = c^2z_{xx} - z_{tt}$ [Ans. $L^*(w) = c^2w_{xx} - w_{tt}$ L is self-adjoint]

iii) $L(z)=c^2 z_{xx}-z_{tt}$ [Ans. $L^*(w)=c^2 w_{xx}-w_{tt}$ L is self-adjoint]

iv) $L(z)=Az_{xx}+bz_{xy}+cz_{yy}+Dz_x+Ez_y+Fz$, where *A, B, C, D, E and F* are functions of x and y only [Ans. $L^*(w)=\frac{\partial^2}{\partial x^2}(Aw)+\frac{\partial^2}{\partial x\partial y}(Bw)+\frac{\partial^2}{\partial y^2}(Cw)$

$$-\frac{\partial}{\partial x}(Dw)-\frac{\partial}{\partial y}(Ew)+F(w)]$$

9. Prove that for the equation $\frac{\partial^2 z}{\partial x\partial y}+\frac{1}{4}x=0$, the Green's function is givent by $w(w,y,\xi,\eta)=J_0\left\{\sqrt{(x-\xi)(y-\eta)}\right\}$, where $J_0(z)$ in the Bessel function of the first kind of order zero.

10. Obtain the Riemanna solution for the equation $\frac{\partial^2 z}{\partial x\partial y}=F(x,y)$

i) $z=f(x)$ or Γ,

ii) $\frac{\partial z}{\partial n}=g(x)$ on Γ, where Γ is the curve $y=x$, $\frac{\partial}{\partial n}$ representing the normal derivative.

[Ans. $[z]_P=\frac{1}{2}\{[zw]_A+[zw]_B\}-\frac{1}{2}\int_\Gamma\left(z\frac{\partial w}{\partial x}dx+w\frac{\partial z}{\partial x}dx\right)$

$$+\frac{1}{2}\int_\Gamma\left(z\frac{\partial w}{\partial y}dy+w\frac{\partial z}{\partial y}dy\right)-\iint_\Sigma wF\,dx\,dy\,]$$

§ 3.7 Summary

The unit has been dealt with partial differential equation of order one. Special attentiaon has been given to second order linear partial differential equations (solutions of these equations with constant coefficients are demonstrated in some details), and their classifications, canonical forms and various methods of solutions, including Rieman's method of solution of hyperbolic equation.

UNIT 4 ❑ ELLIPTIC DIFFERENTIAL EQUATIONS

§ 4.1. Introduction

In Unit—3, we have seen that second order linear partial differential equations can be classified into three types, e.g., elliptic, parabolic and hyperbolic. In this unit, we shall consider Lapace and Poisson equations which are elliptic differential equations occurring most frequently in physical problems.

§ 4.2. The Occurrence of Laplace's and Poisson's Equations

To discuss Laplace and Poisson equations, it is useful to illustrate the theory with reference to some physical problems and we state some branches of physics where the field equations can be reduced to either Laplace's or Poisson's equation.

(a) *Gravitation.* The force of attraction **F** at any point inside or outside the attracting matter can be expressed in terms of a gravitational potential ψ by the equation $\mathbf{F} = \Delta\psi$ and in empty space ψ satisfies Laplace's equation $\Delta^2\psi = 0\left(\Delta^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)$ .But at any point at which the density of gravitating matter is ρ, the potential function ψ satisfies Poisson's equation $\nabla^2\psi = 4\pi\rho$

(b) *Irrotational motion of a perfect fluid.* The velocity q of a perfect fluid for irrotational motion can be expressed in terms of a velocity potential ψ by the equation $q = -\nabla\psi$. In the absence of sources, sinks etc., the function ψ satisfies Laplace's equation $\nabla^2\psi = 0$.

(c) *Magnetostatics.* The magnetic vector H can be expressed in terms of a magnetostatic potential ψ by the equation $H = -\Delta\psi$. If n is the permeability, ψ satisfies the equation $\Delta.(\mu\Delta\psi) = 0$ which reduces to Laplace's equation $\nabla^2\psi = 0$ if μ is constant.

(e) *Steady currents.* Here the conduction current vector j is derived from a potential function ψ by the formula $j = -\sigma\nabla\psi = 0$ where σ is the conductivity. The function ψ satisfies the equation $\nabla =.(\sigma\nabla\psi) = 0$ which reduces to Laplace's equation if a is constant.

(f) *Surface waves in a fluid.* The velovity potential ψ of two-dimensional wave motions of small amplitude in perfect fluid under gravity satisfies the equation $\nabla^2\psi = 0$.

(g) *Steady flow of heat.* For steady flow in the theory of conduction of heat, the temperature ψ is independent of time and satisfies the equation $\nabla.(\kappa \nabla\psi)=0$, where κ is the thermal conductivity. For constant κ, this equation reduces to Laplace's equation $\nabla^2\psi=0$

(h) *Torsion problem in solid mechanics.* For the problems of torsion of cylindrical bars in solid mechanics, the stress function ψ (Called Prandtl's stress function) satisfies the Poisson's equation $\nabla^2\psi=-2$

§ 4. 3. Boundary Value Problems (BVPs)

The function ψ whose analytical form is to be determined for different problems, in addition to satisfying Laplace's or Poisson's equation in a bounded region V also satisfies certain conditions on the boundary 5 of V. Any problem in which we require such a function ψ is called a *boundary value problem (BVP)* for Laplace's or Poisson's equation.

There are two main types of BVPs associated with the names of Dirichlet and Neumann.

(j) *Dirichlet Problem.*

By the *interior Dirichlet problem* we mean : If f is a continuous function prescribed on the bourdary S of a bounded region V, then to determine a function ψ (x, y, z) which satisfies the equation $\nabla^2\psi=0$ at any point within V and the condition $\psi=f$ on S.

The *exterior Dirichlet problem* states : If f is a continuous function prescribed on the boundary S of a finite simply-connected region V, then to determine a function ψ (x, y, z) which satisfies the equation $\nabla^2\psi=0$ outside V and the condition $\psi=f$ on 5.

For instance, to find the steady state temperature distribution within a body where no heat sources or sinks are present and the surface of the body is kept at a prescribed temperature, is an interior Dirichlet problem. On the other hand, determination of potential outside the body, the surface potential of which is prescribed, is an exterior Dirichlet problem.

Dirichlet problem is also known as *first boundary value problem.*

(ii) *Neumann problem.*

By the *interior Neumann problem* we mean : If F is a continnous function defined uniquely at each point of the boundary S of a finite region V, then to find out a function ψ (x, y, z) such that $\nabla^2\psi=0$ within V and its normal derivative $\frac{\partial\psi}{\partial n}=f$ at every point of S.

Similarly, the *exterior Neumann problem* states : If / is a continuous function defined uniquely at each point of the boundary 5 of finite region V, then to determine a function

$\psi(x, y, z)$ such that $\nabla^2\psi = 0$ outside V and its normal derivative $\frac{\partial\psi}{\partial n} = f$ on 5.

Neumann problem is also known as *second boundary value problem.*

Churchill (1954) has analyzed a boundary value problem which is different from those of the above two. This is called *Churchill or third or mixed boundary value problem.* By the *interior Churchill problem* we mean : If f is a continuous function prescribed on the boundary S of a finite region V, then to determine a function $\psi(x, y, z)$ such that $\nabla^2\psi = 0$ within V and $\frac{\partial\psi}{\partial n} + (k+1)\psi = f$ (k being constant) at every point of S. An *exterior churchill problem* can be defined in a similar manner.

The boundary value problems for Poisson's equation can be formulated in a similar way.

§ 4.4. Some Important Mathematical Results

Let V denote a closed regular region enclosed by a closed surface S and ϕ and ψ are two functions of x, y, z defined in V and continuous in $V + S$ together with their partial derivatives of the first order. In addition, we suppose that ψ has second order derivatives in $V+S$. Then putting $F = \phi\nabla\psi$ in Gauss's divergence theorem.

$$\iiint_V \nabla . F\, dv = \iint_S F \cdot \hat{n}\, ds$$

where dv is an element of volume, ds is an element of surface and $\hat{n}$ is the outward drawn unit normal vector, we get

$$\iiint_V \phi\nabla\psi dv = +\iiint_V \nabla\phi . \nabla\psi dv = \iint_S \phi\frac{\partial\psi}{\partial n} ds \qquad (4.1)$$

This is known as *Green's first identity.*

Again, we suppose that ϕ and ψ are both continuously differentialde in V and possess continuous second order derivatives in $V+S$. The interchanging ϕ and ψ in (4.1) we get

$$\iiint_V \psi\nabla\phi dv = +\iiint_V \nabla\psi . \nabla\phi dv = \iint_S \phi\frac{\partial\phi}{\partial n} ds \qquad (4.2)$$

Subtracting this from (4.1) we get

$$\iiint_V (\phi\nabla^2\psi - \psi\nabla^2\phi) = \iint_s (\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n})ds \tag{4.3}$$

This is known as *Green's second identity.*

Harmonic function

A function ψ (*x, y,* z) is said to be *harmonic at a point (x, y, z)* if its second derivatives exist and are continuous and satisfy Laplace's equation $\nabla^2\psi = 0$ throughout some neighbourhood of that point. V|/ is said to be *harmonic in a domain or open continumn* if it is harmonic at all points of that domain. It is said to be *harmonic in a closed region,* if it is continuous and harmonic at all interior points of the region.

A function ψ (*x, y,* z) is said to be *regular at infinity* $r\psi, r^2\frac{\partial\psi}{\partial x}, r^2\frac{\partial\psi}{\partial y}, r^2\frac{\partial\psi}{\partial z}$ are bounded for sufficiently large r where $r^2 = x^2 + y^2 + z^2$.. *If a function is harmonic in an unbounded region, then it must be regular at infinity.*

Some properties of harmonic functions

Harmonic function possesses a number of interesting properties and they are presented in the following theorems.

Theorem 4.1 : *If a harmonic function vanishes everywhere on the boundary, then it is identically zero everrwhere.*

Proof. Let ψ be the harmonic function so that $\nabla^2\psi = 0$ in *V.* Also by the given condition $\psi = 0$ on *S.* Now putting $\phi = \psi$ in Green's first identity (4.1) we get

$$\iiint_V \psi\nabla^2\psi dv = +\iiint_V (\nabla\psi)^2 dv = \iint_s \psi\frac{\partial\psi}{\partial n}ds$$

and using the above facts we have

$$\iiint_V (\nabla\psi)^2 dv = 0$$

which is satisfied only if $\nabla\psi = 0$ ie. $\psi =$ constant in *V.* Since ψ is continuous in *V + S* and ψ in zero on 5, we have $\psi = 0$ in *V.*

Theorem 4.2 : *If a function ψ is harmonic in V and $\frac{\partial\psi}{\partial n} = 0$ on S, then ψ is constant in V.*

Proof. Using Green's first identity and data of the Theorem 4.1, we conclude that

$$\iiint_V (\nabla\psi)^2 dv = 0$$

implying $\nabla\psi = 0$ i.e. ψ is constant in V. Since the value of is not known on the boundary S while $\frac{\partial\psi}{\partial n} = 0$, it is obvious that is constant on S and hence on $V + S$.

Theorem 4.3 : *If the dirichlet problem for a bounded region has a solution, then it is unique.*

Proof : If possible, let ψ_1 and ψ_{2v} be two solutions for the interior Dirichlet proble. Then

$$\nabla^2\psi_1 = 0 \text{ in } V \text{ and } \psi_1 = f \text{ on } S$$

and $$\nabla^2\psi_1 = 0 \text{ in } V \text{ and } \psi_2 = f \text{ on } S.$$

Let $\psi = \psi_1 - \psi_2$ so that $\nabla^2\psi = \nabla^2\psi_1 - \nabla^2\psi_2 = 0$ in V and $\psi = \psi_1 - \psi_2 = f - f = 0$ on S. Hence using theorem 4.1 we have $\psi = 0$ on V +S, i.e. $\psi_1 = \psi_2$ on V+S. The dirichlet interior problem has thus a unique solution.

Theorem 4.4. : *If the Neumann problem for a bounded region has a solution, then it is either unique or differs from one another by a constant.*

Proof : Let ψ_1 and ψ_2 be two distinct solutions of the Neumann problem. Then

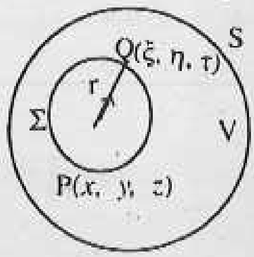
$$\nabla^2\psi_1 = 0 \quad V \text{ and } \frac{\partial\psi_1}{\partial n} = f \quad \text{on } S$$

and $$\nabla^2\psi_2 = 0 \quad V \text{ and } \frac{\partial\psi_2}{\partial n} = f \quad \text{on } S$$

Let so that in $\psi = \psi_1 - \psi_2$ so that $\nabla^2\psi = 0$ in V and $\frac{\partial\psi}{\partial n} = 0$ on S. Hence by Theorem 4.2 we have ψ = constant on V+S. If the constant is zero, then $\psi_1 = \psi_2$ on V+S and so the Neumann problem has a qunique solution. On the other hand, if this constant is nonzero, then solutions of Neumann problem differ from one another by a constant.

The spherical mean

Let V be a bounded region enclosed by a closed surface S and $P(x, y, z)$ be any point in V. Also let $\sum$ be a sphere with centre at P and radius r such that $\sum$ lies entirely within V. Let ψ be a continuous function defined in V. Then the *spherical mean of* ψ, denoted by ψ, is defined by.



$$\psi(r)=\frac{1}{4\pi r^2}\iint_{\Sigma}\psi(Q)d\Sigma$$

where $Q(\xi,\eta,\xi)$ is any variable point on the surface of the sphere Σ and $d\Sigma$ is the surface element of integration. Since for fixed value of r, $\overline{\psi}(r)$ is the average value of ψ over the sphere Σ, it is called the spherical mean.

Taking the origin at P, we have

$$\xi=x+r\sin\theta\cos\phi,\ \eta=y+r\sin\theta\sin\phi,\ \zeta=z+r\cos\theta$$

in spherical polar coordinodes. Hence the spherical mean can be written as

$$\psi(r)=\frac{1}{4\pi r^2}\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\psi(x+r\sin\theta\cos\phi,y+r\sin\theta\sin\phi,z+r\cos\theta)d\theta d\phi$$

Since ψ is continuous on Σ, $\overline{\psi}$ is also a continuous function of r on some interval $0\le r\le R$, because.

$$\overline{\psi}(r)=\frac{1}{4\pi}\iint\psi(Q)\sin\theta d\theta d\phi=\frac{\psi(Q)}{4\pi}\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\sin\theta d\theta d\phi=\psi(Q)$$

and as $r\to 0,\ Q\to P$, we have $\overline{\psi}(r)\to\psi(P)$. Hence $\overline{\psi}$ is continnous in $0\le r\le R$

Mean value theorem for harmonic functions

Theorem 4.5 : *Let ψ be harmonic in a region V and $P(x,y,z)$ be a given point in V. Let Σ be a sphere with centre at P and radius r such that Σ lies entirely in the domain of harmonicity of ψ. Them*

$$\psi(P)=\psi(r)=\frac{1}{4\pi^2}\iint_{\Sigma}\psi(Q)d\Sigma$$

Proof : Since ψ is harmonic in V, its spherical mean $\psi(r)$ is continuous in it and is given by

$$\psi(r) = \frac{1}{4\pi r^2}\iint_{\Sigma} \psi(Q)d\Sigma = \frac{1}{4\pi}\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} \psi(\xi,\eta\xi)\sin\theta d\theta d\phi$$

so that $\dfrac{d\overline{\psi}}{dr} = \dfrac{1}{4\pi}\displaystyle\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\left(\psi_{\xi}\xi_r + \psi_{\eta}\eta_r\right)\sin\theta d\theta d\phi$

$$= \frac{1}{4\pi}\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\left(\psi_{\xi}\sin\theta\cos\phi + \psi_{\eta}\sin\theta\sin\phi + \psi_{\zeta}\cos\theta\right)\sin\theta d\theta d\phi$$

Nothing that the normal $\hat{n}$ on Σ has direction costines si $\sin\theta\cos o$, $\sin\theta\sin\phi$, $\cos\theta$, we may write

$$\frac{d\overline{\psi}}{dr} = \frac{1}{4\pi^2}\iint_{\Sigma}\nabla\psi\cdot\hat{n}r^2\sin\theta d\theta d\varphi \qquad \left[\because \nabla\psi = \hat{i}\psi_{\xi} + \hat{j}\psi_{\eta} + \hat{k}\psi_{\zeta}\right]$$

$$= \frac{1}{4\pi r^2}\iint_{\Sigma}\nabla\psi\cdot\hat{n}ds$$

$$= \frac{1}{4\pi r^2}\iiint_{V}\nabla\cdot\nabla\psi dv$$ [By divergence theoem and v is the volume of the sphere Σ]

$$= \frac{1}{4\pi r^2}\iiint_{V}\nabla^2\psi dv = 0 \qquad [\because \psi \text{ is harmonic}]$$

which gives $\overline{\psi}$ = constant. The continnity of $\overline{\psi}$ at $r = 0$ proves the relation

$$\overline{\psi}(r) = \psi(P) = \frac{1}{4\pi r^2}\iint_{\Sigma}\psi(Q)d\Sigma \tag{4.5}$$

Maximun-minimum principle and consequences

Theorem 4.6 : *Let V be a region bounded by a closed surface S. Let ψ be a function continuous in V + S. and harmonic in V. Further. if ψ is not constant evrywhere on V + S, then the maximum and minimum values of ψ must occur only on the boundary S.*

Proof : Let the harmonic function ψ is not constant everywhere in V + S. If possible, let ψ attain its maximum value M at some interior point P in V. Since ψ has a maximum M at P, we can construct a sphere Σ about P of radius r such that some of the values of ψ on Σ must be less than M. But by the mean value property, the value of ψ at P is the average of the values of ψ on Σ so that it is less than M. This is a contradiction of our assumption that $\psi = M$ at P. Hence ψ must be constant over the entire sphere Σ.

Again, let Q be only other point in V. We connect Q to P by means of an arc lying entirely within V. By covering this are with spheres and using Heine-Borel theorem for choosing a finite number of covering spheres and considering the above argument, we concluede that ψ has the same constant value at Q as at P. Thus ψ cannot attain a maximum value at any point inside of V. Hence ψ attains its maxium value on the boundary S of V.

A similar argument leads us to conclude that ψ attains its minimum value on S.

We now consider some consequences of the maximum-minimum principle by the following theorems.

Theorem 4.7 : *(Stability Theorem) : The solutions of the Dirichlet problem depend continuously on the boundary values.*

Proof : Let ψ_1 and ψ_2 be two solutions of the Dirichlet problem and f_1, f_2 be the values of them on the boundary S of V. Then

$$\nabla^2\psi_1 = 0,\ \nabla^2\psi_2 = 0 \text{ in } V \text{ and } \psi_1 = f_1,\ \psi_2 = f_2 \text{ on } S.$$

Let $\psi = \psi_1 - \psi_2$ so that $\nabla^2\psi = 0$ in V and $\psi = f_1 - f_2$ on S.

Hence the Divichlet problem has the solvtion ψ which takes the value $f_1 - f_2$ on S. Now by the maximum-minimum principle. ψ attains its maximum and minimum values on the boundary S. Thus at any point within V, we have for a given $\in > 0$

$$-\in < \psi_{\min} \le \psi \le \psi_{\min} < \in \text{ i.e., } |\psi_1 - \psi_2| < \in$$

Hence if $|f_1 - f_2| < \in$ on S, then $|\psi_1 - \psi_2| < \in$ in V.

Thus any change in the initial values makes an arbitrary small change in the solution.

Theorem 4.8 : *If a sequence of functions* $\{f_n\}$, *each of which is continuous on V+S and harmonic in V, converges uniformly on S, then it converges uniformly on V +S.*

Proof : Let the sequence $\{f_n\}$, converges uniformly on *S*. Then for given $\in < 0$, we can always find an integer N such that.

$f_n - f_m \angle \in \forall n, m > N.$

Hence, by stability theorem, we have

$f_n - f_m \angle \in \text{in} V, \forall n, m > N$

Thus $\{f_n\}$, converges uniformly on *V + S*.

§ 4.5. Laplace's Equation in Polar Coordinates

In many practical problems, it becomes necessary to write the Laplace's equation in various system of coordinates. For example, if the region be a circle, we use plane polar coordinates ; if it is a cylinder, then cylindrical coordinates are used and if it is in spherical form, the use of spherical polar coordinates are useful.

(a) *Laplace's equation in plane polar coordinates* (r, θ)

Laplace's equation in two-dimensional Cartesian coordinates (r, θ) is

$$\nabla_1^2 \psi = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = \psi_{xx} + \psi_{yy} = 0 \tag{4.6}$$

and the relations between Cartesian and plane polar coordinates are

$$x = r\cos\theta, y = r\sin\theta, \quad \text{i.e.} r^2 = x^2 + y^2, \theta \tan^{-1}\frac{y}{x}.$$

Now $r_x = \cos\theta, r_y = \sin\theta, \theta_x = -\frac{\sin\theta}{r}, \theta_y = \frac{\cos\theta}{r}$

so that $\psi_x = \psi_r r_x + \psi_\theta \theta_x = \psi_r \cos\theta - \psi_\theta \frac{\sin\theta}{r}$

Similarly $\psi_y = \psi_r r_y + \psi_\theta \theta_y = \psi_r \cos\theta + \psi_\theta \frac{\sin\theta}{r}$

Again, $\psi_{xx} = (\psi_x)r_x + (\psi_x)_\theta \theta_x = \psi_\theta\left(\psi_r \cos\theta - \psi_\theta \frac{\sin\theta}{r}\right)_r \cos\theta$

$$+\left(\psi_r \cos\theta - \psi_\theta \frac{\sin\theta}{r}\right)_\theta \left(-\frac{\sin\theta}{r}\right)$$

$$= \psi_{rr} \cos^2\theta - \psi_{\theta r} \frac{\sin\theta\cos\theta}{r} + \psi_\theta \frac{\sin\theta\cos\theta}{r^2} - \psi_{r\theta} \frac{\sin\theta\cos\theta}{r^4} + \psi_r \frac{\sin^2\theta}{r}$$

$$+\psi_{\theta\theta} \frac{\sin^2\theta}{r^2} + \psi_\theta \frac{\sin\theta\cos\theta}{r^2}$$

$$\therefore \psi_{xx} = \psi_{rr} \cos^2\theta - 2\psi_{\theta r} \frac{\sin\theta\cos\theta}{r} + \psi_{\theta\theta} \frac{\sin^2\theta}{r^2} + \psi_r \frac{\sin^2\theta}{r} + 2\psi_\theta \frac{\sin\theta\cos\theta}{r^2}$$

Similarly,

$$\psi_{yy} = \psi_{rr} \sin^2\theta + 2\psi_{\theta r} \frac{\sin\theta\cos\theta}{r} + \psi_{\theta\theta} \frac{\cos^2\theta}{r} + \psi_r \frac{\cos^2\theta}{r} \psi_r \frac{\cos^2\theta}{r} - 2\psi_\theta \frac{\sin\theta\cos\theta}{r^2}$$

Substituting these values of ψ_{xx} and ψ_{yy} in (4.6), Laplace's equation in plane polar coordinates (r,θ) is given by

$$\psi_{rr} + \frac{1}{r}\psi_r + \frac{1}{r^2}\psi_{\theta\theta} = 0, \text{ i.e., } \frac{\partial^2 m}{\partial r^2} + \frac{1}{r}\frac{\partial \psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \theta^2} = 0 \tag{4.7}$$

(b) *Laplace's equation in cylindrical coordinates* (r,θ,z).

Laplace's equation in Cartesian coordinates (x,y,z) is

$$\nabla^2\psi = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} = \psi_{xx} + \psi_{yy} + \psi_{zz} = 0 \tag{4.8}$$

and the relations between Cartesian and cylindrical coordinates (r,θ,z) are

$$x = r\cos\theta,\ y = r\sin\theta,\ \ z = z \ \text{ i.e. } r^2 = x^2 + y^2,\ \theta = \tan^{-1}\frac{y}{x},\ z = z$$

Now, we have $\psi_x = \psi_r r_x + \psi_\theta \theta_x + \psi_z z_x = \psi_r \cos\theta = \psi_\theta \dfrac{\sin\theta}{r}$

Similarly, $\psi_y = \psi_r \sin\theta + \psi_\theta \dfrac{\cos\theta}{r}$ and $\psi_z = \psi_z$.

Also, $\psi_{xx} = (\psi_x)_r r_x + (\psi_x)_\theta \theta_x + (\psi_x)_z z_x$

$$= \left(\psi_r \cos\theta - \psi_\theta \frac{\sin\theta}{r}\right)_r \cos\theta + \left(\psi_r \cos\theta - \psi_\theta \frac{\sin\theta}{r}\right)_\theta \left(-\frac{\sin\theta}{r}\right)$$

i.e., $\psi_{xx} = \psi_{rr} \cos^2\theta - 2\psi_{r\theta} \dfrac{\sin\theta\cos\theta}{r} + \psi_{\theta\theta} \dfrac{\sin^2\theta}{r^2} + \psi_r \dfrac{\cos^2\theta}{r} - 2\psi_\theta \dfrac{\sin\theta\cos\theta}{r^2}$

Similarly, $\psi_{yy} = \psi_{rr} \sin^2\theta + 2\psi_{r\theta} \dfrac{\sin\theta\cos\theta}{r} + \psi_{\theta\theta} \dfrac{\cos^2\theta}{r^2} + \psi_r \dfrac{\cos^2\theta}{r} - 2\psi_\theta \dfrac{\sin\theta\cos\theta}{r^2}$

and $\psi_{zz} = \psi_{zz}$.

Hence using these values of ψ_{xx}, ψ_{yy} and ψ_{zz} in (4.8), we obtain Laplace's equation in cylindrical coordinates (r,θ,z) as

$$\psi_{rr} + \frac{1}{r}\psi_r + \frac{1}{r^2}\psi_{\theta\theta} + \psi_{zz} = 0, \text{ i.e., } \frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2} + \frac{\partial^2\psi}{\partial z^2} = 0 \qquad (4.9)$$

(c) *Laplace's equation in spherical polar coordinates* (r,θ,ϕ)

In Cartesian coordinates (x,y,z), Laplace's equation is given by (4.8). The relations between Cartesian and spherical polar coordinates (r,θ,ϕ) are

$x = r\sin\theta\cos\varphi, z = r\cos\theta$

i.e., $r^2 = x^2 + y^2 + z^2$, $\cos\theta = \dfrac{z}{r}$, $\tan\theta = \dfrac{y}{x}$.

It is easy to verify that

$r_x = \dfrac{x}{r}, r_y = \dfrac{y}{r}, r_z = \dfrac{z}{r},$

$$\theta_x = \frac{\cos\theta\cos\phi}{r}, \theta_y = \frac{\cos\theta\sin\phi}{r}, \theta_z = -\frac{\sin\theta}{r}$$

$$\phi_x = -\frac{\sin\phi}{r\sin\theta}, \ \phi_y = \frac{\cos\phi}{r\sin\theta}, \phi_z = 0.$$

Now, $\psi_x = \psi_r r_x + \psi_\theta \theta_x + \psi_\phi \phi_x = \psi_r \sin\theta\cos\phi + \psi_\theta \dfrac{\cos\theta\cos\phi}{r} - \psi_\phi \dfrac{\sin\phi}{r\sin\theta}$

Similarly, $\psi_y = \psi_r \sin\theta\sin\phi + \psi_\theta \dfrac{\cos\theta\sin\phi}{r} + \psi_\varphi \dfrac{\cos\phi}{r\sin\theta}$

$$\psi_z = \psi_r \cos\theta - \psi_\theta \frac{\sin\theta}{r}$$

Also $\psi_{xx} = (\psi_x)_r r_x + (\psi_x)_\theta \theta_x + (\psi_x)_\varphi \phi_x$

$$= \left(\psi_r \sin\theta\cos\phi + \psi_\theta \frac{\cos\theta\cos\phi}{r} - \psi_\phi \frac{\sin\phi}{r\sin\theta}\right)_r \sin\theta\cos\phi$$

$$+\left(\psi_r \sin\theta\cos\phi + \psi_\theta \frac{\cos\theta\cos\phi}{r} - \psi_\phi \frac{\sin\phi}{r\sin\theta}\right)_\theta \frac{\cos\theta\cos\phi}{r}$$

$$+\left(\psi_r \sin\theta\cos\phi + \psi_\theta \frac{\cos\theta\cos\phi}{r} - \psi_\phi \frac{\sin\phi}{r\sin\theta}\right)_\phi \left(-\frac{\sin\phi}{r\sin\theta}\right)$$

i.e., $\psi_{xx} = \psi_{rr} \sin^2\theta\cos^2\phi + \psi_{\phi\phi} \dfrac{\sin^2\phi}{r^2\sin^2\theta} + \psi_{r\theta} \dfrac{2\sin\theta\cos\theta\cos^2\phi}{r}$

$$-\psi_{r\varphi} \frac{2\sin\phi\cos\phi}{r} - \psi_{\theta\phi} \frac{2\cos\theta\cos\phi\sin\phi}{r^2\sin\theta} + \psi_r \left(\frac{\cos^2\theta\cos^2\phi}{r} + \frac{\sin^2\phi}{r}\right)$$

$$+\psi_\theta \left(\frac{\cos\theta\sin^2\phi}{r^2\sin\theta} - \frac{2\cos\theta\sin\theta\cos^2\phi}{r^2}\right) + \psi_\phi \left(\frac{\sin\phi\cos^2\phi}{r^2} + \frac{\cos^2\theta\cos^2\theta\cos\phi\sin\phi}{r^2\sin^2\theta} + \frac{\sin\phi\cos\phi}{r^2\sin^2\theta}\right)$$

Similarly, $\psi_{yy} = \psi_{rr}\sin^2\theta\sin^2\phi + \psi_{\theta\theta}\dfrac{\cos^2\theta\sin^2\phi}{r^2} + \psi_{\phi\phi}\dfrac{\cos^2\phi}{r^2\sin^2\theta} + \psi_{r\theta}2\sin\theta\cos\theta\sin^2\phi$

$$+\psi_{r\phi}\frac{2\sin\phi\cos\phi}{r} - \psi_{\theta\phi}\frac{2\cos\theta\cos\phi\sin\phi}{r^2\sin\theta} + \psi_r\left(\frac{\cos^2\theta\sin^2\phi}{r} + \frac{\cos^2\phi}{r}\right)$$

$$+\psi_\theta\left(\frac{\cos\theta\cos^2\phi}{r^2\sin\theta} - \frac{2\cos\theta\sin\theta\sin^2\phi}{r^2}\right) - \psi_\phi\left(\frac{\sin\phi\cos\phi}{r^2} + \frac{\cos^2\theta\sin\phi\cos\phi}{r^2\sin\theta} + \frac{\sin\phi\cos\phi}{r^2\sin^2\theta}\right)$$

and $\psi_{zz} = \psi_{rr}\cos^2\theta + \psi_{\theta\theta}\dfrac{\sin^2\theta}{r^2} - \psi_{r\theta}\dfrac{2\sin\theta\cos\theta}{r} + \psi_r\dfrac{\sin^2\theta}{r} + \psi_\theta\dfrac{\cos\theta\sin\theta}{r^2}$

Hence $\nabla^2\psi = \psi_{rr} + \dfrac{1}{r^2}\psi_{\theta\theta} + \dfrac{1}{r^2\sin^2\theta}\psi_{\theta\theta} + \dfrac{2}{r}\psi_r + \dfrac{\cos\theta}{r^2\sin\theta}\psi_\theta$

i.e., $\nabla^2\psi = \dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2\dfrac{\partial\psi}{\partial r}\right) + \dfrac{1}{r^2\sin\theta}\dfrac{\partial}{\partial\theta}\left(\sin\theta\dfrac{\partial\theta}{\partial\theta}\right) + \dfrac{1}{r^2\sin^2\theta}\dfrac{\partial^2\psi}{\partial\phi^2}$

so that Laplace's equation in spherical polar coordinates (r,θ,ϕ) is

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2} = 0 \tag{4.10}$$

§ 4.6. Solution of Laplace's Equation in Two-Dimensions : Separation of Variables Method.

The method of separation of variables can be applied to a large numbe of classical linear homogeneous equations. The choice of the coordinate system depends on the shape of the boundary.

1. Solution of Laplace's equation in Cartesian coordinates (x, y) :

To solve two-dimensidnal Laplace's equation in Cartesian coordinates (x, y) given by

$$\nabla_1^2\psi = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = 0 \tag{4.11}$$

we assume a solution of (4.11) in the form

$$\psi(x,y) = X(x)Y(y). \tag{4.12}$$

Substituting this into (4.11) we obtain

$$\frac{1}{x}\frac{d^2X}{dx^2} = \frac{1}{Y}\frac{d^2Y}{dy^2} = k(\text{say}) \tag{4,13}$$

where k is a separation constant. We shall have the following three cases :

Case (a) : Let $k = p_2 > 0$. p being real. Then we have from (4,13)

$$\frac{d^2X}{dx^2} - p^2X = 0,\ \frac{d^2Y}{dy^2} + p^2Y = 0$$

whose solutions are

$$X(x) = C_1c^{px} + C_2e^{-px},\ Y(y) = C_3\cos py + C_4\sin py$$

where C_1, C_2, C_3 and C_4 are constants. Hence the solution of (4.11) is given by

$$\psi(x,y) = \left(C_1e^{px} + C_2e^{-px}\right)\left(C_3\cos py + C_4\sin py\right) \tag{4.14}$$

Case (b) : Let k = 0. Then (4.13) gives

$$\frac{d^2X}{dx^2} = 0,\ \frac{d^2Y}{dy^2} = 0$$

yielding the solutions

$$X(x) = C_5x + C_6 \text{ and } Y(y) = C_7y + C_8$$

so that the solutions of (4.11) is

$$\psi(x,y) = (C_5x + C_6)(C_7y + C_8) \tag{4.15}$$

Case (e) : Let $k = -p^2 < 0$. Then as in case (a), we obtain the solution of (4.11) in the form.

$$\psi(x,y) = \left(C_9\cos px + C_{10}\sin px\right)\left(C_{11}e^{py} + C_{12}e^{-py}\right). \tag{4.16}$$

In all the above cases, the constants $C_i(i = 1, 2, \ldots\ldots, 12)$ are determined by the use of boundary conditions.

We illustrate the above results by some specific problems.

(a) *Dirichlet problem for a rectangle.*

The Dirichlet probliem for a rectangle is defined as follows :

To solve Laplace's equation $\nabla_{1^2}\psi = 0$ at any point interior to the rectangle $0 \le x \le a$, $0 \le x \le b$ subject to the boundary conditions.

$\psi(x,b) = \psi(a,y) = \psi(o-y) = 0.\ \psi(x,o) = f(x).$

where the function f(x) is supposed to be expansible in Fourier sine series.

This is an interior Dirichlet problem. The Solutions of the equation $\nabla_{1^2}\psi = 0$ are given by (4,14) to (4.16) of which we are to choose the one consistent with the nature of the problem.

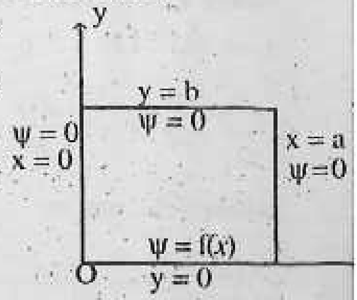


Fig. 4.1

First we consider the solution (4.14). The boundary conditions $\psi(0,y) = 0$ and $\psi(a,y) = 0$ give

$$C_1 + C_2 = 0,\ C_1 e^{ap} + C_2 e^{-ap} = 0\ (\therefore C_3 \cos py + C_4 \sin py \neq 0)$$

yielding $C_1 = C_2 = 0$ so that $\psi(x,y) = 0$ isthe only non-trivial solution. Thus the possibility of the solution (4.14) is ruled out.

Next we consider the solution (4.15) which also yields the non-trivial solution $\psi(x,y) = 0$ and, therefore, this is also ruled out.

Hence the only possible solution is given by (4.16). The boundary condition $\psi(0,y) = 0$ gives $C_9 = 0$ nd the boundary condition $\psi(a,y) = 0$ gives

$$C_{10} \sin pa\left(C_{11} e^{py} + C_{12} e^{-py}\right) = 0$$

For non-trival solution $C_{10} \neq 0$ and so sin $pa = 0$, i.e., $p = \frac{n\pi}{a}(n-1, 2, \ldots\ldots)$.

Therefore, the possible non-trivial solution is given, after using superposition principle, by

$$\psi(x,y) = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{a}\left[a_n e^{n\pi y/a} + b_r e^{-n\pi y/a}\right] \tag{4.17}$$

Now the boundary condition $\psi(x,b) = 0$ gives

$$a_n e^{n\pi b/a} + b_n e^{-n\pi b/a} = 0 \Rightarrow b_n = -\frac{a_n e^{n\pi b/a}}{e^{-n\pi b/a}}(r = 1,2,3\ldots.)$$

so that

$$\psi(x,y)=\sum_{n=1}^{\infty}\frac{a_n}{\exp\left(-\frac{n\pi p}{a}\right)}\sin\frac{n\pi x}{a}\left[\exp\left\{\frac{n\pi(y-b)}{a}\right\}-\exp\left\{-\frac{n\pi(y-b)}{a}\right\}\right]$$

i.e., $\psi(x,y)=\sum_{n=1}^{\infty}A_n\sin\frac{n\pi x}{a}\sinh\left\{\frac{n\pi(y-b)}{a}\right\}$

where $A_n=2a_n\exp\left(\frac{n\pi b}{a}\right)$. Finally, the non-homogeneous boundary condition $\psi(x,0)=f(x)$ gives

$$\sum_{n=1}^{\infty}A_n\sin\frac{n\pi x}{a}\sinh\left(-\frac{n\pi b}{a}\right)=f(x)$$

which is a half-range Fourier series so that

$$A_n\sinh\left(\frac{-n\pi b}{a}\right)=\frac{2}{a}\int_b^a f(x)\sin\frac{n\pi x}{a}dx$$

Hence the required solution of the given Dirichlet problem is

$$\psi(x,y)=\sum_{n=1}^{\infty}A_n\frac{n\pi x}{a}\sinh\left\{\frac{n\pi(y-b)}{a}\right\} \tag{4,18}$$

where

$$A_n=-\frac{2}{a\sinh\frac{n\pi b}{a}}\int_0^a f(x)\sin\frac{n\pi x}{a}dx$$

(b) *Neumann Problem for a rectangle.*

The Neumann problem for a rectangle is defined as follows : To solve Laplace's eauation $\nabla_1^2\psi=0$ at any point interior to the rectangle $0\le x\le a,\ 0\le y\le b$ subject to the boundary conditions.

$$\psi_x(0,y)=\psi_x(a,y)=\psi_y(x,0)=0.\psi_y(x,b)=f(x).$$

The general solutions of the equation $\nabla_1 2\psi = 0$ are given by (4.14) to (4.16) of which we are to choose the one consistent with the nature of the problem and the given boundary conditions as in Dirichlet problem. Here the only suitable solution is given by (4.16), i.e.,

$$\psi(x,y)=(C_1\cos px+C_2\sin px)\left(C_3e^{py}+C_4e^{-py}\right)$$

The boundary condition $\psi_x(0,y)=0$ gives $C_2=0$ and the condition $\psi_x(a,y)=0$ implies $\sin pa=0$, i.e., $p=n\pi/a$ (n = 0, 1, 2, ...). Hence, we have the solution in the form

$$\psi(x,y)=\cos\frac{n\pi x}{a}\left(Ae^{n\pi y/a}+Be^{-n\pi y/a}\right)$$

where $A=C_1C_3$ and $B=C_1C_4$. The boundary condition $\psi_y(x,0)=0$ shows that B = **A** so that

$$\psi(x,y)=2A\cos\frac{n\pi x}{a}\cosh\frac{n\pi y}{a}.$$

Defining 2A = A_n and using the superposition priniciple, we obtain

$$\psi(x,y)=\sum_{n=0}^{\infty}A_n\cos\frac{n\pi x}{a}\cosh\frac{n\pi y}{a}.$$

Finally, the boundary condition $\psi_y(x,b)=f(x)$ gives

$$f(x)=\sum_{n=1}^{\infty}A_n\frac{n\pi}{a}\cos\frac{n\pi x}{a}\sinh\frac{n\pi b}{a}$$

which is the half-range Fourier series. Thefore, we have

$$A_n\cdot\frac{n\pi}{a}\sinh\frac{n\pi b}{a}=\frac{2}{a}\int_0^a f(x)\cos\frac{n\pi x}{a}dx$$

Thus the solution of the Neumann interior problem of a rectangle is given by

$$\psi(x,y) = A_o + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{a}\cosh\frac{n\pi y}{b} \tag{4.20}$$

where A_0 is arbitrary and

$$A_n = \frac{2}{n\pi \sinh\frac{n\pi b}{a}} \int_0^a f(x)\cos\frac{n\pi x}{a}dx \tag{4.21}$$

Examle 4.1 : By separating the variables, show that the equation $\nabla_1 2\psi = 0$ has solutions of the form $A\exp(\pm nz \pm iny)$, where A and n are constants. Deduce that functions of the form

$$\psi(x,y) = \sum_r A_r e^{-\frac{r\pi x}{a}} \sin\frac{r\pi y}{a}, x \geq 0, y \geq 0$$

where A_r's are constants, are plane harmoric functions satisfying the conditions $\psi(x,y) = 0$. $\psi(x,a) = 0$, $\psi(x,y) \to 0$ as $x \to \infty$.

solution. Let $\psi(x,y) = X(x)Y(y)$ be the solution of the equation $\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = 0$ so that

$\frac{1}{X}\frac{d^2X}{dx^2} = -\frac{1}{Y}\frac{d^2Y}{dy^2} = n^2$ (say), which yield the solutions $X = A_1 e^{\pm nx}$ and $Y = A_2 e^{\pm iny}$; Hence $\psi = Ae^{\pm nx \pm iny}$ where $A = A_1A_2$.

Again $\psi(x,y) = \sum_n \left(A_n e^{nx} + B_n e^{-nx}\right)(C_n \cos ny + D_n \sin ny)$

where A_n, B_n, C_n and D_n are constants. Since $\psi(x,y) \to 0$ as $x \to \infty$, we must put A_n=0. Also $\psi(x,0) = 0 \Rightarrow C_n = 0$. Thus

$\psi(x,y) = \sum_n E_n e^{-nx} \sin ny$ (where $E_n = B_n D_n$)

Again $\psi(x,a) = 0 \Rightarrow \sin na = 0 \Rightarrow na = r\pi$ is. $n = \frac{r\pi}{a}$ (r = 0, 1, 2, ...). Hence the sodution of the given equation is.

$$\psi(x,y)=\sum_r A_r e^{-\frac{r\pi x}{a}}\sin\frac{r\pi y}{a}$$

where A_r is a new arbitrary constant.

Example 4.2 : A thin rectangular homogeneous thermally conducted plate occupies the region $0\le x\le a,\ 0\le y\le b$. The edge y = 0 is held at temperature Tx (x − a), where T is a constant and the other edges are maintained at 0°. The other faces are insulated and there is no heat source of sink inside the plate. Find the steady state temperature inside the plate.

Solution : The steady state temperature ψ satisfies Laplace's equation $\nabla_1 2\psi=0$. According to the given conditions, we have

$$\psi(0,y)=0,\ \psi(a,y)=0,\ \psi(x,0)=Tx(x-a),\psi(x,b)=0$$

By separation of variable technique, we find that the solution of the equation in consistent with the given conditions is

$$\psi(x,y)=\left(C_1e^{py}+C_2e^{-py}\right)\left(C_3\cos px+C_4\sin px\right).$$

The condition $\psi(0,y)=0\Rightarrow C_3=0$ and the condition $\psi(a,y)=0\Rightarrow\sin pa=0$, i.e. $p=n\pi la\ (n=0,1,2,....)$. Thus

$$\psi(x,y)=\sum_{n=1}^{\infty}\sin\frac{n\pi x}{a}\left(a_ne^{\frac{n\pi y}{a}}+b_ne^{-\frac{n\pi y}{a}}\right)$$

Again, we find that

$$\psi(x,b)=0\Longrightarrow b_n=-a_n\frac{e^{n\pi b/a}}{e^{-n\pi b/a}}\text{ so that}$$

$$\psi(x,y)=\sum_{n=1}^{\infty}A_n\sin\frac{n\pi x}{a}\sinh\frac{n\pi(y-b)}{a}$$

where $A_n=\dfrac{2a_n}{e^{-n\pi b/a}}$

Again, the nonhomogeneous condition $\psi(x,0) = Tx(x-a)$ leads to

$$Tx(x-a) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{a}\sinh\left(-\frac{n\pi b}{a}\right)$$

which is a half-range Fourier series so that

$$A_n \sinh\left(-\frac{n\pi b}{a}\right) = \frac{2T}{a}\int_0^a x(x-a)\sin\frac{n\pi x}{a}dx$$

$$= \frac{4Ta^2}{n^3\pi^3}\left\{(-1)^n - 1\right\} \qquad \text{[Integrating by parts]}$$

$$= \begin{cases} 0 \text{ if } n \text{ is even} \\ \dfrac{-8Ta^2}{n^3\pi^3} \text{ if } n \text{ is odd.} \end{cases}$$

Hence the required temperature distribution is given by

$$\psi(x,y) = \frac{8Ta^2}{\pi^3}\sum_{n=1}^{\infty}\frac{\operatorname{cosech}\frac{(2n+1)\pi b}{a}}{(2n+1)^3}\sin\frac{(2n+1)\pi x}{a}\sinh\frac{(2n+1)(y-b)\pi}{a}$$

Example 4.3 : Solve the Laplace's equation $\nabla_1 2\psi = 0,\ 0 \le x \le a,\ 0 \le y \le b$ satisfying the boundary conditions

$$\psi(0,y) = 0,\ \psi(x,b) = 0,\ \psi_x(a,y) = T\sin^3\frac{\pi y}{a}$$

Solution : The separation of variable method shows that only acceptable solution of the equation $\nabla_1 2\psi = 0$ is

$$\psi(x,y) = \left(C_1e^{px} + C_2e^{-px}\right)\left(C_3\cos py + C_4\sin py\right)$$

The boundary conditions $\psi(x,0) = 0$ and $\psi(x,b) = 0$ give respectively $C_3 = 0$ and sinpb $= 0$ i.e., $p = n\pi / b$ (n = 0, 1, 2, 3.....), Also the condition $\psi(0,y) = 0$ implies $C_2 = -C_1$. Thus we have

$$\psi(x,y)=2C_1\sinh\left(\frac{n\pi x}{b}\right)\sin\left(\frac{n\pi y}{b}\right)$$

Again the boundary condition $\psi_x(a,y)=T\sin^3\left(\frac{\pi y}{a}\right)$ yields

$$T\sin^3\left(\frac{\pi y}{a}\right)=2C_1\frac{n\pi}{b}\cos\left(\frac{n\pi a}{b}\right)\sin\left(\frac{n\pi y}{b}\right)$$

which gives C_1 and, the refore,

$$\psi(x,y)=\frac{bT}{n\pi}\sec h\left(\frac{n\pi a}{b}\right)\sinh\left(\frac{n\pi x}{b}\right)\sin^3\left(\frac{\pi y}{b}\right)$$

The priniciple of superposition gives the desired solution as

$$\psi(x,y)=\frac{bT}{n\pi}\sum_{n=1}^{\infty}\sec h\left(\frac{n\pi a}{a}\right)\sinh\left(\frac{n\pi x}{b}\right)\sin^3\left(\frac{\pi y}{a}\right)$$

Example 4.4 : Find the solution of Laplace's equation $\nabla_1 2\psi=0$ inthe semiinfinite region bounded by $x\geq 0, 0\leq y\leq 1$ subject to the boundary conditions

$$\left(\frac{\partial\psi}{\partial x}\right)_{x=0}=0,\left(\frac{\partial\psi}{\partial y}\right)_{y=0}=0 \text{ and } \psi(x,1)=f(x)$$

where *f(x)* is assumed to be known.

Solution. Let $\psi(x,y)=X(x)Y(y)$. Then the Laplace's equation $\nabla_1 2\psi=0$ gives

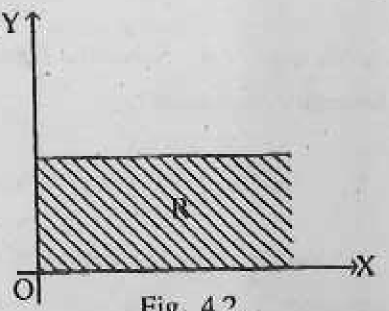


Fig. 4.2

$$\frac{1}{X}\frac{d^2X}{dx^2}+\frac{1}{Y}\frac{d^2Y}{dy^2}=0$$

whence $\frac{d^2X}{dx^2}=-n^2X, \frac{d^2Y}{dy^2}=n^2Y$ (4.22)

where n^2 is a separation constant and its sign is so chosen that the solution for X(x) subjected to the given boundary conditions is bounded. The solutions of the equations (4.22) are

$$X(x)=A_1\cos nx+B_1\sin nx, Y(y)=C_1\cosh ny+D_1\sinh ny,$$

whence

$$\psi(x,y)=(A_1\cos nx+B_1\sin nx)(C_1\cosh ny+D_1\sinh ny) \tag{4.23}$$

The boundary condition $\left(\frac{\partial\psi}{\partial x}\right)_{x=0}=0$ gives $B_1=0$ while the boundary condition $\left(\frac{\partial\psi}{\partial y}\right)_{y=0}=0$ leads to $D_1=0$, Accordingly, we may take the solution $\psi(x,y)$ in the form

$$\psi(x,y)=A\cos nx\cosh ny$$

Since all real positive values of n permissible, the general solution of $\nabla_1^2\psi=0$ subject to the first two given boundary conditions is

$$\psi(x,y)=\int_0^\infty A(n)\cos nx\cosh ny\,dn \tag{4.24}$$

where $A(n)$ is an arbitrary function of n.

Now putting $y=1$ in (4.24) we get

$$f(x)=\int_0^\infty A(n)\cos nx\cosh n\,dx$$

Using the Fourier cosine integral formaula, we have

$$A(n)\cosh n=\frac{2}{\pi}\int_0^\infty f(\xi)\cos n\xi\,d\xi$$

so that $$A(n)=\frac{2}{\pi\cosh n}\int_0^\infty f(\xi)\cos n\xi\,d\xi$$ and hence

$$\psi(x,y)=\frac{2}{\pi}\int_0^\infty\frac{\cos nx\cosh ny}{\cosh n}\left[\int_0^\infty f(\xi)\cos n\xi d\xi\right]dn \tag{4.25}$$

[In particular, if we take

$$f(x)=\begin{cases}1, 0\le x\le 1\\ 0, \quad x>1\end{cases}$$

then $\int_0^\infty f(\xi)\cos n\xi d\xi = \int_0^1 1.\cos n\xi d\xi + \int_1^\infty 0.\cos n\xi d\xi = \frac{\sin n}{n}$

and, therefore, by (4.25)

$$\psi(x,y) = \frac{2}{\pi}\int_0^\infty \frac{\cos nx \cosh ny \sin n}{n\cosh n} dn \,]$$

Example 4.5 : In the theory of elasticity, the stress function $\psi(x,y)$, in the problem of torsion, satisfies the Poisson's equation $\nabla_1^2\psi = 2, 0 \le x \le 4, 0 \le y \le 5$, with the boundary comditions $\psi = 0$ on $x = 0, 4$ and $y = 0, 5$. Find the stress function $\psi(x,y)$.

Solution : We assume a solution of the $\psi = u + v$ of the given equation $\nabla_1^2\psi = 2$, where u is the solution of the Laplace's equation

$$\nabla_1^2 u = 0 \tag{4.26}$$

and v is a particular solutim of the Poisson's eauation

$$\nabla_1^2 v = 2 \tag{4.27}$$

It is customary to assume v in the form

$$v = A + Bx + Cy + Dx^2 + Exy + Fy^2.$$

Substituting this in (4.27) we get $D + F = 1$. We take $D = 1, F = 0$. The other coefficients can be chosen arbitrarily. We, therefore, choose

$$v(x,y) = -4x + x^2$$

so that $v = 0$ on the sides $x = 0$ and $x = 4$.

Now we find u from

$$\nabla_1^2 u = 0, 0 \le x \le 4, 0 \le y \le 5, \tag{4.28}$$

satisfying

$$u(0,y) = -v(0,y) = 0,\ u(4,y) = -v(4,y) = 0$$

$$u(x,y) = -v(x,0) = 4x - x^2, u(x,5) = -v(x,5) = 4x - x^2$$

The use of method of seporation of variables and the superposition principle gives the general solution of (4.28) in the form

$$u(x,y) = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{4}\left[A_n e^{n\pi y/4} + B_n e^{-n\pi y/4}\right] \tag{4.29}$$

The non-homogeneous boundary dondition $u(x,0) = 4x - x^2$ gives

$$4x - x^2 = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{4}, \text{ where } a_n = A_n + B_n$$

Also by using the boundary condition $u(x,5) = 4x - x^2$, we get

$$4x - x^2 = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{4}\left[a_n \cosh\frac{5n\pi}{4} + b_n \sinh\frac{5n\pi}{4}\right] \tag{4.30}$$

where $b_n = A_n - B_n$.

It follows that

$$a_n = \frac{2}{4}\int_0^4 \left(4x - x^2\right)\sin\frac{n\pi x}{4}\,dx$$

$$= \frac{64}{\pi^3 n^3}\left[1 - (-1)^n\right] \qquad \text{(integrating by parts)}$$

$$= \begin{cases} 0 & \text{if } n \text{ is even} \\ \dfrac{128}{\pi^3 n^3} & \text{if } n \text{ is odd} \end{cases}$$

Also from (4.30) we have

$$a_n \cos\frac{5n\pi}{4} + b_n \sin\frac{5n\pi}{4} = \frac{2}{4}\int_0^4 \left(4x - x^2\right)\sin\frac{5n\pi}{4}\,dx = a_n$$

so that $$b_n = \frac{a_n\left(1 - \cos\dfrac{5n\pi}{4}\right)}{\sin\dfrac{5n\pi}{4}}$$

Substituting these values of a_n and b_n in (4.29), we get

$$u(x,y)=\sum_{n=1}^{\infty}\frac{a_n\sin\frac{n\pi x}{4}}{\sinh\frac{5n\pi}{4}}\left[\sinh\frac{(5-y)n\pi}{4}+\sinh\frac{n\pi y}{4}\right]$$

Hence the solution of the given Poisson's equation is obtained as

$$\psi(x,y)=x(x-4)+\frac{64}{\pi^3}\sum_{n=1}^{\infty}\frac{\sin\frac{(2n-1)\pi x}{4}}{(2n-1)^3\sinh\frac{5(2n-1)\pi}{4}}\times$$

$$\left[\sinh\left\{\frac{(2n-1)(5-y)\pi}{4}\right\}+\sinh\left\{\frac{(2n-1)\pi y}{4}\right\}\right]$$

II. Solution of Laplace's equation in plane polar corrdinates (r, θ) :

To solve Lapace's equation

$$\nabla_1^2\psi=\frac{\partial^2\psi}{\partial r^2}+\frac{1}{r}\frac{\partial\psi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2}=0 \tag{4.31}$$

in plane polar co-ordinates (r, θ) by separation of variables technique, we put

$$\psi(r,\theta)=R(r)\Theta(\theta) \tag{4.32}$$

Substituting this in (4.31) we obtain

$$\frac{1}{R}\left[r^2\frac{d^2R}{dr^2}+r\frac{dR}{dr}\right]=-\frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2}=k \quad \text{(say)} \tag{4.33}$$

where k is separation constant. We have the following three cases :

Case (a) : Let $k=p^2>0$. The from (4.33) we have

$$r^2\frac{d^2R}{dr^2}+r\frac{dR}{dr}-p^2R=0 \text{ and } \frac{d^2\Theta}{d\theta^2}+p^2\Theta=0$$

leading to the solutions $R = C_1 r^p + C_2 r^{-p}$ and $\Theta = C_3 \cos p\theta + C_4 \sin p\theta$ respectively. Hence the solution of the equation (4.27) is

$$\psi(r,\theta) = \left(C_1 r^p + C_2 e^{-p}\right)\left(C_3 \cos p\theta + C_4 \sin p\theta\right) \tag{4.34}$$

Case(b) : Let $k = 0$. Then from (4.33) we get

$$r\frac{d^2 R}{dr^2} + \frac{dR}{dr} = 0 \text{ and } \frac{d^2\Theta}{d\theta^2} = 0$$

which have solutions $R = C_5 \ln r + C_6$ and $\Theta = C_7\theta + C_8$ respeotively so that the solution of (4.31) in this case is

$$\psi(r,\theta) = (C_5 \ln r + C_6)(C_7\theta + C_8) \tag{4.35}$$

Case (c) : Let $k = -p^2 < 0$. In this case, the solution of (4.31) is obtained as in case (a) in the form

$$\psi(r,\theta) = \left[C_9 \cos(p\ln r) + C_{10}\sin(p\ln r)\right]\left(C_{11}e^{p\theta} + C_{12}e^{-p\theta}\right). \tag{4.36}$$

In all the above cases, the constants $C_i (i = 1, 2, ..., 12)$ are to be detormined by using suitable boundary conditions.

We illustrate the above results by some problems.

(a) *Interior Dirichlet problem for a circle.*

The Interior Dirichlet problem for a circle is defined as follows : To find the value of ψ in terms of its values on the boundary $r = a$ such that ψ is single-valued and continuous within and on the circular region and satisfies the equation $\nabla_1^2\psi = 0$ for $0 \le r \le a$, $0 \le \theta \le 2\pi$ subject to the boundary condition $u(a,\theta) = f(\theta), 0 \le \theta \le 2\pi, f(\theta)$ being a continuous function of θ.

Since the function θ is single-valued, it must satisfy the periodicity condition

$$\psi(r,\theta + 2\pi) = \psi(r,\theta), 0 \le \theta \le 2\pi \tag{4.37}$$

The general solutions of the equation $\nabla_1^2\psi = 0$ are given by (4.34) to (4.35) Noting that $r = 0$ is a point of the domoin of definition of the problem and ln r is undefined at $r = 0$, the solution (4.34) and (4.35) are ruled out. So the solution must be given by (4.36). Now the periodicity condition (4.37) gives from (4.30)

$$C_3 \cos p\theta + C_4 \sin p\theta = C_3 \cos p(2\pi + \theta) + C_4 \sin p(2\pi + \theta)$$

$$\Rightarrow C_3\left[\cos p\theta - \cos p(2\pi + \theta)\right] + C_4\left[\sin p\theta - \sin p(2\pi + \theta)\right] = 0$$

$$\Rightarrow \sin p\pi\left[C_3 \sin(p\theta + \pi) - C_4 \cos(p\theta + \pi)\right] = 0$$

$$\Rightarrow \sin p\pi = 0$$

$$\Rightarrow p = n \qquad (n = 0, 1, 2, \ldots)$$

Hence by the use of superposition principle we can write the solution (4.36) as

$$\psi(r,\theta) = \sum_{n=0}^{\infty}\left(C_n r^n + D_n r^{-n}\right)\left(A_n \cos n\theta + B_n \sin n\theta\right)$$

Since the solution is to be finite at the origin, must put $D_n = 0$. Thus we have

$$\psi(r,\theta) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n \cos n\theta + b_n \sin n\theta\right) \tag{4.38}$$

where $a_0 = 2A_0, a_n = A_n C_n, b_n = B_n C_n$. The solution (4.38) is a full range Fourier series.

Now the bonudary condition $\psi(a,\theta) = f(\theta) gives$

$$f(\theta) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a^n\left(a_n \cos n\theta + b_n \sin n\theta\right)$$

$$\Rightarrow a_0 = \frac{1}{\pi}\int_0^{2\pi} f(\phi)d\phi,\ a_n = \frac{1}{\pi a^n}\int_0^{2\pi} f(\phi)\cos n\phi\, d\phi,\ b_n = \frac{1}{\pi a^n}\int_0^{2\pi} f(\phi)\sin n\phi\, d\phi$$

Thus we obtain the solution (4.38) in the form

$$\psi(r,\theta) = \frac{1}{\pi}\int_0^{2\pi}(\phi)\left\{\frac{1}{2} + \sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \cos n(\phi - \theta)\right\}d\phi \tag{4.39}$$

Let us put

$$c = \sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \cos n(\phi - \theta) \text{ and } s = \sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \sin n(\phi - \theta)$$

so that

$$c + is = \sum_{n=1}^{\infty}\left\{\left(\frac{r}{a}\right)e^{i(\phi-\theta)}\right\}^n = \frac{\left(\frac{r}{a}\right)e^{i(\phi-\theta)}}{1-\left(\frac{r}{a}\right)e^{i(\phi-\theta)}}\left[\because As\left|\frac{r}{a}<1\right|, \left|e^{i(\phi-\theta)}\right|<1\right]$$

Equating real parts on both sides, we get

$$c = \sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \cos n(\phi-\theta) = \frac{(r/a)\cos(\phi-\theta)-\left(r^2/a^2\right)}{1-(2r/a)\cos(\phi-\theta)+\left(r^2/a^2\right)}$$

Thus we have the required solution (4.39) as

$$\psi(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi}\frac{\left(a^2-r^2\right)f(\phi)}{\left\{a^2-2ar\cos(\phi-\theta)+r^2\right\}}d\phi, r<a \tag{4.40}$$

This is known as *Poisson's integral formula* for a circle and this gives a unique solution for the Dirichlet interior problem.

(b) *Exterior Dirichlet problem for a circle.*

The exterior Dirichlet problem for a circle is defined as follows : To find the value of ψ at any point in the exterior of the circle $r = a$ satisfying $\nabla_1^2\psi = 0, r \geq a, 0 \leq \theta \leq 2\pi$ and $\psi(a,\theta) = f(\theta),\ 0 \leq \theta \leq 2\pi$ where $f(\theta)$ is a continuous function of θ and ψ in bounded as $r \to \infty$.

Here the solution is given by (4.34). Noting that ψ is bounded as $r \to \infty$, we take the solution in the form

$$\psi(r,\theta) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} r^{-n}\left(a_n \cos n\theta + b_n \sin n\theta\right) \tag{4.41}$$

which is a full-range Fourier series in $f(\theta)$, where

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(\phi)d\phi, a_n = \frac{a^n}{\pi}\int_0^{2\pi} f(\phi)\cos n\phi\, d\phi, b_n = \frac{a^n}{\pi}\int_0^{2\pi} f(\phi)\sin n\phi\, d\phi \tag{4.42}$$

Thus the solution (4.41) becomes

$$\psi(r,\theta)=\frac{1}{\pi}\int_0^{2\pi} f(\phi)\left\{\frac{1}{2}+\sum_{n=1}^{\infty}\left(\frac{a}{r}\right)^n \cos n(\phi-\theta)\right\}d\phi$$

Proceeding as in the case of interior Dirichlet problem we have

$$\psi(r,\theta)=\frac{1}{\pi}\int_0^{2\pi}\frac{\left(r^2-a^2\right)f(\phi)}{r^2-2ar\cos(\phi-\theta)+a^2}d\phi, r>a \tag{4.43}$$

which is the required solution of the exterior Dirichlet problem for a circle.

(c) *Interior Neumann problem for a circle.*

The interior Neumann problem for a circle is defined by : To find the value of ψ at any point in the interior of the circle $r=a$ satisfying $\nabla_1^2\psi=0, 0\le r\le a, a\le\theta\le 2\pi$ and $\frac{\partial\psi}{\partial n}=\frac{\partial\psi}{\partial r}=g(\theta)$ on $r=a$, where $g(\theta)$, $0\le\theta\le 2\pi$, is a continuous function of θ.

In this case also, the solution of the equation $\nabla_1^2\psi=0$ is given by (4.34). Moreover, since ψ is bounded at $r=0$, we take the solution in the form

$$\psi(r,\theta)=\sum_{n=0}^{\infty} r^n\left(A_n\cos n\theta+B_n\sin n\theta\right)$$

or, in the form

$$\psi(r,\theta)=\frac{1}{2}a+\sum_{n=1}^{\infty} r^n\left(a_n\cos n\theta+b_n\sin n\theta\right) \tag{4.44}$$

where, without loss of generality, we choose $A_0=\frac{1}{2}a, a_n=A_n, b_n=B_n\ (n>0)$. Noting that $\frac{\partial\psi}{\partial r}=g(\theta)$ on $r=a$, we have

$$g(\theta)=\sum_{n=1}^{\infty} na^{n-1}\left(a_n\cos n\theta+b_n\sin n\theta\right)$$

which is a full-range Fourier series in $g(\theta)$. Hence

$$a_n = \frac{1}{na^{n-1}\pi}\int_0^{2\pi} g(\phi)\cos n\phi\, d\phi,\ b_n = \frac{1}{na^{n-1}\pi}\int_0^{2\pi} g(\phi)\sin\phi\, d\phi$$

Thus, we have from (4.44)

$$\psi(r.\theta) = \frac{1}{2}a + \int_0^{2\pi} g(\phi)\left\{\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n . \frac{a}{n\pi}\cos n(\phi-\theta)d\phi\right\} \tag{4.46}$$

Now we put

$$c = \sum_{n=1}^{\infty}\frac{a}{n\pi}\left(\frac{r}{a}\right)^n \cos\pi(\phi-\theta) \text{ and } s = \sum_{n=1}^{\infty}\frac{a}{n\pi}\left(\frac{r}{a}\right)^n \sin n(\phi-\theta)$$

so that

$$c + is = \frac{a}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left[\frac{r}{a}e^{i(\phi-\theta)}\right]^n = -\frac{a}{\pi}\ln\left[1-\frac{r}{a}e^{i(\phi-\theta)}\right]$$

Equating real parts on both sides, we get

$$c = -\frac{a}{2\pi}\ln\left\{\left(s^2 - 2ar\cos(\phi-\theta) + r^2\right)/a^2\right\}$$

Hence the required solution of the interior Neumann problem is obtained from (4.46) as

$$\psi(r,\theta) = \frac{1}{2}a - \frac{a}{2\pi}\int_0^{2\pi}\ln\left\{1 - 2\frac{r}{a}\cos(\phi-\theta) + \frac{r^2}{a^2}\right\}g(\phi)d\phi$$

Example 4.6. : ψ is function of r and θ satisfying the Laplace's equation $\nabla_1^2\psi = 0$ within the region of the place bounded by $r=a, r=b, \theta=0$ and $\theta=\pi/2$. Its value along the boundary $r=a$ is $\theta\left(\frac{\pi}{2}-\theta\right)$ and along the other boundoris, it is zero. Show that

$$\psi(r,\theta) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(r/b)^{4n-2} - (b/r)^{4n-2}}{(a/b)^{4n-2} - (b/a)^{4n-2}} . \frac{\sin(4n-2)\theta}{(en-1)^3}$$

Solution : By using the method of separation of variables, it follows that the only acceptable solution of the equation $\nabla_1^2\psi = \frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2} = 0$ is

$$\psi(r,\theta) = \left(C_1 r^p + C_2 r^{-p}\right)\left(C_3 \cos p\theta + C_4 \sin p\theta\right)$$

The boundary condition $\psi(r,0)=0$ gives $C_3 = 0$ and the boundary condition $\psi(r,\pi/2)=0$ implies $\sin p\pi/2 = 0$, i.e., $p = 2n, (n = 1,2......)$. Also the boundary condition $\psi(b,\theta)=0$ gives $C_2 = -C_1 b^{4n}$. Hence the possible solution $\psi(r,\theta)$ is given by

$$\psi(r,\theta) = \sum_{n=1}^{\infty} C_n\left(r^{2n} - \frac{b^{4n}}{r^{2n}}\right)\sin 2n\theta$$

To satisfy the boundary condition $\psi(a,\theta) = \theta\left(\frac{\pi}{2} - \theta\right)$ we have

$$\theta\left(\frac{\pi}{2} - \theta\right) = \sum_{n=1}^{\infty} C_n\left(r^{2n} - \frac{b^{4n}}{r^{2n}}\right)\sin 2n\theta$$

which is a Fourier sine series. Thus we have

$$C_n\left(a^{2n} - \frac{b^{4n}}{a^{2n}}\right) = \frac{4}{\pi}\int_0^{\pi/2} \theta\left(\frac{\pi}{2} - \theta\right)\sin 2n\theta\, d\theta$$

$$= \frac{1}{4n^3}\left[1 - (-1)^n\right] \qquad \text{(integrating by parts)}$$

so that

$$C_n = \begin{cases} \dfrac{a^{2n}}{2n^3\left(a^{4n} - b^{4n}\right)} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$$

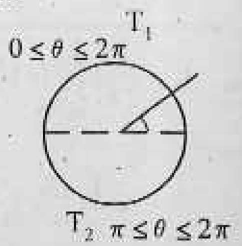
Hence the required solution is

$$\psi(r,\theta)=\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}\left(\frac{a}{r}\right)^{4n-2}\frac{r^{8n-4}-b^{8n-4}}{a^{8n-4}-b^{8n-4}}\sin(4n-2)\theta$$

$$\text{i.e., }\ \psi(r,\theta)=\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(r/b)^{4n-2}-(b/r)^{4n-2}\sin(4n-2)\theta}{(a/b)^{4n-2}-(b/a)^{4n-2}(2n-1)^3}$$

Example 4.7 : A long circular cylinder is made of two halves, the upper half-surface in at temperature T_1 while the lower half is at temperature T_2. Find the steady-state distribution of temperature inside the cylinder.

Solution. Let the z-axis be taken as the axis of the cylinder. Since the cylinder is long enough, the symmetry about the z-axis shows that it has no effect on the temperature distribution ψ. Also, for steady state, ψ satisfies La;lace's equation.



$$\frac{\partial^2\psi}{\partial r^2}+\frac{1}{r}\frac{\partial\psi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2}=0$$

The only acceptable solution of this equation is

$$\psi(r,\theta)=\left(C_1r^{p}+C_2r^{-p}\right)\left(C_3\cos p\theta+C_4\sin p\theta\right)$$

Noting that 0 must be finite at $r=0$, we have $C_2=0$ and, therefore,

$$\psi(r,\theta)=\sum_{p=0}^{\infty}r^{p}\left(A_p\cos p\theta+B_p\sin p\theta\right)$$

Now let $\psi=f(\theta)$ at $r=a, a$ being the radius of the cylinder. Then

$$f(\theta)=A_0+\sum_{p=0}^{\infty}a^{p}\left(A_p\cos p\theta+B_p\sin p\theta\right)$$

Integrating both sides w. r. t θ between the limits $\theta=0$ to $\theta=2\pi$, we get

$$A_0=\frac{1}{2\pi}\int_{1\pi}^{2\pi}f(\theta)d\theta$$

Again multiplying both sides of (4.48) by $\cos p\theta$ and integrating between $\theta = 0$ to $\theta = 2\pi$, we obtain

$$A_p = \frac{1}{\pi a^p} \int_0^{2\pi} f(\theta) \cos p\theta d\theta$$

Similarly, we have $B_p = \frac{1}{\pi a^p} \int_0^{2\pi} f(\theta) \sin p\theta d\theta$

Now we are given that

$$f(\theta) = T_1 \text{ for } 0 < \theta < \pi \text{ (upper half)}$$

$$= T_2 \text{ for } \pi < \theta < 2\pi \text{ (lower half)}$$

Hence $A_0 = \frac{1}{2\pi}\left[\int_0^{\pi} f(\theta) d\theta + \int_{\pi}^{2\pi} f(\theta) d\theta\right] = \frac{1}{2\pi}\left[\int_0^{\pi} T_1 d\theta + \int_{\pi}^{2\pi} T_2 d\theta\right]$

$$= \frac{1}{2}(T_1 + T_2)$$

$$A_p = \frac{1}{\pi a^p}\left[\int_0^{\pi} T_1 \cos p\theta d\theta + \int_{\pi}^{2\pi} T_2 \cos p\theta d\theta\right] = 0$$

$$B_p = \frac{1}{\pi a^p}\left[\int_0^{\pi} T_1 \sin p\theta d\theta + \int_{\pi}^{2\pi} T_2 \sin p\theta d\theta\right] = \frac{(T_1 - T_2)(1 - \cos p\theta)}{p\pi a^p}$$

i.e., $B_p = \frac{2(T_1 - T_2)}{p\pi a^p}$ if p is odd and $B_p = 0$ if p is even.

Thus the reqiured temperature distribution is

$$\psi(r,\theta) = \frac{1}{2}(T_1 + T_2) + \frac{2(T_1 - T_2)}{\pi} \sum_{n=1}^{\infty} \frac{r^{2n-1} \sin(2n-1)\theta}{(2n-1)a^{2n-1}}$$

III. Solution of Laplace's equation in spherical polor coordinates (r, θ)

In the case of axial symmetry about the polar axis $\theta = 0, \psi(r,\theta,\phi)$ is independent of ϕ and Laplace's equation in spherical polar coordinates (r,θ) in given by

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial \psi}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \psi}{\partial \theta}\right) = 0 \tag{4.49}$$

To solve this equation, we put

$$\psi(r,\theta) = R(r)\,\Theta(\theta) \tag{4.50}$$

where R and Θ are functions of r and θ respectively. The function $\Theta(\theta)$ is called *zonal surface harmonic.* Substituting (4.50) into (4.49) we get

$$\frac{1}{R}\frac{d}{dr}\left(r^2 \frac{dR}{dr}\right) = -\frac{1}{\Theta \sin\theta}\frac{d}{d\theta}\left(\sin\theta \frac{d\Theta}{d\theta}\right) = k \tag{4.51}$$

where k is separation constant. We put $k = n(n+1)$ where n is constant.

then we have from (4.51)

$$r^2 \frac{d^2 R}{dr^2} + 2r\frac{dR}{dr} - n(n+1)R = 0$$

whose solution is

$$R(r) = Ar^n + \frac{B}{r^{n+1}} \tag{4.52}$$

Also from (4.51) we have

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta \frac{d\Theta}{d\theta}\right) + n(n+1)\Theta = 0$$

which by the substitution $\mu = \cos\theta$ reduces to

$$\frac{d}{d\mu}\left[\left(1-\mu^2\right)\frac{d\Theta}{d\theta}\right] + n(n+1)\Theta = 0 \tag{4.54}$$

This is well-known *Legendre equation** and has the solution

$$\Theta(\theta) = CP_n(\mu) + DQ_n(\mu) \tag{4.54}$$

The function $P_n(\mu)$ and $Q_n(\mu)$ are called *Legendre functions of the first kind and second kind* respectively.

Using (4.52) and (4.54), the solution of Laplace's equation (4.49) is given by the principle of superposition as

$$\psi(r,\theta) = \sum_{n=0}^{\infty}\left(A_n r^n + \frac{B_n}{r^{n+1}}\right)\left(C_n P_n(\cos\theta) + D_n Q_n(\cos\theta)\right) \tag{4.55}$$

Example 4.8 : A thermally conducting solid bounded by two concentric spheres of radio a and b, $(a < b)$, is such that the internal boundary is kept at temperature $f_1(\theta)$ and the outer boundary at $f_2(\theta)$. Find the steady state temperature in the solid.

Solution : For steady state, the temperature ψ satisfies Laplace's equation (4.49) whose solution is given by (4.55). The boundary conditions $\psi(a,\theta) = f_1(\theta)$ and $\psi(b,\theta) = f_2(\theta)$ give

$$f_1(\theta) = \sum_{n=0}^{\infty}\left(A_n a^n + \frac{B_n}{a^{n+1}}\right)P_n(\cos\theta),\ f_2(\theta) = \sum_{n=0}^{\infty}\left(A_n b^n + \frac{B_n}{b^n}\right)P_n(\cos\theta) \qquad (4.56), (4.57)$$

on the assumptions of f_1 and f_2 to be expansible in series of Legendre polynomials.

In order to find A_n and B_n we use the following orthogonality relation for Legerdre polynomial

$$\int_0^{\pi} P_m(\cos\theta)P_n(\cos\theta)\sin\theta d\theta = \begin{cases} 0 & \text{if } m \neq n \\ \dfrac{2}{2n+1} & \text{if } m = n \end{cases} \tag{4.58}$$

Thus multiplying both sides of (4.56) by $P_m(\cos\theta)\sin\theta$ and integrating, we obtain

$$\int_0^{\pi} f_1(\theta)P_m(\cos\theta)\sin\theta d\theta = \sum_{n=0}^{\infty}\left(A_n a^n + \frac{B_n}{a^{n+1}}\right)\int_0^{\pi} P_n(\cos\theta)P_m(\cos\theta)\sin\theta d\theta$$

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* See Study Material PG(MT) 03 : Group A, Unit–7 Page 141 and 150.

i.e., $$\int_0^{\pi} f_1(\theta)P_n(\cos\theta)\sin\theta d\theta = \left(A_n a^n + \frac{B_n}{a^{n+1}}\right)\cdot\frac{2}{2n+1} \tag{4.59a}$$

Similary, $$\int_0^{\pi} f_2(\theta)P_n(\cos\theta)\sin\theta d\theta = \left(A_n b^n + \frac{B_n}{b^{n+1}}\right)\cdot\frac{2}{2n+1} \tag{4.59b}$$

We let

$$\frac{2n+1}{2}\int_0^{\pi} f_1(\theta)P_n(\cos\theta)\sin\theta d\theta = C_n, \frac{2n+1}{2}\int_0^{\pi} f_2(\theta)P_n(\cos\theta)\sin\theta d\theta$$

so that from (4.59) we get

$$A_n a^n + \frac{B_n}{a^{n+1}} = C_n, \qquad A_n b^n + \frac{B_n}{b^{n+1}} = D_n$$

Solving we have

$$A_n = \frac{C_n a^{n+1} - D_n b^{n+1}}{a^{2n+1} - b^{2n+1}}, \quad B_n = \frac{a^{n+1}b^{n+1}\left(C_n b^n - D_n a^n\right)}{b^{2n+1} - a^{2n+1}} \tag{4.60}$$

Hence the required solution is

$$\psi(r,\theta) = \sum_{n=0}^{\infty}\left(A_n r^n + \frac{B_n}{r^{n+1}}\right)P_n(\cos\theta)$$

where A_n and B_n are given by (4.60).

Example 4.9 : Determine the potential ψ of a grounded conducting sphere in a uniform field E_0 defined by $\nabla_1^2\psi = 0, 0 \le r < a, 0 < \theta < \pi$ subject to the conditions $\psi(a,\theta) = 0$ and $\psi(r,\theta) \to -E_0 r\cos\theta$ as $r \to \infty$

Solution : For axial symmetry, the solution of the equation $\nabla_1^2\psi = 0$ is given by (4.55). Using the boundary condition $\psi(r,\theta) \to -E_0 r\cos\theta$ as $r \to \infty$, we find that

$$\psi(r,\theta) = \sum_{n=0}^{\infty} A_n r^n P_n(\cos\theta) = -E_0 r\cos\theta$$

which is true for $n=1$ only. Hence $A_n = 0$ for $n \geq 2$ and $A_1 = -E_0$. Therefore

$$\psi(r,\theta) = -E_0 r\cos\theta + \sum_{n=1}^{\infty} \frac{B_n}{r^{n+1}} P_n(\cos\theta)$$

Applying the boundary condition $\psi(a,\theta) = 0$, we have

$$0 = -E_0 a\cos\theta + \sum_{n=1}^{\infty} \frac{B_n}{a^{n+1}} P_n(\cos\theta)$$

Multiplying both sides by $P_m(\cos\theta)\sin\theta$ and integrating between the limits π to 0 we get

$$0 = -E_0 a \int_0^{\pi} P_n(\cos\theta)\sin\theta\cos\theta d\theta + \frac{B_n}{a^{n+1}} \cdot \frac{2}{2n+1}$$

i.e.,
$$B_n = \frac{2n+1}{2} E_0 a^{n+2} \int_0^{\pi} P_n(\cos\theta)\sin\theta\cos\theta d\theta$$

where we have used the orthogonality relation (4.58) for Legendre polynomial $P_n(\cos\theta)$. Noting that $P_1(\cos\theta) = \cos\theta$, we have, $B_n = 0$ for $n > 1$ and $B_1 = E_0 a^3$ (by orthogonal property). Hence the required potential is

$$\psi(r,\theta) = -E_0 r\cos\theta + \frac{E_0 a^3}{r^2}\cos\theta$$

§ 4.7. Solution of Laplace's Equation in Three-Dimensions : Separation of Variables Method.

I Solution of Laplace's equation in Cartesian coordinates (x, y, z).

Laplace's equation in three-dimensions in Cartesian coordinates (x, y, z) is given by

$$\nabla^2 \psi = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial y^2} = 0 \tag{4.61}$$

To solve this equation by separation of variables technique, we put

$$\psi(x,y,z) = X(x)Y(y)Z(z) \tag{4.62}$$

Substituting this in (4.61) we get

$$\frac{1}{Y}\frac{d^2Y}{dy^2} + \frac{1}{Z}\frac{d^2Z}{dz^2} = -\frac{1}{X}\frac{d^2X}{dx^2} = \lambda_1^2, \text{ (say)} \tag{4.63}$$

where λ_1^2 is a separation constant. Thus we have

$$\frac{d^2X}{dx^2} + \lambda_1^2 X = 0$$

whose solution is $X(x) = C_1\cos\lambda_1 x + C_2\sin\lambda_1 x$.

Again, from (4.63), it follows that

$$\frac{1}{Z}\frac{d^2Z}{dz^2} - \lambda_1^2 = -\frac{1}{Y}\frac{d^2Y}{dy^2} = \lambda_2^2, \text{ (say)}$$

so that $\dfrac{d^2Y}{dy^2} + \lambda_2^2 Y = 0$ and $\dfrac{d^2Z}{dz^2} - \lambda_3^2 Z = 0$ (4.64)

where $\lambda_3^2 = \lambda_1^2 + \lambda_2^2$ and λ_2^2 is a separation constant. Solutions of the equations (4.64) are

$$Y(y) = C_3\cos\lambda_2 y + C_4\sin\lambda_2 y,\ Z(z) = C_5\cosh\lambda_3 z + C_6\sinh\lambda_3 z.$$

Hence the general solution of the euqation (4.61) in Cartesian coordinates is

$$\psi(x,y,z) = (C_1\cos\lambda_1 x + C_2\sin\lambda_1 x)(C_3\cos\lambda_2 y + C_4\sin\lambda_2 y)(C_5\cosh\lambda_3 z + C_6\sinh\lambda_3 y). \tag{4.65}$$

Example 4.10 : Find the potential $\psi(x,y,z)$ in a rectangular box defined by $0 \le x \le a$, $0 \le y \le b$, $0 \le z \le c$, if the potential is zero on all sides and the bottom, while $\psi = f(x,y)$ onthe top $z = c$ of the box.

Solution : The potential distribution in the rectangular box satisfies the Laplace's equation $\nabla^2\psi = 0$. The boundary conditions are :

$$\psi(0,y,z) = \psi(a,y,z) = 0$$

$$\psi(x,0,z) = \psi(x,b,z) = 0$$

$$\psi(x,y,0) = 0,\ \psi(x,y,c) = f(x,y).$$

where $f(x,y)$ is assumed to be expansible in double Fourier series. The general solution of Laplace's equation is given by (4.61).

The boundary conditions $\psi(o,y,z)=0$

$\psi(a,y,z)=0$ give respectively $C_1=0$ and $\lambda_1=\frac{m\pi}{a}$ $(m=1,2,.....)$. Also the boundary conditions $\psi(x,0,z)=0$ and $\psi(x,b,z)=0$ give respectively $C_3=0$ and $\lambda_2=\frac{n\pi}{b}(n=1,2,......)$. The condition $\psi(x,y,0)=0$ gives $C_5=0$. Further we note that

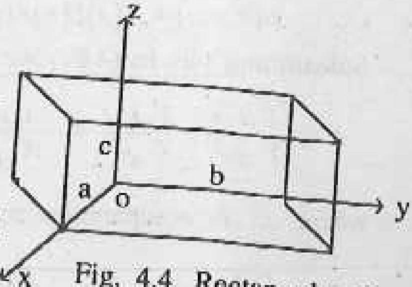


Fig. 4.4. Rectangular Box

$$\lambda\frac{2}{3}=\lambda_1^2+\lambda_2^2=\pi^2\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)=\lambda_{mn}^2(\text{say}).$$

so that $\lambda_3=\pi\sqrt{\frac{m^2}{a^2}+\frac{n^2}{b^2}}=\lambda_{mn}$

Thus, by using the principle of superposition, the solution for $\psi(x,y,z)$ is given by

$$\psi(x,y,z)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}C_{mn}\sin\frac{m\pi x}{a}.\sin\frac{n\pi y}{b}\sinh\lambda_{mn}z \tag{4.66}$$

Noting the boundary condition $\psi(x,y,c)=f(x,y)$, we have

$$f(x,y)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}C_{mn}\sin\frac{m\pi x}{a}.\sin\frac{n\pi y}{b}\sinh\lambda_{mn}c$$

which is a double Fourier sine series. Hence we have

$$C_{mn}\sinh\lambda_{mn}c=\frac{4}{ab}\int_0^a\int_0^b f(x,y)\sin\frac{m\pi x}{a}.\sin\frac{n\pi y}{b}.dx\,dy \tag{4.67}$$

Therefore, the required potential is given by (4.66), when C_{mn} is obtained from (4.67).

II Solution of Laplace's equation in cylindrical Co-ordirates (r, q, z).

The Laplace's equation is cylirdrical coordinates (r,θ,z) is given by (4.9) as

$$\nabla^2\psi=\frac{\partial^2\psi}{\partial r^2}+\frac{1}{r}\frac{\partial\psi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2}+\frac{\partial^2\psi}{\partial z^2}=0 \tag{4.68}$$

We assume its solution in the form

$$\psi(r,\theta,z)=R(r)\Theta(\theta)Z(z). \tag{4.69}$$

Substituting this in (4.68) we get

$$\frac{1}{R}\left(\frac{d^2R}{dr^2}+\frac{1}{r}\frac{dR}{dr}\right)+\frac{1}{r^2\Theta}\frac{d^2\Theta}{d\theta^2}+\frac{1}{Z}\frac{d^2Z}{dz^2}=0 \qquad (4.70)$$

Let $\frac{1}{Z}\frac{d^2Z}{dz^2}=m^2$ whose solution is $Z=e^{\pm mz}$

Also putting $\frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2}=-n^2$ we get $\frac{d^2\Theta}{d\theta^2}+n^2\Theta=0$ whose sloution is $\Theta=e^{\pm in\theta}$

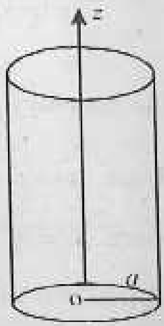


Fig. 4.5 circular Cylinder

Now the equation (4.70) gives

$$\frac{d^2R}{dr^2}+\frac{1}{r}\frac{dR}{dr}+\left(m^2-\frac{n^2}{r^2}\right)R=0$$

which is Bessel's equation and its solution is

$$R(r)=C_1J_n(mr)+C_2Y_n(mr).$$

Hence the most general solution of (.470) is

$$\psi(r,\theta,z)=\{C_1J_n(mr)+C_2Y_n(mr)\}(C_3\cos n\theta+C_4\sin m\theta) \qquad (4.71)$$

$$\left(C_5e^{mz}+C_6e^{-mz}\right)$$

Example 4.11. Find the potential ψ inside the cylinder $0\le r\le a,\ 0\le\theta\le 2\pi,\ 0\le z\le h$, if the potential on the top $z=h$ and on the lateral surface $r=a$ in held at zero, while on the base $z=0$, the potential is given by $\psi(r,\theta,0)=\psi_0\left(1-\frac{r^2}{a^2}\right)$, where ψ_0 is a constant; r,θ,z are cylindrical polar coordinates.

solution : Here the potential ψ must be single-valued and satisfy the Lapace's equation $\nabla^2\psi=0$. The boundary conditions are $\psi=0$ on $z=h$ $\psi=0$ on $r=a$ and

$$\psi=\psi_0\left(1-\frac{r^2}{a^2}\right) \text{ on } z=0.$$

In cylindrical coordinates, the general solution of Laplace's equation is given by (4.71) as

$$\psi(r,\theta,z) = \left[C_1 J_n(mr) + C_2 Y_n(mr)\right]\left[C_3 \cos n\theta + C_3 \cos n\theta + C_4 \sin n\theta\right]\left[C_5 e^{mz} + C_6 e^{-mz}\right]$$

(4.72)

Since $Y_n(mr) \to \infty$ as $r \to 0$, we must put $C_2 = 0$. Also the face $z = 0$ has the potential $\psi_0\left(1-\frac{r^2}{a^2}\right)$ which is a function of r only and is independent of θ and, therefore, the potential inside the cylinder will be independent of θ. This is possible provided $n = 0$. Thus the general solution for the potential can be written as

$$\psi(r,z) = J_0(mr)\left(Ae^{mz} + Be^{-mz}\right)$$

The boundary condition $\psi = 0$ on $z = h$ gives $Ae^{mh} + Be^{-mh} = 0$, i.e., $B = -\frac{Ae^{mh}}{e^{-mh}}$

Hence $\psi(r,z) = A_1 J_0(mr)\sin wm(z-h)$

where $A_1 = A / e^{-mh}$. The boundary condition $\psi = 0$ on the lateral surface $r = a$ implies that $J_0(ma) = 0$ which has infinitely many positive roots ξ_n, say, so that $\xi_n = ma$ is $m = \xi_n / a$.

$$\psi(r,z) = \sum_{n=1}^{\infty} A_n J_0\left(\xi_n \frac{r}{a}\right) shinh\left\{\frac{\xi_n(z-h)}{a}\right\}$$

Again the boundary condition $\psi = \psi_0\left(1-\frac{r^2}{a^2}\right)$ on $z = 0$ gives

$$\psi_0\left(1-\frac{r^2}{a^2}\right) = \sum_{n=1}^{\infty} A_n \sinh\left(-\frac{\xi_n h}{a}\right) J_0\left(\xi_n \frac{r}{a}\right)$$

which is a Fourier-Bessel series. Multiplying both sides by $rJ_0\left(\xi_p \frac{r}{a}\right)$ and integrating we get

$$\psi_0 \int_0^a \left(1-\frac{r^2}{a^2}\right) rJ_0\left(\xi_p \frac{r}{a}\right) dr = \sum_{n=1}^{\infty} A_n \sinh\left(-\frac{\xi_n h}{a}\right) \int_0^a rJ_n\left(\xi_p \frac{r}{a}\right) J_0\left(\xi_p \frac{r}{a}\right) dr$$

Using the orthogonal property of Bessel functions, viz,

$$\int_0^a xJ_n(a_i x) J_n(a_j x)\, dx = \begin{cases} 0 \text{ if } i = j \\ \frac{a^2}{2} J_{n+1}^2(a_i) \text{ if } = j \end{cases}$$

where α_i, α_j are the zeros of $J_n(x)=0$, we get

$$\psi_0 \int_0^a \left(1-\frac{r^2}{a^2}\right) r J_0\left(\xi_n \frac{r}{a}\right) dr = \frac{a^2}{2} \sum_{n=1}^{\infty} A_n \sinh\left(-\frac{\xi_n h}{a}\right) J_1^2(\xi_n)$$

giving $$A_n = \frac{2\psi_0}{a^2 \sinh\left(-\frac{\xi_n h}{a}\right) J_1^2(\xi_n)} \int_0^a \left(1-\frac{r^2}{a^2}\right) r J_0\left(\xi_n \frac{r}{a}\right) dr$$

Noting the relations $\int x J_0(x) dx = x J_1(x)$ and $\int x^2 J_1(x) dx = x^2 J^2(x)$, we obtain after integrating by parts,

$$A_n = \frac{4\psi_0 J_2(\xi_n)}{\xi_n^2 \sinh\left(-\xi_n \frac{h}{a}\right) J_1^2(\xi_n)}$$

Again the recurrence relation $J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x} J_n(x)$ gives for $n = 1$ and $x = \xi_n$,

$$J_0(\xi_n) + J_2(\xi_n) = \frac{2}{\xi_n} J_1(\xi_n), \text{ i.e. } J_2(\xi_n) = \frac{2}{\xi_n} J_1(\xi_n), \text{ since } J_0(\xi_n) = 0$$

Therefore,

$$A_n = \frac{8\psi_0}{\xi_n^3 \sinh\left(-\xi_n \frac{h}{a}\right) J_1(\xi_n)}$$

Thus the required potential inside the cylinder is given by

$$\psi(r,z) = 8\psi_0 \sum_{n=1}^{\alpha} \frac{J_0\left(\xi_n \frac{r}{a}\right) \sinh\left\{\frac{\xi_n(z-h)}{a}\right\}}{\xi_n^3 J_1(\xi_n) \sinh\left(-\frac{\xi_n h}{a}\right)}$$

III. Solution of Laplace's equation in spherical polar coordinates (r, θ, ϕ).

The Laplace's equation in spherical polar coordinates (r, θ, ϕ) is given by (4.10) as

$$\nabla^2 \psi = \frac{\partial}{\partial r}\left(r^2 \frac{\partial \psi}{\partial r}\right) + \frac{1}{\sin\theta} \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \psi}{\partial \theta}\right) + \frac{1}{\sin^2\theta} \frac{\partial^2 \psi}{\partial \phi^2} = 0 \qquad (4.73)$$

Let the solution be given by

$$\psi(r,\theta,\phi)=R(r)F(\theta,\phi) \tag{4.74}$$

Substituting this in (4.73) we get

$$F\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)+\frac{R}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial F}{\partial\theta}\right)+\frac{R}{\sin^2\theta}\frac{\partial^2 F}{\partial\phi^2}=0$$

$$\text{or, }\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)=-\frac{1}{F\sin\theta}\left\{\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial F}{\partial\theta}\right)+\frac{1}{\sin\theta}\frac{\partial^2 F}{\partial\phi^2}\right\}=-\lambda(\text{say}) \tag{4.75}$$

where λ is a separation parameter. Therefore,

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)+\lambda=0 \text{ i.e. } r^2\frac{d^2R}{dr^2}+2r\frac{dR}{dr}+\lambda R=0 \tag{4.76}$$

Le $R=r^m$ be a solution of this equation. Then, when $\lambda=-n(n+1)$, we have $m(m=1)+2m-n(n+1)=0\Rightarrow m=n,-(n+1)$. Thus, the solution of the equation (4.76) is

$$R(r)=C_1r^n+C_2/r^{n+1}$$

Again, for $\lambda=-n(n+1)$, we get from (4.75)

$$\frac{1}{\sin\theta}\left\{\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial F}{\partial\theta}\right)+\frac{1}{\sin\theta}\frac{\partial^2 F}{\partial\phi^2}\right\}+n(n+1)F=0 \tag{4.78}$$

Let the solution of this equation be

$$F(\theta,\phi)=\Theta(\theta)\Phi(\phi)$$

so that from (4.78), we obtain

$$-\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2}=\frac{\sin\theta}{\Theta}\left\{\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)+n(n+1)\sin^2\theta\Theta\right\}=m^2(\text{say}) \tag{4.79}$$

where m^2 is another separation constant. Then

$$\frac{d^2\Phi}{d\phi^2}+m^2\Phi=0$$

which has solution $\Phi=C_3\cos m\phi+C_4\sin m\phi$, provided $m\neq 0$. IF $m=0$, the solution is independent of ϕ, and we have the axisymmetric (case vide § 4.6 III).

Again, from (4.79) we have

$$\frac{\sin\theta}{\Theta}\left\{\frac{d}{d\theta}\left(\sin\theta\frac{d\theta}{d\theta}\right)+n(n+1)\sin^2\theta.\Theta\right\}=m^2$$

which, on substituting $\cos\theta=\mu$, reduces to

$$\left(1-\mu^2\right)\frac{d^2\Theta}{d\mu^2}-2\mu\frac{d\Theta}{d\mu}+\left\{n(n+1)-\frac{m^2}{1-\mu^2}\right\}\Theta=0$$

This is the well-known associated Legendre equation and has the general solution

$$\Theta(\mu)=C_5P_n^m(\mu)+C_6Q_n^m(\mu\theta),-1\le\mu\le1$$

i.e $\quad \Theta(\theta)=C_5P_n^m(\cos\theta)+C_6Q_n^m(\cos\theta), o\le\theta\le\pi$

where p_n^m and Q_n^m are associated Legendre functions of the first and second kind respectively. Now the finction $Q_n^m(\cos\theta)$ has a singularity at $\theta=0$. So we choose $C_6=0$.

Therefore, the general solution of the Laplace's equation (4.73) is given by the principle of superposition as

$$\psi(r,\theta,\phi)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left(C_1r^n+\frac{C_2}{r^{n+1}}\right)(C_3\cos m\phi+C_4\sin m\phi)P_n^m(\cos\theta) \qquad (4.79)$$

In particular, for the axisymmetric case, the general solution is

$$\psi(r,\theta,\phi)=\sum_{n=0}^{\infty}\sum_{n=0}^{\infty}\left(C_1r^n+\frac{C_2}{r^{n+1}}\right)P_n^m(\cos\theta)$$

Interior Dirichlet problem for a sphere.

The interior Dirichlet problem for a sphere is defined by : To find the value of ψ at any point in the interior of the sphere $r=a$ such that

$$\nabla^2\psi=0,\ 0\le r\le a,\ 0<\theta<\pi,\ 0\le\varphi\le2\pi$$

and $\psi(a,\theta,\phi)=f(\theta,\phi)$ on $r=a$.

Since $r=0$ is a point within the sphere, we must put $C_2=0$ in the general solution (4.79) of the Laplace's equation. Thus, we have

$$\psi(r,\theta,\phi)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}r^n\left[A_{mn}\cos m\phi+B_{mn}\sin m\phi\right]P_n^m(\cos\theta) \tag{4.80}$$

$$0\le r\le a$$

where A_{nm}, B_{nm} are new constants after adjustment. Tus use of the boundary condition $\psi(r,\theta,\phi)=F(\theta,\phi)$ on $r=a$ gives

$$f(\theta,\phi)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}a^n P_n^m(\mu)\left[A_{mn}\cos m\phi+B_{mn}\sin m\phi\right] \tag{4.81}$$

where $\mu=\cos\theta$ and it is assumed that $f(\theta,\phi)$ is expansible in series of associated Legendre function. Multiplying both sides of (4.81) by $P_n^m(\mu)\cos m\phi$ and performing double integration with respect to $\mu(-1\le\mu\le1)$ and $\phi(0\le\phi\le2\pi)$ we get

$$\int_{-1}^{1}\int_{0}^{2\pi}f(\theta,\phi)P_n^m(\mu)\cos m\phi\, d\mu\, d\phi$$

$$=a^n A_{mn}\int_0^{2\pi}\cos^2 m\phi\left[\int_{-1}^{1}\left\{P_n^m(\mu)\right\}^2 d\mu\right]d\phi=\frac{2\pi a^n A_{mn}(m+n)!}{(2n+1)(n-m)!}$$

This implies that

$$A_{mn}=\frac{(2n+1)(n-m)!}{2\pi a^n(n+m)!}\int_0^{2\pi}f(\theta,\phi)P_n^m(\mu)\cos m\phi\, d\mu\, d\phi \tag{4.82a}$$

Similarly, multiplying both sides of (4.81) by $P_n^m(\mu)\sin m\phi$ and integrating we get

$$B_{mn}=\frac{(2n+1(n-m)!}{2\pi a^n(n+m)!}\int_0^{2\pi}f(\theta,\phi)P_n^m(\mu)\sin\phi p\, d\mu\, d\phi \tag{4.82b}$$

Using (4.82b) we get

$$\psi(r,\theta,\phi)=\frac{1}{2\pi}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(2n+1)(n-m)!}{(n+m)!}\left(\frac{r}{a}\right)^n X$$

$$\times\int_{-1}^{1}\int_{0}^{2\pi} f(\eta,\chi)P_n^m(\mu)P_n^m(\mu')\left[\cos m\chi\cos m\eta+\sin m\chi\sin m\eta\right]d\eta d\chi$$

i.e. $$\psi(r,\theta,\phi)=\frac{1}{2\pi}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(2n+1)(n-m)!}{(n+m)!}\left(\frac{r}{a}\right)^n\times$$

$$\times\int_{-1}^{1}\int_{0}^{2\pi} f(\eta,\chi)P_n^m(\cos\chi)P_n^m(\cos\eta)\cos m(\chi-\eta)d\eta d\chi$$

which is the required solution.

§ 4.8. Helmholtz Equation

An equation of the form $(\nabla^2-k^2)\psi=0$ where k is constant (real of complex) is known as *Helmholtz equation.*

I. Solution of Helmholtz equation in cylindrical coordinates.

For a long cylinder with axial symmetry, the Helmholtz equation can be witten as (since ψ is independent of θ and z)

$$\frac{d^2\psi}{dr^2}+\frac{1}{r}\frac{d\psi}{dr}-k^2\psi=0 \tag{4.84}$$

This is modified Bessel's equation of order zero and has the solution

$$\psi(r)=AI_0(kr)+BK_0(kr) \tag{4.85}$$

where I_0 K_0 are modified Bessel functions of oreder zero and of the first and second kine respectively.

II. Solution of Selmhotlz equation in spherical coordinates

In the case of spherical symmetry so that $\psi=\psi(r)$, the Helmholtz equation is given by

$$\frac{d^2\psi}{dr^2}+\frac{2}{r}\frac{d\psi}{dr}-k^2\psi=0 \tag{4.86}$$

Putting $\psi=\frac{2}{\sqrt{r}}F(r)$, the above equation becomes

$$\frac{d^2F}{dr^2}+\frac{1}{r}\frac{dE}{dr}-\left(k^2+\frac{1}{4r^2}\right)F=0$$

which is modified Bessel functions of order $\frac{1}{2}$. The solution of this equation is

$$F(r)=AI_{1/2}(kr)+BK_{1/2}(kr)$$

so that $\psi(r)=r^{\frac{1}{2}}\left[AI_{1/2}(kr)+BK_{\frac{1}{2}}(kr)\right]$ (4.87)

where $I_{\frac{1}{2}}$ and $K_{\frac{1}{2}}$ are modified Bessel functions of order $\frac{1}{2}$ of first and second kind respectively, is the solution of Helmoltz equation (4.86)

Exercises

1. Show that in cylindrical coordinates (r,θ,z), Laplace's equation has solution of the form $R(r)e^{\pm imz\pm\theta}$, where $R(r)$ is a solution of the Bessel's equation

$$\frac{d^2R}{dr^2}+\frac{1}{2}\frac{dR}{dr}+\left(m^2-\frac{n^2}{r^2}\right)R=0$$

If the solution tends to zero as $z\longrightarrow\infty$ and is finite when $r=0$, show that in the usual notiations of Bessal function, the appropriate solutions are made up of terms of the form

$$J_n(mr)e^{-mz\pm in\theta}$$

2. A conducting cylinder $r=a$ is placed in a uniform electrostatic field whose intensity E is in the direction of negative x-axis. The portion $-\frac{\pi}{2}<\theta<\frac{\pi}{2}, r=a$ of the cylinder is at a constant potential V and the portion $\frac{\pi}{2}<\theta<\frac{3\pi}{2}, r=a$ is earth connected (i.e., at zero potential). Determine the eelectrostatic potential at the end point $P(r,\theta)$ external to the cylinder.

$$\left[\text{Ans. Potential at } (r,\theta) = \frac{1}{2}V + E\cos\theta\left(r - \frac{a^2}{r}\right) + \frac{2V}{\pi}\left(\frac{a\cos\theta}{r} - \frac{a^3}{3r^3}\cos 3\theta\right) + \ldots\right]$$

3. A solid right circular cylinder is bounded by the surfaces $r = a, z = \pm h$, the system of coordinates being (r,θ,z). Find the steady temperature $\psi(r,z)$ at an internal point (r,θ,z) if $\psi = 0$ on $r = a$, $\psi = T_1$ on $z = h$ and $\psi = T_2$ on $z = -h$

$$\left[Ans.\ \psi(r,z) = \frac{2}{a}\sum_{n=1}^{\infty}\frac{\left\{\left(T_2 e^{-\lambda_n h} - T_1 e^{\lambda_n h}\right)e^{\lambda_n z} + \left(T_1 z^{-h_n h} - T_2 e^{\lambda_n 1}\right)e^{-\lambda_n Z}\right\}J_0(\lambda_n r)}{\lambda_n J_1(\lambda_n a)\left(e^{2\lambda_n h} - e^{-2\lambda_n h}\right)}\right.$$

λ_n being ther oots of the equation $J_0(\lambda a) = 0]$

4. A homogeneous thermally conducting cylinder occupies the region $0 \le r \le a. 0 \le \theta \le 2\pi, 0 \le z \le h$, where (r,θ,z) are cylindrical coordinates. The top $z = h$ and the lateral surface $r = a$ are held at 0°C while the base $z = 0$ is held at 100°C. Assuming that there are no sources of heat generation within the cylinder, find the steady temerature distribution within the cylinder.

[Ans. Temperature $= 200\sum_{n=1}^{\infty}\frac{J_0(\xi_n r/a)\sinh\{\xi_n(z-h)/a\}}{\xi_n J_1(\xi_n)\sinh(-\xi_n h/a)}$, ξ_n being the roots of the equation $J_0(\lambda a) = 0$.]

5. Solve the partical differential equation

$$\frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2} = 0$$

subject to the conditions $\frac{\partial\psi}{\partial r} = 0$ at $r = a$ and $\frac{\partial\psi}{\partial r} \to U\cos\theta, \frac{1}{r}\frac{\partial\psi}{\partial r} \to -U\sin\theta$ as $r \to \infty$.

[Ans. $\psi(r,\theta) = Ur\left(1 + \frac{a^2}{r^2}\right)\cos\theta + k$, k being constant.]

6. Find the steady state temperature distribution in a semi-circular plate of radius a insulated on both the faces with its curved boundary kept at a constnat temperature U_0 and its bounding diameter kept at zero temperature.

[Ans. Temperature $= \dfrac{4U_0}{\pi}\sum_{n=0}^{\infty}\dfrac{1}{(2n+1)}\left(\dfrac{r}{a}\right)^{2n+1}\sin(2n+1)\theta$.

7. In a soild sphere of radius a, the surface is maintained at the temperature given by

$$f(\theta)=\begin{cases} k\cos\theta, 0\le\theta\le\pi/2 \\ 0 \quad ,\dfrac{\pi}{2}<\theta<\pi \end{cases}$$

Prove that the steady state temperature within the solid is

$$\psi(r,\theta)=k\left[\frac{1}{4}P_0(\cos\theta)+\frac{1}{2}\left(\frac{r}{a}\right)P_1(\cos\theta)+\frac{5}{16}\left(\frac{r}{a}\right)^2 P_2(\cos\theta)-\frac{3}{32}\left(\frac{r}{a}\right)^4 P_4(\cos\theta)+....\right]$$

8. Find the electrostatic potential ψ for the spherical shell bounded by the concentric sphere $r=a, r=b(0<a<b)$, if the inner and outer surfaces are kept at potentials V_1 and $V_2(V_1\ne V_2)$.

[Ans. $\psi=\dfrac{ab}{b-a}\left[V_1\left(\dfrac{1}{r}-\dfrac{1}{b}\right)+V_2\left(\dfrac{1}{a}-\dfrac{1}{r}\right)\right]$

9. The problem of axisymmeric fluid in a semi-infinite or in a finite circular pipe of radius a is described as follows in cylindrical coordinates :

Partial Differential Equation $\nabla^2\psi=0\; 0<r<a$

Boundary conditions : $\dfrac{\partial\psi}{\partial r}=0$ at r = 0 and

$\dfrac{\partial\psi}{\partial z}=0, \dfrac{\partial\psi}{\partial r}=V(z)$ at $r=a$.

Show that the speed of suction $\left(V(z)=\left(\dfrac{\partial x}{\partial r}\right)_{r=a}\right)$ is given by

$$V(z)=-\sum_{n=1}^{\infty}\alpha_n(A_n\cosh\alpha_n z+B_n\sinh\alpha_n z)J_1(\alpha_n a)$$

where α_n are the roots of the equation $J_0(\alpha a)=0$

10. Consider the Cauchy problem for the Laplace equation $\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}=0$ subject to

$\psi(x,0)=0, \frac{\partial\psi}{\partial y}(x,0)=\frac{1}{x}\sin nx,$ where n is a positive integer. Show that its solution is

$$\psi(x,y)=\frac{1}{n^2}\sin nx, \sinh ny.$$

11. Let f(x) and g(x) be analytic and let $\psi_1(x,y)$ be the solution of the Cauchy problem described by

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}=0 \text{ subject to } \psi(x,0)=f(x), \frac{\partial\psi}{\partial y}(x,0)=g(x)$$

and let $\psi_2(x,y)$ be the solution of the above partial differential equation subject to

$$\psi(x,0)=f(x), \frac{\partial\psi(x,0)}{\partial y}=g(x)+\frac{1}{n}\sin nx \text{ show that}$$

$$\psi_2(x,y)-\psi_1(x,y)=\frac{1}{n^2}\sin nx \sinh ny$$

12. Solve the Possion's equation

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}=2, 0\le x\le 1, 0\le x\le 1$$

with the boundary condition $\psi=0$ on sides $x=0, x=1, y=0, y=1.$

[Ans. $\psi(x,y)=x(x-1)+\frac{8}{\pi^3}\sum_{n=1}^{\infty}\left[\frac{\sinh(2n-1)(1-y)\pi+\sin(2n-1)\pi y}{(2n-1)^3\sinh(2n-1)\pi}\right]\sin(2n-1)\pi x$]

§ 4.9. Summary

Mentioned has been made of different fields such as gravitation, electrostatics, fluid mechanics etc. in which elliptic differential equations (Laplace and Possion) recur naturally. Boundary value problems (Dirichlet and Neumann) have been discussed. Laplace's equation has been solved by the method of separation of variables in different system of coordinates, e.g., Cartesian, cylindrical and spherical polar coordinates. Possion's equation has been solved in a particular type of problem. The teachnique has also been considered for the solution of Helmholtz equation.

UNIT : 5 ❑ PARABOLIC DIFFERENTIAL EQUATIONS

§ 5.1. Introduction

The diffusion phenomena such as conduction of heat in solids, diffusion of vorticity for viscous fluid flow etc. are all governed by parabolic type differential equation of the form

$$\kappa\nabla^2 T = \frac{\partial T}{\partial t} \tag{5.1}$$

where κ is a constant. The equation (5.1) is known as *heat conduction or diffusion equation.* In this unit, we shall consider various properties and tecniques for solving this type of parabolic differential equation. Firstly, we outline some circumstances in which the solution of such equations is of importance.

§ 5.2. The Occurence of the Diffusion Equation

We now indicate some instances of the occurrence and derivation of diffusion equation from the basic concepts of theoretical physics.

(a) The conduction of heat in solids. Let T (r, t) be the temperature at a point in a homogeneous isotropic solid. Then the rate of flow of heat per unit area across any plane is

$$q = -k\frac{\partial T}{\partial n}, \tag{5.2}$$

where k is the thermal conductivity of the solid and the operator $\frac{\partial}{\partial n}$ denotes the differentiation along the normal to the plane. Now the flow of heat through a small element of volume $d\tau$, the variation of T is governed by the equation

$$\rho c\frac{\partial T}{\partial t} = div\left(k \text{ grad } T\right) + H(r, T, t), \tag{5.3}$$

where ρ is the density, c the specific heat of the solid and H (r, T, t) $d\tau$ is the amount of heat generated in the element $d\tau$ per unit time at the points with position vector r. The term H (r, T, t) may arise because the solid undergoes radioactive decay or absorbing radiation or there is generation or absorption of heat due to chemical reaction. If the conductivity k is constant throughout the body, then writing

$$\kappa = \frac{k}{\rho c}, Q(r,T,t) = \frac{H(r,T,t)}{\rho c}$$

we have from (5.3)

$$\frac{\partial T}{\partial t} = \kappa \nabla^2 T + Q(r,T,t) \tag{5.4}$$

(b) Diffusion in isotropic substances. The process of diffusion in physical chemistry leads to the equation of concentrations with a single phase and is governed by the laws connecting the rate of flow of the diffusing substance with concentration gradient. Thus it c be the concentration of the diffusing substance, then the diffusion current vector J is governed by the Fick's law of diffusion

J = – D grad c,

D being the diffusion coefficient of the substance. The eqation of continuity for the diffusing substance is given by

$$\frac{\partial c}{\partial t} + div J = 0$$

i.e., $\frac{\partial c}{\partial t} + div(D\ grad\ c)$ (5.6)

In most cases, the diffusion coefficient D depends on the concentration c and the space variables. However, if D is constant, then the above equation reduces to

$$\frac{\partial c}{\partial t} = D\nabla^2 c \tag{5.5}$$

(c) The diffusion of vorticity. If a viscous find of density ρ and coefficient of viscosity μ be started from rest into motion, the vorticity ξ (= curl q, where q is the velocity of the fluid) is governed by the diffusion equation

$$\frac{\partial \xi}{\partial t} = \upsilon \nabla^2 \xi \tag{5.7}$$

where $\pi = \mu / \rho$ is the kinematic viscosity

(d) Conducting media. For the propagation of long waves in a good conductor,the components of the electric field vector E satisfy the equation of the form

$$\frac{\partial E}{\partial t} = \upsilon \nabla^2 E \tag{5.7}$$

where $\upsilon = c^2 / 4\pi\sigma\mu$, c is the velocity of light in free space, σ is the conductivity of the medium and μ is the permeability.

(e) The slowing down of neutrons in matter. Under certain circumstances, the one-dimensional transport equations for slowing down of neutrons can be reduced to the form

$$\frac{\partial \chi}{\partial \theta} = \frac{\partial^2 \chi}{\partial z^2} + T(z,\theta) \tag{5.8}$$

where $\chi(z,\theta)$ is the number of neutrons per unit time which rach the age θ and the function $T(z,\theta)$ is a function of the number of neutrons produced per unit time per unit volume.

§ 5.3. Boundary Conditions.

The heat conduction equation may have numerous solutions unless we specify a set of initial and boundary conditions. The boundary conditions are mainly of three types which we explain briefly.

Boundary Condition I : The temperature is prescribed all over the boundary surface.

Here the temperature T = G (r, t) which is some prescribed function on the boundary. This type of boundary condition is called *Dirichlet condition.* The boundary conditions depend on the specific problem under investigation. Sometimes G may be function of position r only or a function of time t only or a constant. In particular, if G (r, t) = 0 on the boundary, it is called a *homogeneous boundary condition.*

Boundary condition II : The flux of heat is prescribed on the surface of the boundary.

Here the flux of heat i.e. the normal derivative $\frac{\partial T}{\partial n} = f(r,t)$ on the surface.

This is called *Neumann condition.* In particular, if f (r, t) = 0, then this homogeneous boundary condition is called *insulated boundary condition* in which case the heat flux across the boundary surface is zero.

Boundary condition III : A linear combination of the temperature and its normal derivative is prescribed on the boundary.

Here we have

$$k\frac{\partial T}{\partial n} + hT = G(r,t)$$

where k and h are constants. This is called *Robin's condition.* In this case the boundary surface dissipates heat by convection. Following Newton's law of cooling which states that the rate of heat transferred from the body to the surroundings is proportional to the temperature difference between the body and the surroundings, we have

$$k\frac{\partial T}{\partial n}=h(T-T_a)$$

T_a being the temperature fo the surroundings.

In particular, when

$$k\frac{\partial T}{\partial n}-hT=0$$

i.e. for homogeneous boundary condition, the heat is convected by dissipation from the boundary surface into a surrounding medium maintained at zero temperaure.

In addition to the boundary conditions, the initial condition for temperature is to be prescribed to solve the diffusion equation.

§ 5.4 Elementary Solutions of the Diffusion Equation

Let us consider the one-dimensional diffusion equation.

$$\kappa\frac{\partial^2 T}{\partial x^2}=\frac{\partial T}{\partial t}, -\infty\angle x\angle\infty, t>0. \quad (5.9)$$

Putting $$T(x,t)=\frac{1}{\sqrt{4\pi\kappa t}}\exp\{-(x-\xi)^2/4\kappa t\}$$

so that $$\frac{\partial T}{\partial t}=\frac{1}{\sqrt{4\pi\kappa t}}\left\{\frac{(x-\xi)^2}{4\kappa t^2}-\frac{1}{2t}\right\}\exp\{-(x-\xi)^2/4\kappa t\}$$

and $$\frac{\partial^2 T}{\partial x^2}=\frac{1}{\sqrt{4\pi\kappa t}}\left\{-\frac{1}{2\pi\kappa t}+\frac{(x-\xi)^2}{4\kappa^2 t^2}-\frac{1}{2t}\right\}\exp\{-(x-\xi)^2/4\kappa t\}$$

ξ being a real constant, we find that (5.10) is a solution of the equation (5.9). The function (5.10), known as the *kernel*, is the elementary or the fundamental solution of the diffusion equation for the infinite interval. For $t > 0$, the kernal $T(x, t)$ is the analytic function of x and t and it is positive for all x. Thus the entire x-axis is the region of influence for the diffusion equation. It may noted that $T \to 0$ as $|x| \to \infty$.

To get an insight into the nature of the solution, we consider the one dimensional heat equation for infinite region subjected to an initial temperature f(x). Thus the problem is described by

$$\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2}, -\infty \angle x \angle \infty, t > 0. \tag{5.11}$$

and $$T(x, o) = f(x), , -\infty \angle x \angle \infty, t > 0. \tag{5.12}$$

Following the method of sepeartion of variable, we put

$$T(x, t) = X(x)\beta(t).$$

Substituting this in (5.11) we get

$$\frac{1}{X}\frac{d^2 X}{dx^2} = \frac{1}{\kappa\beta}\frac{d\beta}{dt} = \lambda \tag{5.13}$$

where λ is a separation constant. The solution for β is $\beta = ce^{\kappa\lambda t}$. If $\lambda>0$, then β and, therefore, T grows exponentially with time which is unrealistic from the physical point of view, Thus we assume that f (x) $\rightarrow$ 0 as |x| $\rightarrow$ and |T (x, t)| < M as |x| $\rightarrow$. So, for T (x, t) to remain bounded, λ should be negative, ie. we take $\lambda = -\mu^2$. Hence, we have

$$\beta(t) = ce^{-\kappa\mu^2 t}$$

Again from (5.13) we have

$$\frac{d^2 X}{dx^2} + \mu^2 X = 0$$

Whose solution is $X = c_1 \cos\mu x + c_2 \sin\mu x$

Thus the solution of (5.11) is given by

$$T(x, t, \mu) = (A\cos\mu x + B\sin\mu x)e^{-\kappa\mu^2 t}$$

where A and B are arbitrary constants. Since f(x) is in general non-periodic, we may consider Fourier integral instead of Fourier series in our present case. Also, since A and B are arbitrary, we may consider them as functions of μ. Moreover as we have no boundary condition which limit out choice of m, we should consider all possible values. Thus the principle of superposition gives this summation of all the product solutions as the relation.

$$T(x,t) = \int_0^\infty T(x, t, \mu)d\mu) = \int_0^\infty [A(\mu)\cos\mu x + B(\mu)\sin\mu x]e^{-\kappa\mu^2 t} d\mu \tag{5.15}$$

which is the solution of (5.11). Using the initial condition (5.12) we have

$$T(x,0) = f(x) = \int_0^\infty [A(\mu)\cos\mu x + B(\mu)\sin\mu x] d\mu \tag{5.16}$$

Now the Fourier integral theorem

$$f(u)=\frac{1}{\pi}\int_{\infty}^{\infty}\left[\int_{\infty}^{\infty}f(x)\cos\omega(u-x)dx\right]$$

gives
$$f(x)=\frac{1}{\pi}\int_{0}^{\infty}\left[\int_{-\infty}^{\infty}f(y)\cos\mu(x-y)d\mu\right]$$

$$=\frac{1}{\pi}\int_{0}^{\infty}\left[\int_{0}^{\infty}f(y)(\cos\mu x.\cos\mu y+\sin\mu x\sin\mu y)dy\right]d\mu$$

$$=\frac{1}{\pi}\int_{0}^{\infty}\left[\cos\mu x\int_{-\infty}^{\infty}f(y)(\cos\mu ydy+\sin\int_{-\infty}^{\infty}f(y)\sin\mu ydy\right]d\mu \tag{5.17}$$

Let
$$A(\mu)=\frac{1}{\pi}\int_{-\infty}^{\infty}f(y)\cos\mu ydy\ B(\mu)=\frac{1}{\pi}\int_{\infty}^{\infty}f(y)\sin\mu ydy$$

Then we can write (5.17) in the form

$$f(x)=\int_{-\infty}^{\infty}[A(\mu)\cos\mu x+B(\mu)\sin\mu x]d\mu$$

Comparing this with (5.16), we write the relation (5.16) as

$$T(x,0)=f(x)=\frac{1}{\pi}\int_{0}^{\infty}\left[\int_{-\infty}^{\infty}f(y)\cos\mu(x-y)dy\right]d\mu \tag{5.18}$$

Then from equation (5.15) we have

$$T(x,t)=\frac{1}{\pi}\int_{0}^{\infty}\left[\int_{-\infty}^{\infty}f(y)\cos\mu(x-y)e^{-\kappa\mu^2 t}dy\right]d\mu$$

$$=\frac{1}{\pi}\int_{-\infty}^{\infty}f(y)\left[\int_{0}^{\infty}e^{-\kappa\mu^2 t}\cos\mu(x-y)dy\right]d\mu \tag{5.19}$$

[Assuming that the interchange of the order of integration is valid]

Noting that $\int_{0}^{\infty}e^{-z^2}\cos(2bz)bz=\frac{\sqrt{\pi}}{2}e^{-b^2}$ and putting $z=\mu\sqrt{\kappa t}, b=\frac{x-y}{2\sqrt{\kappa t}}$ where b is real, we have $\int_0^\infty e^{-\kappa\mu^2 t}\cos\mu(x-y)d\mu = \dfrac{\sqrt{\pi}}{2\sqrt{\kappa t}}e^{-(x-y)^2/4\kappa t}$

Hence from (9.19) we obtain

$$T(x,t) = \frac{1}{\sqrt{4\kappa\pi t}}\int_{-\infty}^{\infty} f(y)e^{-(x-y)^2/4\kappa t}dy$$

Thus is $f(y)$ is bounded for all real values of y,k (5.20) gives the required solution of the problem described.

Exapmle 5.1. : In a one dimensional infinite solid $-\infty < x < \infty$, the surface a < x< b is initially at temperature T_0 and at zero temperature outside the surface. Show that

$$T(x,t) = \frac{T_0}{2}\left[erf\left(\frac{b-x}{\sqrt{4\kappa t}}\right) - erf\left(\frac{a-x}{\sqrt{4\kappa t}}\right)\right]$$

where *erf* is error function.

Solution : Hence we are to solve the equation $\dfrac{\partial T}{\partial t} = \kappa\dfrac{\partial^2 T}{\partial x^2}, -\infty < x < \infty$, subject to the initial conditions $T(x,0) = a < x < b$ and $T(x\,0) =$ Outside the region.

Now the solution of the above equation is given by (5.20)

$$T(x,t) = \frac{1}{\sqrt{4\pi\kappa t}}\int_{-\infty}^{\infty} f(\xi)e^{-(x-\xi)^2 4\kappa t}d\xi$$

$$= \frac{T_0}{\sqrt{4\pi\kappa t}}\int_{-\infty}^{\infty} e^{-(x-\xi)^2/4\kappa t}d\xi$$

$$= \frac{T_0}{\sqrt{\pi}}\int_{(a-x)/\sqrt{4kt}}^{(b-x)/\sqrt{4kt}} e^{-\eta^2}d\xi \qquad \left[\text{Puting } -\frac{x-\xi}{\sqrt{4\kappa t}} = \eta\right]$$

$$= \frac{T_0}{2}\left[\frac{2}{\sqrt{\pi}}\int_0^{(b-x)/\sqrt{4\kappa t}} e^{-\eta^2 d\eta} - \frac{2}{\sqrt{\pi}}\int_0^{(a-x)/\sqrt{4\kappa t}} e^{-\eta^2 d\eta}\right]$$

Hence $\quad T(x,t) = \dfrac{T_0}{2}\left[erf\left(\dfrac{b-x}{\sqrt{4\kappa t}}\right) - erf\left(\dfrac{a-x}{\sqrt{4\kappa t}}\right)\right].$

55. Solution of diffusion Equation in One-Dimension : Separation of variables method.

We now find out solution of diffusion equation in one-dimension by the separation of variables technique.

I. Cartesian coordinates.

We consider the one-dimensional diffusion equation $\frac{\partial T}{\partial t}=\kappa\frac{\partial^2 T}{\partial x^2}$. To solve this equation, we put $T(x,t)=X(x)Y(t)$ so that this equation gives

$$\frac{1}{X}\frac{d^2X}{dx^2}=\frac{I}{\kappa Y}\frac{d^2Y}{dt}=\lambda,(\text{say})$$

where is a separation constant. Then

$$\frac{d^2X}{dx^2}-\lambda X=0 \text{ and } \frac{dY}{dt}-\kappa\lambda Y=0 \tag{5.21}$$

Hence three distinct cases arise :

Case (a) : Let $\lambda>0$ and $\lambda=\alpha^2$. Then the solutions of the equations (5.21) are

$$X=C_1e^{\alpha x}+C_2e^{-\alpha x},\ Y=C_3c^{\kappa\alpha^2 t}$$

and hence the required solution of the diffusion equation is

$$T(x,t)=(Ae^{\alpha x}+Be^{-\alpha x})\,e^{\kappa\alpha^2 t} \tag{5.22}$$

where $A=C_1C_3$, $B=C_2C_3$.

Case (b) : Let $\lambda=0$ Then the solution of (5.21) are $X=C_1x+C_2$, $y=C_3$ so that

$$T(x,t)=Ax+B \tag{5.23}$$

Case (c) : *Let* $\lambda=0$ and $\lambda=\alpha^2$. Then the solutions of the equations (5.2) are

$$X=C_1\cos\alpha+C_2\sin\alpha x \text{ and } Y=C_3c^{-\kappa\alpha^2 t}$$

Hence
$$T(x,t)=(A\cos\alpha x+B\sin\alpha x)c^{-\kappa\alpha^2 t} \tag{5.24}$$

Example 5.2. : Solve the one-dimensional diffusion equation in the region $0\le x\le\pi$, $T\ge0$

subject to the conditions

(i) T remains finit as $t \to \infty$,

(ii) $T=0$ if $x=0$ and r for all t

(iii) $At\ t=0, T=\begin{cases} x, & 0 \le x \le \pi/2 \\ \pi - x, & \frac{\pi}{2} \le x \le \pi \end{cases}$

Solution : Solutions of the one-dimensional diffusion equation are given by (5.22) to (5.24). Since by (i), T must be finite as $t \to \infty$, the solution (5.22) is rejected. Again, according to (ii), the solution (5.23) gives B = 0, A= 0 so that T (x, t) = 0, for all t and, therefore, it is a trival solution. Hence we consider the solution (5.24), i.e.

$$T(x,t) = (A\cos\alpha x + B\sin\alpha x)e^{-\kappa\alpha^2 t}$$

The boundary coolition (i) gives A = 0 and sin $\alpha\pi = 0$ i.e. $\alpha = n$. Thus the solution, by the superposition principle, is of the form

$$T(x,t) = \sum_{n=1}^{\infty} B_n \sin nx.e^{-\kappa n^2 t}$$

Initially, $$T(x,0) = \sum_{n=1}^{\infty} B_n \sin nx$$

which is a half-range Fourier-since series and, therefore,

$$B_n = \frac{2}{\pi}\int_0^{\pi} T(x,0)\sin nx dx$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2} x \sin nx dx + \int_{\pi/2}^{\pi} (\pi - x)\sin nx dx\right]$$

$$= \frac{4}{n^3\pi}\sin(n\pi/2)$$

Thus the required solution is

$$T(x,t) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\sin nx.\sin(n\pi/2)}{n^2}e^{-\kappa n^2 t}$$

Example 5.3 : a conducting bar of uniform cross-section lies along the x-axis with ends at $x=0$ and $x=L$ It is kept initially at temperture 0° and its leteral surface is insulted. There are o heat sources in the bar. The end x = 0 is kept at 0° and heart is suddenly applied at the end x = L so that there is a constant hear flux q_0 at x= L. Find the temperature distribution in the bar for t > 0.

Solution : The given initial boundary value problem can be described as follows :

To solve the heat conduction equation $\frac{\partial T}{\partial t}=k\frac{\partial^2 t}{\partial x^2}$ subject to the boundary conditions

$T(0,t)=0, \frac{\partial T(L,t)}{\partial x}=q_0$ for t>0 and initial condition $T(x,0)=0, 0\le x\le L$

Prior to sudden application of heat to the end x = L and t = 0, the heat flow is indapendet to time (i.e , stady state condition). Let $T(x,t)=T_x(x,t)$ where $\mathbf{T}_s$ represents the steady part and T_1 is the transient part of the solution.

Now for steady state, the governing equation is $\frac{\partial^2 T_x}{\partial x^2}=0$ having the general solution $T_s=Cx+D$. Using the given boundary conditions. $T_x=0$ at $x=0$ and $\frac{\partial T_x(L)}{\partial x}=q_0$, we get $C=q_0$ and D = 0. Hence, the steady state solution is $T_s=q_0x$

For transient part, T_1, the governing equation is

$$\frac{\partial T_1}{\partial t}=k\frac{\partial^2 T_1}{\partial x^2}$$

The boundary conditions are : $T_1(0,t)=0, \frac{\partial \mathrm{T}_1(L,t)}{\partial x}=0,\ t>0$

and the initial condition is $T_1(x,\ 0)=-Ts(x)=-q_0x$.

Now the solution of the above diffusion equation in conformity with the given conditions is

$$T_1(x,t)=\left(A\cos\alpha x+B\sin\alpha x)e^{-\kappa\alpha^2 t}\right)$$

The boundary conditions give A = 0 and $\alpha=(2n-1)\pi/2L,\ n=1,2,3,....$ Thus using the superposition principle we get

$$T_1(x,t)=\sum_{n=1}^{\infty}B_n\sin\frac{(2n-1)\pi x}{2L}e^{-\frac{\kappa(2n-1)^2\pi^2}{4L^2}}$$

Applying the initial condition , we get

$$T_1(x,0)=-q_0x\sum_{n=1}^{\infty}Bn\sin\frac{(2n-1)\pi x}{2L}$$

Multiplying both side by sin $\frac{(2n-1)\pi x}{2L}$ and integrating between 0 and L and noting that

$$\int_0^L B_n\sin\frac{(2n-1)\pi x}{2L}\sin\frac{(2m-1)\pi x}{2L}dx=\begin{cases}0,m\neq n\\ B\,L\\ 2,m=n\end{cases}$$

we have after integrating by parts

$$-q_0\cdot\frac{4L^2}{(3n-1)^2\pi^2}(-1)^{n-1}=B_nL/2$$

so that $$B_n=\frac{8Lq_8(-1)^{n-1}}{(2n-1)^2\pi^2}$$

Hence the required temperature distribution is

$$T(x,t)=q_0x+\frac{8Lq_0}{\pi^2}\sum_{n-1}^{\infty}\left[\frac{(-1)^n}{(2n-1)^2}\sin\frac{(2n-1)\pi x}{2L}e^{-\kappa\frac{(2n-)^2\pi^2}{2L^3}}\right]$$

II. Plane polar coordinates.

Here the diffusionequation is

$$\frac{\partial^2T}{\partial r^2}+\frac{1}{r}\frac{\partial t}{\partial r}=\frac{1}{\kappa}\frac{\partial T}{\partial t}$$

where T = T (r, t).

Let its solution is s $T(r,t)=R(r)\tau(t)$ so that the above equation gives

$$\frac{1}{R}\left(\frac{d^2R}{dr^2}+\frac{1}{r}\frac{dr}{dr}\right)=\frac{1}{\kappa\tau}-\frac{d\tau}{\kappa\tau}=-\alpha^2\ \text{(say)}$$

Then we have

$$\frac{d^2R}{dr^2}+\frac{1}{r}\frac{dr}{dr}+\alpha^2R=0\ \text{and}\ \frac{d\tau}{dt}+\alpha^2\kappa\tau=0$$

Solutions of these equations are

$$R(r)=C_1J_0(\alpha r)+C_2Y_0(\alpha t) \text{ and } \tau(t)=C_3e^{-\kappa\alpha^2 t}$$

Hence the solution of the diffusion equation is

$$T(r,t)=\{AJ_0(\alpha J_0(\alpha r)+BY_0(\alpha r)\}e^{-\kappa\alpha^2 t}$$

where $A=C_1C_3,\ B=C_2C_3$.

Example 5.4 : If T (r, t) stisfied the equations.

(i) $\dfrac{\partial^2 T}{\partial r^2}+\dfrac{1}{r}\dfrac{\partial T}{\partial r}=\dfrac{1}{\kappa}\dfrac{\partial T}{\partial t},\ 0\le r\le a,\ t>\upsilon$

(ii) $T(r,0)=f(r), 0\le r\le a,\ t>\upsilon$

(iii) $\dfrac{\partial T}{\partial r}+hT=0$ at r = a, $\forall t>0$

where f(r) is assumed to be expansible in Bessel series, show that

$$T(r,t)=\frac{2}{a^2}\sum_i \frac{\xi_i^2 e^{-\kappa\xi_i^2 t} J_0(\xi_i r)}{(h^2+\xi_i^2)[J_0(\xi_i a]^2}\int_0^a uf(u)J_0(\xi_i u)du$$

where the sum is taken over the positive roots $\xi_1,\xi_2,\xi_3,\cdots$ of the equation $hJ_0(a\xi)-\xi J_1(a\xi)$

Solution : Proceeding as above, we find, by the use of superposition principle and noting that is finite at r = 0, the solution of the given diffusion equation as

$$T(r,t)=\sum_i A_iJ_0(\xi r)e^{-\kappa\xi^2 t} \tag{5.26}$$

where Ai are constants and ξ^2 is the separation constant.

Now, $T(r,0)=f(r)=\sum_i A_iJ_0(\xi r)$

Also the boundary condition (III) gives

$$\xi J_0(\xi a)+hJ_0(\xi a)=0 \Rightarrow hJ_0(\xi a)=\xi J_1(\xi a). \tag{5.27}$$

Let the roots of the equation be given by $\xi_1,\xi_2,\xi_3\cdots$ Then

$$f(r)=\sum_{i=1}^{\infty} A_iJ_0(\xi_i r)$$

Multiplying both sides by $rJ_0(\xi_j r)$ and integration w.n.t.r. from 0 to a, we get

$$\int_0^a rf(r)J_0(\xi_j r)dr = \sum_{i=1}^{\infty} A_j \int_0^a rJ_0(\xi_j r)dr$$

since $\int_0^a J_o(\xi_i r)J_0(\xi_j r)dr = \begin{cases} 0 \; for \;\; \xi_i \neq \xi_j \\ \frac{1}{2}a^2[J_0^2(\xi_i a) + J_1^2(\xi_i a)] for \; \xi_i = \xi_j \end{cases}$

we have $\int_0^a rf(r)J_0(\xi_i r)dr = \frac{1}{2}a^2 A_i[J_0^2(\xi_i a) + J_i^2(\xi_i a)]$

$$= \frac{1}{2}a^2 A_i J_0^2(\xi_i a).\frac{h^2+\xi_i^2}{\xi_i^2}$$

so that $A_i = \frac{2\xi_i^2}{a^2(h^2+\xi_i^2)J_0^2(\xi_i a)}\int_0^a rf(r)J_0(\xi_i r)dr$

Hence from (5.26), the required solution of the given diffusion equation is

$$T(r,t) = \frac{2}{a^2}\sum_i \frac{\xi_i^2}{(h^2+\xi_i^2)J_0^2(\xi_i a)}\int_0^a uf(u)J_0(\xi_i u)du$$

III. Spherical polar coordinates

The diffusion equation in spherical polar coordinates

$$\kappa\left(\frac{\partial^2 T}{dr^2} + \frac{2}{r}\frac{\partial T}{\partial r}\right) = \frac{\partial T}{\partial t}$$

where T is assumed to be independet of θ and ϕ

Let its solution be given by T (r,t) = R(r) τ (t). Then the above equation gives

$$\frac{1}{R}\left(\frac{d^2 R}{dr^2} + \frac{2}{r}\frac{dR}{dr}\right) = \frac{1}{\kappa\tau}\frac{d\tau}{dt} = -\alpha^2 \;\; \text{(asy)}$$

where α^2 is a separation constant. Then we have

$$\frac{d^2 R}{dr^2} + \frac{2}{r}\frac{dR}{dr} + \alpha^2 R = 0 \qquad (5.28)$$

and $$\frac{dT}{dt}=-\kappa\alpha^2\tau \tag{5.29}$$

The solution of the equation (5.29) is $\tau(t)=A_1e^{-\kappa\alpha^2 t}$

To solve the equation (5.28), we put S = rR. Then it is easy to see that the equation (5.28) reduces to

$$\frac{d^2S}{dr^2}+\alpha^2S=0$$

whose solution is $S=B_1\cos\alpha r+C_1\sin\alpha r$

Thus the solution of the given diffusion equation is

$$T(r,t)=\frac{e^{-\kappa\alpha^2 t}}{r}(A\cos\alpha r+B\sin\alpha r) \tag{5.30}$$

where $A=A_1B_1$, and $B=A_1C_1$

Example 5.5. : A homogeneous solid sphere of radius *a* has the initial temperature distribution f(r), $0\le r\le R$, where r is the distance measured from the centre. The surface temperature is maintained at 0°. Show that th temperature T(r, t) in the sphere is the solution of

$$\kappa\left(\frac{\partial^2T}{\partial r^2}+\frac{2}{r}\frac{\partial T}{\partial r}\right)=\frac{\partial T}{\partial t}$$

and hence find the solution.

Solution : The temperature distribution in a solid sphere is governed by the parabolic heat equation

$$\kappa\nabla^2T=\frac{\partial T}{\partial t}$$

which in view of the symmetry of the sphere reduces to

$$\kappa\left(\frac{\partial^2T}{\partial r^2}+\frac{2}{r}\frac{\partial T}{\partial r}\right)=\frac{\partial T}{\partial t}$$

Proceeding as above, we see that the solution of this equation, by using superposition principle, is given by

$$T(r,t)=\sum_{\alpha}\frac{1}{r}(A_\alpha\cos\alpha r+B_\alpha\sin r)e^{-\kappa\alpha^2 t}$$

since T is be bounded at r = 0, so we must have $A_\alpha = 0$ also the boundary condition $T(a, t) = 0$ gives sin $\alpha a = 0$, i.e. $\alpha = n\pi / a, (n = 1, 2, 3, ...)$. Hence, we have

$$T(r,t) = \sum \frac{B_n}{r} \sin(n\pi r / a) e^{-\kappa n^2 \pi^2 t / a^2}$$

Finally, applying the initial condition $T(r,0) = f(r)$, we have

$$rf(r) = \sum_{n=1}^{\infty} B_n \sin(n\pi r / a)$$

which is a half-range Fourier series, Therefore,

$$B_n = \frac{2}{a} \int_0^a rf(r) \sin(n\pi r / a) dr \qquad (5.31)$$

Thus the temperature in the sphere is given by

$$T(r,t) = \frac{1}{r} \sum_{n-1} B_n \sin(n\pi r / a) e^{-\kappa n^2 \pi^2 t / a^2}$$

where B_n is given by (5.31).

§ 5.6. Solution of diffusion equation in Two-dimensions : Separation of Variables Method.

1. Cartesian coordinates (x, y) :

The diffusion equation in two-dimensional Cartsian coordinates (x, y) is given by

$$\kappa\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) = \frac{\partial T}{\partial t} \qquad (532)$$

To solve this equation, we put. $T(x, y, t) = X(x)Y(y)\tau(t)$ Then the equation (5.32) gives

$$\frac{1}{X}\frac{d^2 X}{dx^2} + \frac{1}{Y}\frac{d^2 Y}{dy^2} = \frac{1}{\kappa}\frac{1}{\tau}\frac{d\tau}{dt} = -\alpha^2, \text{ (say)} \qquad (5.33)$$

where is a seperation constant. then $\tau = A_1 e^{-\kappa\alpha^2 t}$

Again, from (5.33) we have

$$\frac{1}{X}\frac{d^2 X}{dx^2} = -\left(\frac{1}{Y}\frac{d^2 Y}{dy^2} + \alpha^2\right) = -p^2 \ (say)$$

so that $\frac{d^2X}{dx^2}+p^2X=0$ and $\frac{d^2Y}{dy^2}+q^2Y=0$ which have solutions

$X=A\cos px+B\sin px$, and $Y=C_1\cos qy+D_1\sin qy$ where $q^2=\alpha^2-p^2$.

Thus the general solution of the equation (5.32) is

$$T(x,y,t)=(A\cos px+B\sin px)(C\cos qy+D\sin qy)e^{-\kappa\alpha^2 t} \tag{5.34}$$

where $C=C_1A_1$, $D=D_1A_1$

Example 5.6 : The boundaries of the rectangel $0\le x\le a$, $0\le y\le b$ are maintained at zero temperature. If a t = 0, the temperature T has the prescribed value f (x, y), show that for t > 0, the temperature at a point within the rectangle is given by

$$T(x,y,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}F(m,n)\sin\left(\frac{m\pi x}{a}\right)e^{-\kappa\alpha_{mn}^2 t}$$

where $f(x,y)$ is supposed to be expansible in double Fourier series and

$$F(m,n)=\frac{4}{ab}\int_0^a\int_0^b f(x,y)\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)dxdy, \alpha_{mn}^2=\pi^2\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)$$

Solution : The governing equation is the heat conduction equation (diffusion equation) given by (5.32) as

$$\kappa\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right)=\frac{\partial T}{\partial t}$$

whose solution as obtained in (5.34) is

$$T(x,y,t)=(A\cos px+B\sin px)(C\cos qx+D\sin qx)e^{-\kappa\alpha^2 t}$$

where $\alpha^2=p^2+q^2$.

Now the boundary conditions $T(o,y,t)=0$ and $T(x,o,t)=0$ respectively imply that A = 0 and C= 0 so that

$$T(x,y,t)=BD\sin px\sin qye^{-\kappa\alpha^2 t}.$$

Also the boundary conditions $T(x,y,t)=0$ gives $\sin pa=0$ i.e. $p=\frac{m\pi}{a}$, $=1,2,\ldots$ and the condition $T(x,b,t)=0$ shows $\sin qb=0$ i.e. $q=\frac{n\pi}{b}, n=1,2.....$

Hence by superposition principle we have

$$T(x,y,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}A_{mn}\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)e^{-\kappa\pi^2\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)}$$

i.e.
$$T(x,y,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}A_{mn}\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)e^{-\kappa\alpha^2{}_{mn}t}$$

Finally using the initial condition $T(x,y,0)=f(x,y)$, we have

$$f(x,y)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}A_{mn}\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)$$

which is a double Fluier series, so that

$$A_{mn}=\frac{4}{ab}\int_0^a\int_0^b f(x,y)\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)dxdy$$

Hence the required solution is

$$T(x,y,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}F(m,n)\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)e^{-\kappa\alpha^2{}_{mn}t}$$

where $F(m,n)=\frac{4}{ab}\int_0^a\int_0^b f(x,y)\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right), \alpha_{mn}^2=\pi^2\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)$

II. Plane polar coordinates (re)

Here the diffusion equation is

$$\kappa\left(\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}+\frac{1}{r^2}\frac{\partial^2 T}{\partial\theta^2}\right)=\frac{\partial T}{\partial t}$$

We put $T(r,\theta,t)=R(r).\Theta(\theta)\tau(t)$ into equation and obtain

$$\frac{1}{R}\frac{d^2R}{dr^2}+\frac{1}{Rr}\frac{dR}{dr}+\frac{1}{r^2\Theta}\frac{d^2\Theta}{d\theta^2}=\frac{1}{\kappa\tau}\frac{d\tau}{dt}=\lambda^2.\ \text{(say)} \tag{5.36}$$

where λ^2 is a separation constant. Then it follows that

$$\tau(t) = Ex^{-\kappa\lambda^2 t}$$

E. being integration constant. Also from (5.36) we have

$$r^2\left(\frac{1}{R}\frac{d^2R}{dr^2}+\frac{1}{Rr}\frac{dR}{dr}+\lambda^2\right)=-\frac{1}{\Theta}\frac{d^3\Theta}{d\theta^2}=\Omega^2 \text{ (say)}$$

where Ω^2 is a separation parameter. This leads to the two equations

$$\frac{d^2R}{dr^2}+\frac{1}{r}\frac{dR}{dr}+\left(\lambda^2-\frac{\Omega^2}{r^2}\right)R=0 \text{ and } \frac{d^2\Theta}{d\theta^2}+\Omega^2\Theta=0$$

which have solutions

$$R(r)=AJ_\Omega(\lambda r)+BY_\Omega(\lambda r) \text{ and } \Theta(\theta)=C_1\cos\Omega\theta+D_1\sin\Omega\theta$$

Hence the solution of the equation (5.35) is given by

$$T(r,0,t)=\{AJ_\Omega(\lambda r)+BY_\Omega(\lambda r)\}(C\cos\Omega\theta+D\sin\Omega\theta)e^{-\kappa\Omega^2 t} \quad (5.37)$$

where $C=C_1E$ and $D=D_1E$

Example 5.7. : Find the temperature in a long cylindrical region bounded by the planes r - *a*, $\theta = 0$ and $\theta = \pi$ which are maintained at zero temperature and its initial temperature is. $f(r,\theta)$

Solution : The solution of the heat conduction (diffusion) equation is given by (5.37) as

$$T(r,0,t)=\{AJ_\Omega(\lambda r)+BY_\Omega(\lambda r)\}(C\cos\Omega\theta+D\sin\Omega\theta)e^{-\kappa\Omega^2 t}$$

Since T must be finite at r = 0 where $Y_\Omega(\lambda r)$ is undefined, we must put $B=0$. The boundary condition $T = 0$ at $\theta = 0$ and $\theta = \pi$ give C = 0.and $\sin\Omega\pi=0$ i.e. $\Omega=n(n=1,2,3,....)$ Also the boundar condition $T = 0$ at $r = a$ gives $J_\Omega(\lambda a)=0$ i.e. $J_n(\lambda a)=0$ Let λn_1, λn_2, λn_3 .... be the roots of this equation. Thus the solution for the temperature is given, by the use of superposition principle, as

$$T(r,\theta,t)=\sum_{i=1}^{\infty}\sum_{n=1}^{\infty}A_{ni}J_n(\lambda_i r)\sin n\theta e^{-\kappa\lambda^2{}_{n,i}t}$$

The initial condition $T(r,\theta,0)=f(r,\theta)$ gives

$$T(r,\theta)=\sum_{i=n}^{\infty}\sum_{n=1}^{\infty}A_{ni}J_n(\lambda_i r)\sin n\theta$$

Multiplying both sides by $r\,J_n(\lambda_j r)\sin p\theta$ and performing double integration with respect to r and θ for $0 < r < a$ and $0 < \theta < \pi$ respectively, we get

$$\int_0^a\int_0^a r(r(r,\theta),J_n(\lambda_j r)\sin p\theta d\theta$$

$$=\sum_{m-1}^{\infty}\sum_{n-1}^{\infty}A_{n,j}\int_0^a rJ_n(\lambda_i r)J_n(\lambda_j r)\left\{\int_0^\pi \sin p\theta\sin n\theta d\theta\right\}dr$$

noting that $\int_0^\pi \sin p\theta\sin n\theta d\theta=\begin{cases}\pi/2 & for\ p=n\\ 0 & forp\neq n\end{cases}$

and $\int_0^\pi rJ_n(\lambda_i r)J_n(\lambda_j r)dr=\begin{cases}\dfrac{a^2}{2}J^2(\lambda_1 a) & for\ i=j\\ 0 & for\ i\neq j\end{cases}$

we have $\int_0^a\int_0^\pi r\,f(r.\theta)J_n(\lambda_i r)dr=\dfrac{\pi a^2}{4}J_n^2(\lambda_i a)A_{nj}$

so that $A_{nj}=\dfrac{4}{\pi a^2 J_n^2(\lambda_1 a)}\int_0^a\int_0^\pi rf(r,\theta)j_n(\lambda_i r)\sin n\theta d\theta dr.$

Hence the required solution for the temperature is

$$T(r,\theta,t)=\frac{4}{\pi a^2}\sum_{i=1}^{\infty}\sum_{n=1}^{\infty}A_{ni}\frac{B_{n,j}J_n(\lambda_i r)}{J_n^2(\lambda_i a)}\sin n\theta e^{-\kappa\lambda^2 t}$$

where $B_{n,j}=\int_0^a\int_0^\pi r\,f(r,\theta),J_n(\lambda_i r)\sin n\theta\,d\theta\,dr$

III. Spherical polar coordinates (r, θ, ϕ) with axial symmetry

In this case, the diffnsion equation is

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial T}{\partial\theta}\right)=\frac{1}{\kappa}\frac{\partial T}{\partial t}, \tag{5.38}$$

since for axial symmetry, T is independent of ϕ.

We put $T(r,\theta,t)=R(r)\Theta(\theta)\tau(t)$. Then the equation (5.38) gives

$$\frac{1}{r^2 R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)+\frac{1}{r^2\sin\theta}\cdot\frac{1}{\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)=\frac{1}{\kappa\tau}\frac{d\tau}{dt}=-\lambda^2,\ \text{say} \tag{5.39}$$

where $-\lambda^2$ is a separation parameter. Then $\tau(t)=Ee^{-\kappa\lambda^2 t}$

Again from (5.39) we have

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)+\frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)=-\lambda^2 r^2$$

$$\text{or,}\quad \frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)+\lambda^2 r^2=-\frac{1}{\Theta\sin\theta}\frac{d}{d\Theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)=k$$

where k is a separation constant, We put $k=n(n+1)$. Then the above equation leads to the two equations.

$$\frac{d^2R}{dr^2}+\frac{2}{r}\frac{dR}{dr}+\left(\lambda^2-\frac{n(n+1)}{r^2}\right)R=0 \tag{5.40}$$

$$\text{and}\quad \frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)+n(n+1)\Theta=0 \tag{5.41}$$

To solve the equation (5.40) we put $R(r)=(\lambda r)-\frac{1}{2}S(r)$. Then the equation (5.40) reduces to the Bessel equation

$$\frac{d^2S}{dr^2}+\frac{1}{r}\frac{dS}{dr}+\left\{\lambda^2-\frac{\left(n+\frac{1}{2}\right)^2}{r^2}\right\}S=0$$

which has solution $S(r)AJ_{n+\frac{1}{2}}(\lambda r)+BY_{n+\frac{1}{2}}(\lambda r)$ so that

$$R(r)=(\lambda r)^{-\frac{1}{2}}\left[AJ_{n+\frac{1}{2}}(\lambda r)+BY_{n+\frac{1}{2}}(\lambda r)\right]$$

Again putting $\cos\theta = \mu$, the equation (5.41) reduces to Legendre equation

$$\frac{d}{d\mu}\left[\left(1-\mu^2\right)\frac{d\Theta}{d\theta}\right] + n(n+1)\Theta = 0$$

Which has solution $\Theta(\theta) = A_1 P_n(\mu) + B_1 Q_n(\mu)$ where $P_n(\mu)$ and $Q_n(\mu)$ are the Legendre functions of the first kind and second kind respectively.

Thus the solution of the equation (5.38), by using the principle of superposition, is

$$T(r,\theta,t) = \sum_{n=1}^{\infty}(\lambda r)^{-\frac{1}{2}}\left[A_n J_{n+\frac{1}{2}}(\lambda r) + B_n Y_{n+\frac{1}{2}}(\lambda r)\right]$$

$$\left[C_n P_n(\cos\theta) + D_n Q_n(\cos\theta)\right]e^{-\kappa\lambda^2 t} \qquad (5.42)$$

Example 5.8 : Find the temperature is a sphere of radius a, when its surface is kept at zero temperature and its initial temperature is $f(r,\theta)$.

Solution : Noting that $Y_{n+\frac{1}{2}}(\lambda r)$ and $Q_n(\cos\theta)$ are unbounded at $r=0$ and $\theta = \frac{\pi}{2}$ respectively, we write the general solution of (5.42) of (5.38) in the form

$$T(r,\theta,t) = \sum_{\lambda,n} A_{n,i}(\lambda r)^{-\frac{1}{2}} J_{n+\frac{1}{2}}(\lambda r) P_n(\cos\theta) e^{-\kappa\lambda^2 t}$$

Applying boundary condition $T(r,\theta,t) = 0$ at $r = a$ we get $J_{n+\frac{1}{2}}(\lambda a) = 0$

Let $\lambda_1, \lambda_2, \ldots$ are the roots of this equation. Then we have

$$T(r,\theta,t) = \sum_{i=1}^{\infty}\sum_{n=1}^{\infty} A_{n,i}(\lambda_i r)^{-\frac{1}{2}} J_{n+\frac{1}{2}}(\lambda_i r) P_n(\cos\theta) e^{-\kappa\lambda_i^2 t}$$

Applying the initial condition $T(r,\theta,0) = f(r,\theta)$ we get

$$f(r,\theta) = \sum_{i=1}^{\infty}\sum_{n=1}^{\infty} A_{n,i}(\lambda_i r)^{-\frac{1}{2}} J_{n+\frac{1}{2}}(\lambda_i r) P_n(\cos\theta)$$

Multipling both sides by $P_m(\cos\theta)d(\cos\theta)$ and integrating between -1 to 1, we get

$$\int_{-1}^{1} f(r,\theta)\, P_m(\cos\theta)d(\cos\theta) = \sum_{i=1}^{\infty}\sum_{n=1}^{\infty} A_{n,i}(\lambda_i r)^{\frac{1}{2}}$$

$$\int_{-1}^{1} P_m(\cos\theta) P_n(\cos\theta) d(\cos\theta)$$

i.e. $\int_{-1}^{1} f(r,\theta)P_n(\cos\theta)d(\cos\theta) = \sum_{i=1}^{\infty}\sum_{n=1}^{\infty} A_{n,i}(\lambda_i r)^{-\frac{1}{2}} J_{n+\frac{1}{2}}(\lambda_i r)\cdot\frac{2}{2n+1}$, if $n = m$ where we have used the orthogonal property of Legendre polynomials.

Again, multiplying both sides by $r^{3/2} J_{n+\frac{1}{2}}(\lambda_j r)$ and integratng with respect to r between the limits 0 to a, we have

$$\int_0^a r^{3/2} J_{n+\frac{1}{2}}(\lambda_j r)\left[\int_{-1}^{1} f(r,\theta) P_n(\cos\theta) d(\cos\theta)\right] dr$$

$$= \sum_{i=1}^{\infty}\sum_{n=1}^{\infty} A_{n,i}(\lambda_i)^{-\frac{1}{2}} \frac{2}{2n+1}\int_0^a r J_{n+\frac{1}{2}}(\lambda_i r) J_{n+\frac{1}{2}}(\lambda_j r) dr$$

$$= A_{n,i}\lambda_i^{-\frac{1}{2}} \frac{2}{2n+1}\cdot\frac{a^2}{2}\cdot J_{n+\frac{1}{2}}(\lambda_i a) \ \text{ for } i = j$$

Thus

$$A_{n,i} = \frac{2n+1}{a^2}\frac{\lambda_i^{\frac{1}{2}}}{J_{n+\frac{1}{2}}(\lambda_i a)}\int_0^a\int_{-1}^{1} r^{\frac{3}{2}} J_{n+\frac{1}{2}}(\lambda_i r) P_n(\mu) f(r,\theta) d\mu dr$$

Hence the required solution is

$$T(r,\theta,t) = \sum_{i=1}^{\infty}\sum_{n=1}^{\infty}\frac{(2n+1)B_{n,i}}{a^2 J^2{}_{n+\frac{1}{2}}(\lambda_i a)} r^{-\frac{1}{2}} J_{n+\frac{1}{2}}(\lambda_i r) P_n(\mu) e^{-\kappa\lambda_i^2 t}$$

where

$$B_{n,i} = \int_0^a\int_{-1}^{1} r^{3/2} J_{n+\frac{1}{2}}(\lambda_i r) P_n(\mu) f(r,\theta) d\mu dr \ \text{ and } \ \mu = \cos\theta$$

§ 5.7. Solution of Diffusion Equation in Three-Dimension : Separation of Variables Method.

I Cartesian corrdinates (x,y,z)

The diffusion equation is given by

$$\kappa\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}\right)=\frac{\partial T}{\partial t} \tag{5.43}$$

Let $T(x,y,z,t)=X(x)Y(y)Z(z)\tau(t)$. Then the equation (5.43) gives

$$\frac{1}{X}\frac{d^2X}{dx^2}+\frac{1}{Y}\frac{d^2Y}{dy^2}+\frac{1}{Z}\frac{d^2Z}{dz^2}=\frac{1}{\kappa\tau}\frac{d\tau}{dt}=-\lambda^2 \text{ (say)} \tag{5.44}$$

where $-\lambda^2$, is a separation constant. Thus $\tau(t)=Ge^{-\kappa\lambda^2 t}$

Now form (5.44) we have,

$$\frac{1}{X}\frac{d^2X}{dx^2}+\frac{1}{Y}\frac{d^2Y}{dy^2}=-\frac{1}{Z}\frac{d^2Z}{dz^2}-\lambda^2=-\mu^2 \text{ (say)} \tag{5.45}$$

where $-\mu^2$ is a separation constant, Then

$$\frac{d^2Z}{dz^2}+\gamma^2 Z=0$$

where $\gamma^2=\lambda^2-\mu^2$. The solution of the is equation is $Z(z)=E_1\cos\gamma z+F_1\sin\gamma z$.

Again from (5.45) we get

$$\frac{1}{X}\frac{d^2X}{dx^2}=-\frac{1}{Y}\frac{d^2Y}{dy^2}-\mu^2=-\alpha^2 \text{ (say)} \tag{5.46}$$

where $-\alpha^2$ is a separation constant. Then we obtain from (5.46) the equations

$$\frac{d^2X}{dx^2}+\alpha^2X=0 \text{ and } \frac{d^2Y}{dy^2}+\beta^2Y=0$$

where $\beta^2=\mu^2-\alpha^2$ The solutions of these equations are

$$X(x)=A\cos\alpha x+B\sin\alpha x, Y(y)=C\cos\beta y+D\sin\beta y$$

respectively,

Hence the solution of the equations (5.43) is given by

$$T(x,y,z,t)=(A\cos\alpha x+B\sin\alpha x)(C\cos\beta y+D\sin\beta y)(E\cos\gamma z+F\sin\gamma z)e^{-\kappa\lambda^2 t} \quad (5.47)$$

where $E=E_1G, F=F_1G$ and $\lambda^2=\alpha^2+\beta^2+\gamma^2$

Example 5.9 : The faces of the solid parallelopided $0\le x\le a, 0\le y\le b, 0\le z\le c$ are kept at zero temperature. If initially, the temperature of the solid is given by $T(x,y,z,0)=f(x,y,z)$, show that at time $t>0$

$$T(x,y,z,t)=\frac{8}{abc}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{q=1}^{\infty}F(m,n,q)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\sin\frac{q\pi z}{c}e^{-\mu^2 t}$$

where

$$F(m,n,q)=\int_0^a\int_0^b\int_0^c f(x,y,z)\sin\frac{m\pi x}{a}\cdot\sin\frac{n\pi y}{b}\cdot\sin\frac{q\pi z}{c}\,dx\,dy dz$$

and $\mu^2=\kappa\pi^2\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}+\frac{q^2}{c^2}\right)$

Solution : The solution of the heat conduction equation (diffusion equation) (5.43) is given by (5.47) as

$$T(x,y,z,t)=(A\cos\alpha x+B\sin dx)(C\cos\beta y+D\sin\beta y)(E\cos\gamma z+F\sin\gamma z)e^{-\kappa\lambda^2 t}$$

where $\lambda^2 = \alpha^2 + \beta^2 + \gamma^2$.

Now the boundary conditions $T(0,y,z)=0 T(x,0,z)=0$ and $T(x,y,0)=0$ give $A=C=E=0$ so that

$$T(x,y,z,t) = K\sin\alpha x\sin\beta y\sin\gamma z e^{-\kappa\gamma^2 t}$$

where $K=BDF$. The boundary conditions $T(a,y,z)=0, T(x,b,z)=0, T(x,y,c)=0$ give respectively $\sin\alpha a=0$, $\sin\beta b=0$, $\sin\gamma c=0$ leading to $\alpha=\frac{m\pi}{a}, \beta=\frac{n\pi}{b}, \gamma=\frac{q\pi}{c}$ where $m=1,2,3,\ldots$; $n=1,2,3,\ldots$ and $q=1,2,3,\ldots$. Thus by the use of superposition principle, we have the solution

$$T(x,y,z,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{q=1}^{\infty} K_{mnq}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\sin\frac{q\pi z}{c}e^{-\mu^2 t}$$

where $\mu^2 = \kappa\lambda^2 = \kappa\pi^2\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}+\frac{q^2}{c^2}\right)$

Again the initial condition $T(x,y,z,0)=f(x,y,z)$ leads to

$$f(x,y,z) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{q=1}^{\infty} K_{mnq}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\sin\frac{q\pi z}{c}$$

which is a triple Fourier series, so that

$$K_{mnq} = \frac{1}{a}\cdot\frac{1}{b}\cdot\frac{2}{c}\int_0^a\int_0^b\int_0^c f(x,y,z)\sin\frac{m\pi x}{a}\cdot\sin\frac{n\pi y}{b}\cdot\sin\frac{q\pi z}{c}\,dx\,dydz$$

Hence the required solution is

$$T(x,y,z,t) = \frac{8}{abc}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{q=1}^{\infty} F(m,n,q)\sin\frac{m\pi x}{a}\cdot\sin\frac{n\pi y}{b}\cdot\sin\frac{q\pi z}{c}e^{-\mu^2 t}$$

where $F(m,n,q) = \int_0^a\int_0^b\int_0^c f(x,y,z)\sin\frac{m\pi x}{a}\cdot\sin\frac{n\pi y}{b}\cdot\sin\frac{q\pi z}{c}\,dx\,dydz$

II. Cylindrical coordinates (r,θ,z)

In cylindrical corrdinates (r,θ,z) the diffusion equation is

$$\kappa\left(\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}+\frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2}+\frac{\partial^2 T}{\partial z^2}\right)=\frac{\partial T}{\partial t} \tag{5.48}$$

We assume separation of variables in the form $T(r,\theta,z,t)=R(r)\Theta(\theta).Z(z).\tau(t)$ Substituting this in (5.48) we get

$$\frac{1}{R}\frac{d^2 R}{dr^2}+\frac{1}{Rr}\frac{dR}{dr}+\frac{1}{r^2\Theta}\frac{d^2\Theta}{d\theta^2}+\frac{1}{Z}\frac{d^2 Z}{dz^2}=\frac{1}{\kappa}\frac{1}{\tau}\frac{dt}{dt}=-\lambda^2 \text{ (say)} \tag{5.49}$$

where $-\lambda^2$ is a separation constant Then $\tau(t)=Ge^{-\kappa\lambda^2 t}$

Now from (5.49) we have

$$\frac{1}{R}\frac{d^2 R}{dr^2}+\frac{1}{Rr}\frac{dR}{dr}+\frac{1}{r^2\Theta}\frac{d^2\Theta}{d\theta^2}+\lambda^2=-\frac{1}{Z}\frac{d^2 Z}{dz^2}=-\mu^2, \text{ (say)} \tag{5.50}$$

where $-\mu^2$ is a separation constant. Then we have

$$\frac{d^2 Z}{dx^2}-\mu^2 Z=0$$

whose solution is $Z(z)=E_1 e^{\mu z}+F_1 e^{-\mu z}$

Again from (5.40), we obtain

$$\frac{1}{R}\frac{d^2 R}{dr^2}+\frac{1}{Rr}\frac{dR}{dr}+\alpha^2 r^2=-\frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2}=\upsilon^2 \text{ (say)} \tag{5.51}$$

where $\alpha^2=\lambda^2+\mu^2$ and υ^2 is a separation constant. Therefore this leads us

$$\frac{d^2 R}{dr^2}+\frac{1}{r}\frac{dR}{dr}+\left(\alpha^2-\frac{\upsilon^2}{r^2}\right)R=0$$

and
$$\frac{d^2\Theta}{d\theta^2}+\upsilon^2\Theta=0$$

whose solutions are

$$R(r) = AJ_\upsilon(\alpha r) + BY_\upsilon(\alpha r)$$

and $$\Theta(\theta) = C\cos\upsilon\theta + D\sin\upsilon\theta,$$

respectively and $J_\upsilon(\alpha r), Y_\upsilon(\alpha r)$ are Bessel functions of order υ of the first and second kind respectively. Hence the solution of the diffusion equation (5.48) is

$$T(r,z,t) = \{AJ_\upsilon(\alpha r) + B\gamma_\upsilon(\alpha r)\}.\ (C\cos\upsilon\theta + D\sin\upsilon\theta)\left(Ee^{\mu z} + Fe^{-\mu z}\right)e^{-\kappa\lambda^2 t} \quad (5.52)$$

where $E = E_1 G$ and $F_1 = FG$

Example 5.10 : The cylinder bounded by the planes $z = 0$ and $z = l$ and the curved surface $r = a$ in cylindrical coordinates (r, θ, z) has its plane faces maintained at zero temperature and the curved surface at the temperature $f(z)$. Show that the steady temperature distribution within the cylinder is given by

$$T(r,z) = \sum_{n=1}^{\infty} A_n \frac{I_0(n\pi r / l)}{I_0(n\pi a / l)} \sin(n\pi z / l)$$

where $A_n = \frac{2}{l}\int_0^l f(z)\sin(n\pi z / l)dz$ and I_0 is modified Bessel function of first kind of order zero.

Solution : Since the temperature is steady, so $\frac{\partial T}{\partial t} = 0$. Also the temperature is symmetric about the z-axis and, therefore, it is independent of θ. Thus the heat conduction equation (5.48) in cylindrical coordinates reduces to

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} = 0 \quad (5.53)$$

We assume that $T(r,z) = R(r)Z(z)$. Then the above equation gives

$$\frac{1}{R}\left(\frac{d^2 R}{dr^2} + \frac{1}{r}\frac{dR}{dr}\right) = -\frac{1}{Z}\frac{d^2 Z}{\partial Z^2} = \lambda^2, \text{ (say)}$$

where λ^2 is a separation parameter. Thus we have two equations

$$\frac{d^2R}{dr^2}+\frac{1}{r}\frac{dR}{dr}-\lambda^2R=0 \text{ and } \frac{d^2Z}{\partial Z^2}+\lambda^2Z=0$$

which have solutions $R(r)=AI_0(\lambda r)+BK_0(\lambda r)$ and $Z=C\cos\lambda z+D\sin\lambda z$ respectively. Here I_0 and K_0 are modified Bessel functions of the first and second kind respectively, each of order zero. Hence the solution of the equation (5.53) is given by

$$T(r,z)=\{AI_0(\lambda r)+BK_0(\lambda r)\}(C\cos\lambda z+D\sin\lambda z)$$

Now $K_0(\lambda r)$ has a singularity at $r=0$. So we choose $B=0$. Also the boundary conditions $T(r,0)=0$ and $T(r,l)=0$ give $C=0$ and $\sin\lambda l=0$ respectively and therefore $\lambda=\frac{n\pi}{l}(n=1,2,3,...)$. Thus we have by superposition principle

$$T(r,z)=\sum_{n=1}^{\infty}B_nI_0\left(\frac{n\pi r}{l}\right)\sin\left(\frac{n\pi z}{l}\right)$$

Again the boundary condition $T(a,z)=f(z)$ on the curved surface leads us to

$$f(z)=\sum_{n=1}^{\infty}B_nI_0\left(\frac{n\pi r}{l}\right)\sin\left(\frac{n\pi z}{l}\right)$$

Multiplying both sides by $\sin\left(\frac{n\pi z}{l}\right)$ and integrating w.r.t.z between 0 to l,

we have

$$B_nI_0\left(\frac{n\pi a}{l}\right)=\frac{2}{l}\int_0^l f(z)\sin\left(\frac{n\pi z}{l}\right)dz$$

Hence the required solution is

$$T(r,z)=\sum_{n=1}^{\infty}A_n\frac{I_0\left(\frac{n\pi r}{l}\right)}{I_0\left(\frac{n\pi a}{l}\right)}\sin\left(\frac{n\pi z}{l}\right)$$

where $A_n = \frac{2}{l}\int_0^l f(z)\sin\left(\frac{n\pi z}{l}\right)dz$

III. Spherical palar coordinates (r, θ, z)

The diffusion equation in spherical polar corrdinates (r,θ,z) is given by

$$\kappa\left[\frac{\partial^2 T}{\partial r^2}+\frac{2}{r}\frac{\partial T}{\partial r}+\frac{1}{r^2\sin\theta}\frac{\partial}{d\theta}\left(\sin\theta\frac{\partial T}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 T}{\partial\phi^2}\right]=\frac{\partial T}{\partial t} \qquad (5.54)$$

Let its solution be $T(r,\theta,z,t)=R(r)\Theta(\theta)\Phi(\phi)\tau(t)$. Substituting this in (5.54) we get

$$\frac{1}{R}\frac{d^2R}{dr^2}+\frac{2}{r}\frac{dR}{dr}+\frac{1}{r^2\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{r\theta}\right)$$

$$+\frac{1}{r^2\sin^2\theta}\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2}=\frac{1}{\kappa}\frac{1}{\tau}\frac{d\tau}{dt}=-\lambda^2 \text{ (say)} \qquad (5.55)$$

where $-\lambda^2$ is a separation parameter. Thenwe have $\tau(t)=Ge^{-\kappa\lambda^2 t}$

Also (5.55) gives

$$r^2\sin^2\theta\left[\frac{1}{R}\left(\frac{d^2R}{dr^2}+\frac{2}{r}\frac{dR}{dr}\right)+\frac{1}{r^2\sin\theta}\frac{1}{\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)+\lambda^2\right]$$

$$=-\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2}=\upsilon^2 \text{ say} \qquad (5.46)$$

where υ^2 is a separation parameter. This gives

$$\frac{d^2\Phi}{d\phi^2}+\upsilon^2\Phi=0$$

whose solution is $\Phi(\phi)=E_1\cos\upsilon\phi+F_1\sin\upsilon\phi$

Again from (5.56), we derive

$$\frac{r^2}{R}\left(\frac{d^2R}{dr^2}+\frac{2}{r}\cdot\frac{dR}{dr}\right)+\lambda^2r^2=\frac{\upsilon^2}{\sin^2\theta}-\frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)=n(n+1)\text{ (say)}$$

so that, on rearrangement, we have

$$\frac{d^2R}{dr^2}+\frac{2}{r}\cdot\frac{dR}{dr}+\left\{\lambda^2-\frac{n(n+1)}{r^2}\right\}R=0 \tag{5.57}$$

and
$$\frac{d^2\Theta}{d\theta^2}+\cot\theta\frac{d\Theta}{d\theta}+\left\{n(n+1)-\frac{\upsilon^2}{\sin^2\theta}\right\}\Theta=0 \tag{5.58}$$

To solve the equation (5.57) we put $R(r)=(\lambda r)^{-\frac{1}{2}}S(r)$. Then the equation provides us

$$\frac{d^2S}{dr^2}+\frac{1}{r}\frac{dS}{dr}+\left\{\lambda^2-\frac{\left(n+\frac{1}{2}\right)^2}{r^2}\right\}S(r)=0$$

whose solution is

$$S(r)=AJ_{n+\frac{1}{2}}(\lambda r)+BY_{n+\frac{1}{2}}(\lambda r)$$

so that $R(r)=(\lambda r)^{-\frac{1}{2}}\left[AJ_{n+\frac{1}{2}}(\lambda r)+BY_{n+\frac{1}{2}}(\lambda r)\right]$

Also we put $\cos\theta=\mu$ in equation (5.58) to reduce it to the form

$$\left(1-\mu^2\right)\frac{d^2\Theta}{d\mu^2}-2\mu\frac{d\Theta}{d\mu}+\left\{n(n+1)-\frac{\upsilon^2}{1-\mu^2}\right\}\Theta=0$$

This is *associated Legendre differential equation* whose solution is

$$\Theta=CP_n^{\upsilon}(\mu)+DQ_n^{\upsilon}(\mu)$$

where P_n^{υ} and Q_n^{υ} are *associated Legendre functions of degree n and of order* υ *of first and second kind* respectively.

Thus the general solution of the equation (5.54) is

$$(\lambda r)^{-\frac{1}{2}}\left[AJ_{n+\frac{1}{2}}(\lambda r)+BY_{n+\frac{1}{2}}(\lambda r)\right]\left[CP_n^{\upsilon}(\cos\theta)+\right.$$

$$+DQ_n^{\mu}(\cos\theta)\big]\left(E\cos\upsilon\phi+F\sin\upsilon\phi\right)e^{-\kappa\lambda^2 t} \tag{5.59}$$

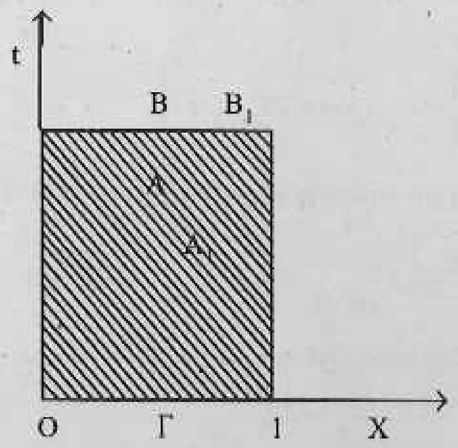
§ 5.8. Maximum-Minimum Priniciple and its Consequences.

***Theorem 5.1 (Maximum-Minimum Principle)** : Let $T(x,t)$ be a continuous function of x and t and is a solution of the diffusion equation*

$$\kappa\frac{\partial^2 T}{\partial x^2}=\frac{\partial T}{\partial t}$$

for $0\leq x\leq L, 0\leq t\leq \tau$ where $\tau>0$ is a fixed time. Then the function $T(x,t)$ attains its maximum and minimum values at time $t=0$ or at the end points $x=0$ and $x=L$ at some time t in the interval $0\leq t\leq \tau$

Proof. Consider the region $0\leq x\leq L, 0\leq t\leq \tau$ of the (x,t) plane, where τ is some chosen time. Then as in Figure 5,1, $T(x,t)$ is given in darker horizontal and vertical line; the darker portion of the boundary is denoted by Γ. Since the direction of heat flow between two bodies (or parts of the same body) in contact is always from a higher to a lower temperature, we expect that the temperature $T(x,t)$ in the shaded region attains its maximum of Γ.



To prove this, we assume the contrary, i.e, we can find a point (x_0,t_0) which is either an interior point A or an upper boundary point B, such that $T(x_0,t_0)$ is greater than the least upper bound of $T(x,t)$ on Γ. Let us define an auxiliary function $\psi(\lambda,t)$ by

$$\psi(x,t)=T(\lambda,t)-\in(t,t_0),$$

where $\in > 0$ is a constant. Since $\psi(x_0, t_0) = T(x_0, t_0)$ which exceeds by some definite amout the greatest value of $T(x,t)$ on Γ, $\in$ can be chosen so small that $\psi(x_0, t_0) >$ Max $\psi(x,t)$ on Γ. Thus $\psi(x,t)$ attains its maximum not on Γ, say at A_1 or B_1. At this maximum point, we have

$\psi_{xx} \le 0, \psi \ge 0$ so that $T_{xx} \le 0, T_t > 0$

But this contradicts the requirement that $T_t = \kappa T_{xx}$ at this point; hence the maximum principle is proved.

Similarly, we can prove the minimum principle.

Theorem 5.2 Uniqueness theorem : *It $T(x,t)$ is a solution of the diffusion equation*

$$\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2}, \; 0 \le x \le L, 0 \le t \le \tau \tag{5.61}$$

subject to the initial condition $T(x,0) = f(x)$ and the boundary conditions $T(0,t) = g_1(t)$ and $T(L,t) = g_2(t)$, where $f(x)$, $g_1(t)$ and $g_2(t)$ are continuous functions on their definitions, then the solution in uniques.

Proof : It possible, let $T_1(x,t)$ and $T_2(x,t)$ are two solutions of the diffusion equation (5.61) satisying the given initial and boundary conditions. Now the function $\vartheta(x,t) = T_2(x,t) - T_1(x,t)$ is also a solution of the equation (5.61) in $0 \le x \le L, 0 \le t \le \tau$ and is a continuous function of x and t. Also $\vartheta(x,0) = 0$ in $0 \le x \le L$ and $\vartheta(0,t) = \vartheta(L,t) = 0$ in $0 \le t \le \tau$ Hence $\vartheta(x,t)$ satisfies the condition of maximum-minimum principle (Theorem 5.1) and, therefore, $\vartheta(x,t) = 0$ for $0 \le x \le L$ and $0 \le t \le \tau$. This implies that $T_2(x,t) - T_1(x,t) = 0$, i.e. $T_2(x,t) = T_1(x,t)$ Hence the equation (5.61) has a unique solution.

Another important consequence of the maximum-minimum principle is the stability property which we state in the following theorem without proof.

Theorem 5.3 (Stability property) : *The solution $T(x,t)$ of the diffusion equation for Dirichlet conditions defined by*

$$\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2}, \; 0 \le x \le L, \; 0 \le t \le \tau$$

$T(x,0) = f(x), 0 \le x \le L$ and $T(o,t) = g_1(t), T(L,t) = g_2(t), 0 \le t \le \tau$ depends continuously on the inital and boundary conditions.

Exercises

1. A uniform rod of length L whose surface is thermally insulated is initially at temperature $T = T_0$. At time $t = 0$, one end is suddenly cooled to $T = 0$ and subsequently maintained at this temperature; the other end remains thermally insulated. Find the temperature distribution $T(x,t)$.

$$\left[\text{Ans. } T(x,t) = \frac{4T_0}{\pi}\sum_{n=0}^{\infty}\frac{1}{2n+1}\sin\left(\frac{2n+1}{2L}\pi x\right)e^{-\kappa\left(\frac{2n+1}{2L}\right)^2\pi^2 t}\right]$$

2. Find the solution of the one-dimensional diffusion equation diffusion equation satisfying the inital condition $T(x,0) = x(a-x), 0 < x < a$, the regularity T is bounded as $t \to \infty$ and the boundary condition $\frac{\partial T(0,t)}{\partial x} = \frac{\partial T(a,t)}{\partial x} = 0, \forall t$

$$\left[\text{Ans. } T(x,t) = \frac{a^2}{6} - \frac{4a^2}{\pi^2}\sum_{n=\text{even}}\frac{1}{n^2}\cos\frac{n\pi x}{a}e^{-\kappa\frac{n^2\pi^2 t}{a^2}}\right]$$

3. Determine the temperature $T(r,t)$ in the infinite cylinder $0 \le r \le a$ when the initial temperature is $T(r,0) = f(r)$ and the surface $r = a$ is maintained at 0° temperature.

$$\left[\text{Ans. } T(r,t) = \frac{2}{a^2}\sum_{n=1}^{\infty}\frac{J_0(\xi_n r)}{J_1^2(\xi_n a)}e^{-\kappa\xi_n^2 t}\left\{\int_0^a u f(u)\, J_0(\xi_n u)du\right\},\right.$$

$\xi_n (n = a,2,3,....)$ being the roots of the equation $J_0(\xi a) = 0]$

4. A circular gylinder of radius a has its surface kept at a constant temperature T_0. If the initial temperature is zero throughout the cylinder, prove that for $t > 0$

$$T_0\left\{1 - \frac{2}{a}\sum_{n=1}^{\infty}\frac{J_0(\xi_n r)}{\xi_n J_1(\xi_n a)}e^{-\kappa\xi_n^2 t}\right\}$$

where $\xi_n (n = 1,2,3....)$ are the roots of the quation $J_0(\xi a) = 0$

5. A conducting bar of uniform cross-section lies along the x-axis, with its ends at $x=0$ and $x=l$. The lateral surface is insulated. There are no heat sources within the body. The ends are also insulated. The initial temperature is $lx-x^2, 0 \le x \le l$. Find the temperature disbribution in the bar for $t>0$.

$$\left[\text{Ans. } T(x,t)=\sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{l}\right)e^{-\kappa n^2\pi^2 t/l^2},\ A_n=\frac{2}{l}\int_0^l \left(l\lambda - x^2\right)\cos\frac{n\pi x}{l}\,dx=\ldots\right]$$

6. Solve the equation $\frac{\partial T}{\partial t}=\frac{\partial^2 T}{\partial x^2}$ satisfying the conditions $T(0,t)=T(1,t)=0\forall t$ and

$$T(x,0)=\begin{cases} 2x, 0\le x\le \frac{1}{2} \\ 2(j-x), \frac{1}{2}\le x\le 1 \end{cases}$$

$$\left[\text{Ans. } T(x,t)=\frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{2}\sin(n\pi x)e^{-n^2\pi^2 t}\right]$$

7. A homogeneous solid sphere of radius a has the initial temperature distribution $f(r), 0\le r\le a$ where r is the distance measured from the centre. The surface temperature is maintained at 0°. Show that the temperature in the sphere for $t>0$ is given by

$$T(x,t)=\frac{1}{r}\sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{a}r\right)e^{-\kappa n^2\pi^2 t/a^2}$$

where $B_n=\frac{2}{a}\int_0^a rf(r)\sin\left(\frac{n\pi}{a}r\right)dn$

8. The ends A and B of a rod, 10 cm in lenght are kept at temperature 0°C and 100°C respectively until the steady state condition prevails. Suddenly the temperature at the end A is increased to 20°C and the end B is decreased to 60°C. Find the temperature distribution in the rod at time t.

$$\left[\text{Ans. } T(x,t)=4x+20-\frac{1}{5}\sum_{n=1}^{\infty}\left[(-1)^n\frac{800}{n\pi}-\frac{200}{n\pi}\right]\sin\left(\frac{n\pi x}{10}\right)e^{-k\frac{n^2\pi^2 t}{100}}\right]$$

9. The edges $x=0,a$ and $y=b$ of the rectangle $0\le x\le a,\ 0\le y\le b$ are maintained at zero temperature while the temperature along the edge $y=0$ is made to vary according to the rule $T(x,0,t)=f(x),\ 0\le x\le a\ \ t>0$. If the initial temperature in the rectangle is zero, find the temperature at any subsequent time t, and deduce that the steady-state temperature is

$$\frac{2}{a}\sum_{m=1}^{\infty}\frac{\sinh[m\pi(b-y)/a]}{\sinh(m\pi b/a)}\sin\left(\frac{m\pi x}{a}\right)\int_0^a f(u)\sin\left(\frac{m\pi u}{a}\right)du$$

§ 5.9 Summary.

The occurrence of parabolic differential equations (diffension/heat conduction) in various fields are mentioned. The separation of variables technique has been applied to solve the diffusion (or heat conduction) equation in different system of coordinates, i.e., Cartesian, cylindrical and spherical polar coordinates. In each system, different problems have been discussed to clarify the technique. Lastly, the maximum-minimum principle and its consequences are also considered.

UNIT 6 ❑ HYPERBOLIC DIFFERENTIAL EQUATIONS

§ 6.1 Introduction

One of the most important homogeneous hyperbolic differential equation is the wave given by

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}=\nabla^2 u \qquad (6.1)$$

where c is the *wave speed*. The differential equation (6.1) is of great use in physics and engineering. The solutions of wave equation are called *wave function*. In this unit, we consider various properties and techniques for solving hyperbolic differential equation of this type.

§ 6.2. The Occurrence of the Wave Equation

We indicate several kinds of situations which arise in physics and engineering and can be discussed by means of the theory of wave equation.

(*a*) *Transverse vibrations of a string.* Let a string of uniform linear density ρ be stretched to a uniform tension T and, in the equilibrium position, the string coincides with the x-axis. Now, if the string is disturbed slightly from its equilibrium position, the transverse displacement $y(x,t)$ satisfies the wave equation

$$\frac{1}{c^2}\frac{\partial^2 \xi}{\partial t^2}=\frac{\partial^2 \xi}{\partial x^2} \qquad (6.2)$$

where $c^2=T/\rho$. At any fixed point $x=a$ of the string, $y(a,t)=0$ for all values of t.

(*b*) *Longitudinal vibrations of a bar.* Let a uniform elastic bar of uniform cross-section whose axis is along the direction of x-axis, be stressed in such a way that each point of a typical cross-section of the bar takes the same displacement $\xi(x,t)$. The ξ satisfies the wave equation

$$\frac{1}{c^2}\frac{\partial^2 \xi}{\partial t^2}=\frac{\partial^2 \xi}{\partial x^2} \qquad (6.3)$$

where $c^2=E/\rho$, E being the Young's modulus and ρ, the density of the material of the bar.

(c) Longitudinal Sound Waves. If plane sound propagate in a horn, of cross-section A(x) and abcissa x in such a way that the section has the same longitudinal displacement $\xi(x,t)$, then ξ satisfies the partial differential equation

$$\frac{\partial}{\partial x}\left\{\frac{1}{A}\frac{\partial}{\partial x}(A\xi)\right\}=\frac{1}{c^2}\frac{\partial^2\xi}{\partial t^2} \qquad (6.4)$$

which reduce to (6.3) for uniform cross-section. In this equation $c^2=\left(\frac{dp}{d\rho}\right)_0$, the suffix indicating that the quantity is to be evaluated in the equilibrium state.

(d) Electrical signals in cables. If in a long insulated cable, the resistance and the leakage parameter are both zero, then te voltage V(x,t) and the current $z(x,t)$ both satisfy the one-dimensional wave equation, the wave velocity c being defined by $c^2=\frac{1}{LC}$, where L is the inductance and C the capacity per unit length.

(e) Transverse vibrations of a membrane. Let a thin elastic membrane of uniform density σ be stretched to a tension T and the membrane coincides with the xy plane in the equilibrium position. Then for small transverse vibrations of the membrane, the transverse displacement $z(x,y,t)$ (assumed small) of any point (x,y) at time t is given by the two-dimensional wave equation

$$\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=\frac{1}{c^2}\frac{\partial^2 z}{\partial t^2} \qquad (6.5)$$

where the wave velocity c is defined by $c^2=T/\sigma$

(f) Sound waves in space. Suppose that because of the passage of a sound wave, the gas at a point (x,y,z) at time t has velocity $v=(u,v,w)$ and p,ρ denote the pressure and density at that point. If the motion of the gas is irrotational then $v=-\operatorname{grad}\phi$ and that the function ϕ satisficcs the wave equation

$$\nabla^2\phi=\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} \qquad (6.6)$$

where $c^2=(dp/d\rho)_0$ the suffix zero indicates that the quantity is to be evaluated in the equilibrium state.

(g) *Electromagnetic waves.* Let $\boldsymbol{H}$ be the magnetic field and $\boldsymbol{E}$, the electric field. If we write $\boldsymbol{H}$=curl$\boldsymbol{A}$, $\boldsymbol{E}=-\frac{1}{c}\frac{\partial A}{\partial t}-\text{grad}\,\phi$, then Maxwall equations

$$\text{div}\boldsymbol{E}=4\pi\rho,\ \text{div}\boldsymbol{H}=0,\ \text{curl}\boldsymbol{E}=-\frac{1}{c}\frac{\partial H}{\partial t},\ \text{curl}\boldsymbol{H}=\frac{4\pi i}{c}+\frac{1}{c}\frac{\partial E}{\partial t}$$

are identically satisfied provided $\boldsymbol{A}$ and ϕ satisfy the equations

$$\nabla^2 A=\frac{1}{c^2}\frac{\partial^2 A}{\partial t^2}-\frac{4\pi i}{c},\quad \nabla^2\phi=\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}-4\pi\rho$$

which, in the absence of charges ρ or currents i, reduce to wave equation. In the above equations, c denotes the velocity of light.

(h) *Elastic waves in solids.* Let $v=(u,v,w)$ be the displacement vector at the point (x,y,z) of a particle of the elastic solid. If we write $v=\text{grad}\,\phi+\text{curl}\,\psi$, in the absence of a body forces, then ϕ and the components of ψ each satisfies a wave equation.

§ 6.3. One-dimensional Wave Equation

Let us consider the one-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2}-c^2\frac{\partial^2 u}{\partial x^2}=0 \tag{6.7}$$

we now find the solution of the equation (6.7) by different methods and different system of coordinates.

I. Solution by canonical reduction

We choose the characteristic lines $\xi=x+ct$, $\eta=x-ct$ so that

$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial\xi}\frac{d\xi}{dx}+\frac{\partial u}{\partial\eta}\frac{\partial\eta}{\partial x}=\frac{\partial u}{\partial\xi}+\frac{\partial u}{\partial\eta}=\left(\frac{\partial}{\partial\xi}+\frac{\partial}{\partial\eta}\right)u$$

Similarly, $\frac{\partial u}{\partial t}=c\left(\frac{\partial}{\partial\xi}-\frac{\partial}{\partial\eta}\right)u$

Also, $\frac{\partial^2 u}{\partial x^2}=\left(\frac{\partial}{\partial \xi}-\frac{\partial}{\partial \eta}\right)^2 u$ and $\frac{\partial^2 u}{\partial t^2}=c^2\left(\frac{\partial}{\partial \xi}-\frac{\partial}{\partial \eta}\right)^2 u$

Substituting these in (6.7) we get

$$\frac{\partial^2 u}{\partial \xi \partial \eta}=0 \tag{6.8}$$

Integrating we have $u(\xi,\eta)=\phi(\xi)+\psi(\eta)$, where ϕ and ψ are arbitrary functions. Expressing ξ and η in terms of the variables defined above, we have

$$u(x,t)=\phi(x+ct)+\psi(x-ct) \tag{6.9}$$

The two terms in (6.9) can be interepreted as waves travelling to the right and left respectively.

For an arbitrary real parameter k,

$$u(x,t)=\phi\{k(x+ct)\}+\psi\{(x-ct)\} \tag{6.10}$$

is also a solution of the one-dimensional wave equation (6.7). Further if $\omega = kc$, then also $u(x,t)=\phi(kx+\omega t)+\psi(kx-\omega t)$ (6.11)

is a solution of (6.7). Here $kx+\omega t$ is called the *phase* for the right travelling wave and $x \pm ct$ are the *characteristics* of the one-dimensional wave equation.

II. The initial value problem : D'Alembert's solution.

Consider the Cauchy type initial value problem described by

$$\frac{\partial^2 u}{\partial x^2}=c^2\frac{\partial^2 u}{\partial t^2}, \quad -\infty < x < \infty, \quad t \geq 0 \tag{6.12}$$

subject to initial conditions

$$u(x,0)=\eta(x) \text{ and } \frac{\partial u}{\partial t}(x,0)=v(x) \tag{6.13}$$

where the curve on which the initial values $\eta(x)$ and $v(x)$ are prescribed is the x-axis. The functions $\eta(x)$ and $v(x)$ are assumed to be twice continously differentiable. We shall consider the solution for the motion of a string.

Let $u(x,t)$ denotes the displacement for any x and t. Then the initial values of the displacement and velocity are prescribed to be $\eta(x)$ and $v(x)$ respectively. Now we know that the general solution of the equation (6.12) is given by (vide equation (6.9))

$$u(x,t)=\phi(x+ct)+\psi(x-ct) \tag{6.9}$$

where ϕ and ψ are arbitrary functions. Using the intial conditions (6.13) we have

$$\phi(x)+\psi(x)=\eta(x), \quad c[\phi'(x)-\psi'(x)]=v(x) \tag{6.14}$$

so that the second equation gives on integration

$$\phi(x)-\psi(x)=\frac{1}{c}\int_0^x v(\xi)d\xi \tag{6.15}$$

The first equation of (6.14) and (6.15) provide us

$$\phi(x)=\frac{1}{c}\eta(x)+\frac{1}{2c}\int_0^x v(\xi)d\xi, \quad \psi(x)=\frac{1}{2}\eta(x)-\frac{1}{2c}\int_0^x v(\xi)d\xi \tag{6.16}$$

This is known as *D'Alembert's solution of the one-dimensional wave equation.* If the string is released from rest, $v \equiv 0$, and the then the equation (6.16) becomes

$$u(x,t)=\frac{1}{2}[\eta(x+ct)+\eta(x-ct)] \tag{6.17}$$

This shows that the subsequent displacement of the string is produced by two pulses of 'shape' $u=\frac{1}{2}\eta(x)$, each moving with velocity c, one to the right and the other to the left.

Example 6.1 : A tighty stretched homogeneous string of length L with its fixed ends at $x = 0$ and $x = L$ executes transverse vibrations. Motion is started with zero initial velocity by displacing the string into the form $f(x)=a\,sin^2\,\pi x$. Find the deflection $u(x,t)$ at any time t.

Solution. Following D'Alembert's solution, the required deflection is

$$u(x,t)=\frac{1}{2}[f(x+ct)+f(x-ct)]$$

$$=\frac{1}{2}a[sin^2\,\pi(x+ct)+sin^2\,\pi(x-ct)]$$

$$=\frac{1}{2}a\left[2-\cos 2\pi(x+ct)-\cos 2\pi(x-ct)\right]$$

$$=\frac{1}{2}a\left[1-\cos 2\pi x\cos 2\pi ct\right]$$

III. The Riemann-Volterra solution

Let us put $y = ct$. Then the equation (6.12) reduces to

$$\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 u}{\partial y^2} \tag{6.18}$$

Let $u=\eta(x,y),\ \frac{\partial u}{\partial \eta}=v(x,y)$ on C (6.19)

and Γ, the projection of a strip C on the xy plane, is a curve with the equation $u(x,y)=0$.

Suppose that we wish to find the value of $u(x,y)$ of the wave function u at a point P with coordinates (x,y) of the wave function u at a point P with coordinates (x,y). Then the characteristics of the equation (6.18) through the point P are given by

$$x+y=\bar{x}+\bar{y},\quad x-y=\bar{x}-\bar{y} \tag{6.20}$$

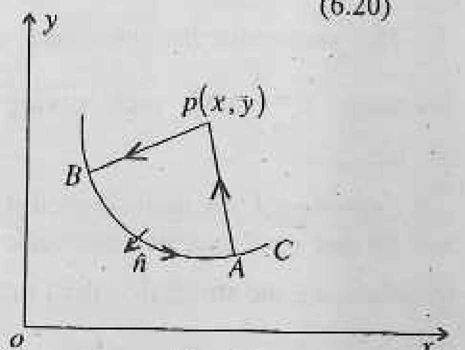


Fig. 6.1

Let the first line intersects the curve C at the point A and the second one intersects C at B. If we let

$$L=\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial y^2}$$

then, since L is self-adjoint, we have

$$\iint_{\Sigma}(wLu-uLw)\,dxdy$$

$$=\iint_{\Sigma}\left[\frac{\partial}{\partial x}\left(w\frac{\partial u}{\partial x}\right)-\frac{\partial}{\partial x}\left(u\frac{\partial w}{\partial x}\right)+\frac{\partial}{\partial y}\left(-w\frac{\partial u}{\partial y}\right)+\frac{\partial}{\partial y}\left(u\frac{\partial w}{\partial y}\right)\right]dx\,dy$$

$$=\iint_{\Sigma}\left[\frac{\partial}{\partial x}\left(w\frac{\partial u}{\partial x}-u\frac{\partial w}{\partial x}\right)+\frac{\partial}{\partial y}\left(-w\frac{\partial u}{\partial y}+u\frac{\partial w}{\partial y}\right)\right]dx\,dy$$

$$= \int_{C'} \left[U\cos(\eta,x) + V\cos(\eta,y) \right] ds \qquad (6.21)$$

where $U = w\frac{\partial u}{\partial x} - u\frac{\partial w}{\partial x}$, $V = -w\frac{\partial u}{\partial y} + u\frac{\partial w}{\partial y}$ (6.22)

and C' is the closed path *ABPA* eclosing the area Σ. Now the Green's function w must satisfy the conditions (see Unit 3, Section 3.6)

(*i*) $Lw = 0$

(*ii*) $\frac{\partial w}{\partial n} = 0$ on *AP* and *BP*

(*iii*) $w = 1$ at the point *P*

It is obvious that these conditions are satisfied if we take $w = 1$. Usig this and the fact that $Lu = 0$, we see that the equation (6.21) reduces to

$$\left(\int_{AP} + \int_{PB} + \int_{BA} \right) \left[\frac{\partial u}{\partial x}\cos(\eta,x) - \frac{\partial u}{\partial y}\cos(\eta,y) \right] ds = 0 \qquad (6.23)$$

Now along the characteristic *PA*, which has the equation $x + y = \bar{x} + \bar{y}$, we have

$$\cos(\eta,x) = \frac{1}{\sqrt{2}}, \quad \cos(\eta,y) = \frac{1}{\sqrt{2}}, \quad ds = -\sqrt{2}dx = \sqrt{2}\,dy$$

so that $\int_{AP} \left[\frac{\partial u}{\partial x}\cos(\eta,x) - \frac{\partial u}{\partial y}\cos\eta(\eta,y) \right] ds = -\int_A^P \left(\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy \right) = u_A - u_P$

Similarly, on the characteristic *PB* having the equation $x - y = \bar{x} - \bar{y}$, we have

$$\cos(\eta,x) = -\frac{1}{\sqrt{2}}, \quad \cos(\eta,y) = \frac{1}{\sqrt{2}}, \quad ds = -\sqrt{2}dx = -\sqrt{2}dy$$

so that the value of the integral along *PB* is $U_B - U_P$. Substituting these values in (6.23), we get

$$U_P = \frac{1}{2}(U_A + U_B) - \frac{1}{2}\int_{AB} \left[\frac{\partial u}{\partial y}\cos(\eta,x) - \frac{\partial u}{\partial y}\cos(\eta,y) \right] ds \qquad (6.24)$$

as the solution of Cauch problem.

For instance, if we are given that $u=\eta(x), \dfrac{\partial u}{\partial y}=v(x)$ on $y=0$ (6.25)

then if P is the point (x,y), it follows that A is $(x+y,0)$ and B is $(x-y,0)$.

Thus we have

$$U_A=\eta(x+y), U_B=\eta(x-y)$$

and

$$\int_{AB}\left[\frac{\partial u}{\partial x}\cos(\eta,x)-\frac{\partial u}{\partial y}\cos(\eta,y)\right]dx$$

$$=-\int_{x-y}^{x+y} v(\xi)d\xi$$

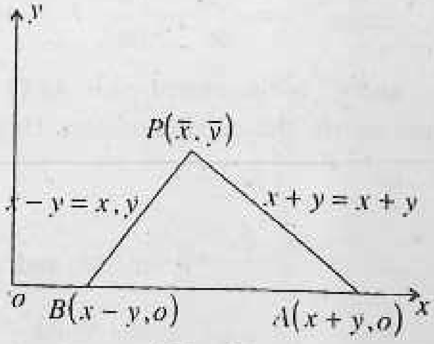


Fig. 6.2

In terms of the original variables x and t, we have, there fore, from (6.24)

$$u(x,t)=\frac{1}{2}\left[\eta(x+ct)+\eta(x-ct)\right]=\frac{1}{2c}\int_{x-ct}^{x+ct} v(\xi)d\xi \tag{6.26}$$

Example 6.2 : sove the Cauchy problem, described by the inhomogeneous wave equation

$$\frac{\partial^2 u}{\partial t^2}-c^2\frac{\partial^2 u}{\partial t^2}=F(x,t) \tag{6.27}$$

subject to the initial conditions

$$u(x,0)=\eta(x), \frac{\partial u(x,0)}{\partial t}=v(x)$$

Hints for solution. Proceed exactly along the same lines as above by introducing an extra term $\iint_{\Sigma} F(x,y)dxdy$ and finally derive the result

$$u(x,t)=\frac{1}{2}\left[\eta(x+ct)+\eta(x-ct)\right]+\frac{1}{2c}\int_{x-ct}^{x+ct} v(\xi)d\xi$$

$$-\frac{1}{2c}\int_0^t\int_{x-ct}^{x+ct} F(\xi,\tau)d\xi d\tau \tag{6.28}$$

IV. Solution by separation of variable method

(a) Cartesian coordinates :

Consider the one-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, -\infty < x < \infty, t > 0 \tag{6.12}$$

To solve this equation by separation of variable technique, we put

$$u(x,t) = X(x)T(t) \tag{6.29}$$

Putting this into (6.12) we have

$$\frac{1}{X}\frac{d^2X}{dx^2} = \frac{1}{c^2T}\frac{d^2T}{dt^2} = k, \text{ a separation constant.} \tag{6.30}$$

Case (i) : Let $k = \lambda^2 > 0$. Then from (6.30), we get

$$\frac{d^2X}{dx^2} - \lambda^2 X = 0, \frac{d^2T}{dt^2} - c^2\lambda^2 T = 0$$

whose solutions are $X = c_1 e^{\lambda x} + c_2 e^{-\lambda x}, T = c_3 e^{c\lambda t} + c_4 e^{-c\lambda t}$ so that

$$u(x,t) = \left(c_1 e^{\lambda x} + e_2 e^{-\lambda x}\right)\left(c_3 e^{c\lambda t} + c_4 e^{-c\lambda t}\right)$$

is the solution of the wave equation (6.12).

Case (ii) : Let $k = 0$. Then equations (6.30) give

$$\frac{d^2X}{dx^2} = 0, \frac{d^2T}{dt^2} = 0$$

whose solutions are $X = c_5 x + c_6$ and $T = c_7 t + c_8$ so that

$$u(x,t) = (c_5 x + c_6)(c_7 t + c_8) \tag{6.32}$$

Case (iii) : Let $k = -\lambda^2 < 0$. Then equations (6.30) lead to

$$\frac{d^2X}{dx^2} + \lambda^2 X = 0, \frac{d^2T}{dt^2} + c^2\lambda^2 T = 0$$

whose solutions are $X = c_9 \cos\lambda x + c_{10} \sin\lambda x,\ T = c_{11} \cos c\lambda t + c_{12} \sin c\lambda t$ so that

$$u(x,t) = (c_9 \cos\lambda x + c_{10} \sin\lambda x)(c_{11} \cos c\lambda t + c_{12} \sin c\lambda t) \tag{6.33}$$

As an illustration, consider a thin homogeneous perfectly flexible string under unifrom tension lying along the x-axis in its equilibrium position. The ends of the string are fixed at $x = 0$ and $x = L$. The string is pulled aside a short distance and released. If there are no external forces which correspond to the case of free vibrations, the subsequent motion of the string is described by the solution $u(x,t)$ of the following problem :

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0,\ 0 \le x \le L,\ t > 0 \tag{6.34}$$

subject to the boundary conditions

$$u(0,t) = 0,\ u(L,t) = 0,\ t > 0$$

and initial conditions

$$u(x,0) = \eta(x), \frac{\partial u}{\partial t}(x,0) = v(x) \tag{6.36}$$

For the case (i) in which $k = \lambda^2 > 0$, the solution is given by (6.31). Using the boundary conditions (6.35) we find that

$$c_1 + c_2 = 0,\ c_1 e^{\lambda L} + c_2 e^{-\lambda L} = 0$$

which possess a non-trivial solution iff

$$\begin{vmatrix} 1 & 1 \\ e^{\lambda L} & e^{-\lambda L} \end{vmatrix} = 0 \Rightarrow e^{2\lambda L} = 1 \Rightarrow \lambda = 0 \quad (\because L \ne 0)$$

which is a contradiction that $\lambda \ne 0$. Hence the solution (6.31) is not acceptable.

It we consider *the case (ii)* for which $k = 0$, the solution is given by (6.32) so that the boundary conditions (6.35) lead to

$$c_6(c_7 t + c_8) = 0 \cdot c_5 L(c_7 t + c_8) = 0 \Rightarrow c_5 = 0, c_6 = 0 \quad (\because L \ne 0)$$

so that $u = 0,\ \forall t$, which is a trivial solution and is therefore, not acceptable.

Lastly, *for the case (iii)* where $k = -\lambda^2 < 0$, the solution is given by (6.33). Using the boundary conditions (6.35), we have $c_9 = 0$ and $\sin\lambda L = 0 \Rightarrow \lambda = \frac{n\pi}{L}, n = 1,2,3,...$ as the eigen values. Hence the possible solution is

$$u_n(x,t) = \sin\frac{n\pi x}{L}\left(A\cos\frac{n\pi ct}{L} + B\sin\frac{n\pi ct}{L}\right), n = 1,2,3,... \tag{6.37}$$

where $A = C_{10}c_{11} + c_{10}c_{12}$. Using the superposition principle, we have

$$u(x,t) = \sum_{n=1}^{\infty}\sin\frac{n\pi x}{L}\left(A_n\cos\frac{n\pi ct}{L} + B_n\sin\frac{n\pi ct}{L}\right) \tag{6.38}$$

The initial conditions give

$$u(x,0) = \eta(x) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L}, \frac{\partial u(x,0)}{\partial t} = v(x)\sum_{n=1}^{\infty} B_n \sin\frac{\eta\pi x}{L}$$

which are half-range Fourier sine series. Hence

$$A_n = \frac{2}{L}\int_0^L \eta(x)\sin\frac{n\pi x}{L}dx, B_n = \frac{2}{L}\int_0^L v(x)\sin\frac{n\pi x}{L}dx \tag{6.39}$$

Hence the required physically meaningful solution is given by (6.38) where A_n and B_n are given by (6.39). $u_n(x,t)$ as given by (6.37) are called *normal modes of vibration* and $w_n = \frac{n\pi c}{L}$, $n = 1,2,3,.......$, are called *normal frequencies*.

Example 6.3. A string of length L is released from rest in the position $y = f(x)$. Show that the total energy of the string is

$$\frac{\pi^2 T}{4L}\sum_{n=}^{\infty} n^2 k_n^2 \quad \text{where} \quad k_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx,$$

T being the tension in the string.

If the mid-point of a string is pulled aside through a small distance and then released, show that in the subsequent motion the fundamental mode (i.e. $n = 1$) contributes $8/\pi^2$ of the total energy.

Solution. Here the partial differental equation is

$$\frac{\partial^2 y}{\partial t^2} - c^2 \frac{\partial^2 y}{\partial x^2} = 0.$$

The boundary conditions are $y(o,t) = y(L,t) = 0$ and the initial conditions are

$$y(x,o) = f(x), \quad \frac{\partial y(x,o)}{\partial t} = 0$$

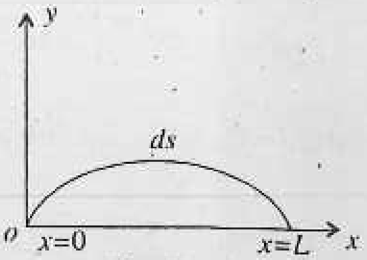


Fig. 6.3

Following the separation of variable method and using the superposition principle, we find that the general solution of the wave equation is

$$y(x,t) = \sum_{n=1}^{\infty} k_n \sin\frac{n\pi x}{L} \cos\frac{n\pi ct}{L} \tag{6.40}$$

where $k_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx.$

Now the kinetic energy of an element dx of the string executing small transverse vibrations is $\frac{1}{2}\rho dx\left(\frac{\partial y}{\partial t}\right)^2$ so that the total kinetic energy is

$$= \frac{1}{2}\int_0^L \rho\left(\frac{\partial y}{\partial t}\right)^2 dx$$

$$= \frac{T}{2}\int_0^L \frac{1}{c^2}\left(\frac{\partial y}{\partial t}\right)^2 dx \quad [\because c^2 = T/\rho, \rho \text{ being the linear unifrom density}]$$

Since $ds^2 = dx^2 + dy^2$ i.e. $ds\sqrt{1+\left(\frac{dy}{dx}\right)^2} \approx \left[1+\frac{1}{2}\left(\frac{dy}{dx}\right)^2\right]$, hence the stretch in the string is $ds - dx = \frac{1}{2}\left(\frac{\partial y}{\partial x}\right)^2 dx$

Also the potential energy of this element is $\frac{1}{2}T\left(\frac{\partial y}{\partial x}\right)^2 dx$ so that the total potential energy is $=\frac{T}{2}\int_0^L\left(\frac{\partial y}{\partial x}\right)^2 dx$

Thus the total energy E is obtained as

$$E=\frac{1}{2}T\int_0^L\left[\left(\frac{\partial y}{\partial x}\right)^2+\frac{1}{c^2}\left(\frac{\partial y}{\partial t}\right)^2\right]dx \tag{6.41}$$

$$=\frac{1}{2}T\int_0^L\sum_{n=1}^{\infty}k_n\frac{n^2\pi^2}{L^2}\left[\cos^2\frac{n\pi x}{L}\cos^2\frac{n\pi ct}{L}+\sin^2\frac{n\pi x}{L}\sin^2\frac{n\pi ct}{L}\right]dx$$

$$=\frac{1}{2}T\cdot\sum_{n=1}^{\infty}k_n\cdot\frac{n^2\pi^2}{L^2}\cdot\frac{1}{2}L\left(\cos^2\frac{n\pi ct}{L}+\sin^2\frac{n\pi ct}{L}\right) \qquad \text{[By using (6.40)]}$$

$$=\frac{\pi^2 T}{4L}\sum_{n=1}^{\infty}n^2k_n^2$$

Now the transverse motion of the string is described by the equation

$$y(x,t)=\sum_{n=1}^{\infty}k_n\sin\frac{n\pi x}{L}\cos\frac{n\pi ct}{L}$$

where $k_n=\frac{2}{L}\int_0^L f(x)\sin\frac{\eta\pi x}{L}dx$

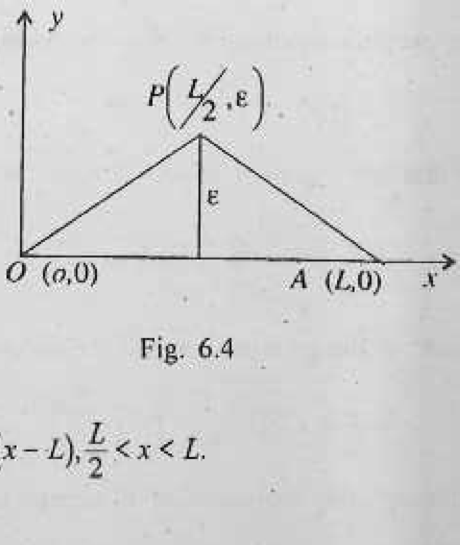


Fig. 6.4

But the equation of the line OP is $y=\frac{2\varepsilon}{L}x$, $0<x<\frac{L}{2}$ and that of the line PA is $y=-\frac{2\varepsilon}{L}(x-L), \frac{L}{2}<x<L.$

Hence

$$k_n=\frac{2}{L}\left[\int_0^{L/2}\frac{2\varepsilon}{L}x\sin\frac{n\pi x}{L}dx-\frac{2\varepsilon}{L}\int_{L/2}^{L}(x-L)\sin\frac{n\pi x}{L}dx\right]$$

$$= \frac{8\varepsilon}{n^2\pi^2}\sin\frac{n\pi}{2}, \text{ for } = \text{odd}$$

so that from (6.42) we get

$$E = \frac{\pi^2 T}{4L}\sum_{n=odd} n^2 . \frac{64\varepsilon^2}{n^4\pi^4} = \frac{16T\varepsilon^2}{L\pi^2}\sum_{n=odd}\frac{1}{n^2} = \frac{16T\varepsilon^2}{L\pi^2}.\frac{\pi^2}{8}.$$

Thus $E = \frac{16T\varepsilon^2}{L\pi^2}\left(\pi^2/8\right)$ while the total energy due to the fundamental mode (i.e., $n=1$) is $\frac{16T\varepsilon^2}{L\pi^2}$.

(b) Cylindrical cooredinates.

In cylindrical coordinates with u depending only on r and t, the one-dimentional wave equation is given by

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \tag{6.43}$$

For periodic solution in time, we assume

$$u(r,t) = F(r)e^{i\omega t} \tag{6.44}$$

so that the equation (6.43) reduces to

$$\frac{d^2F}{dr^2} + \frac{1}{r}\frac{dF}{dr} + \frac{\omega^2}{c^2}F = 0$$

which is Bessel equation and the solution is

$$F = AJ_0\left(\omega r/c\right) + BY_0\left(\omega r/c\right) \tag{6.45}$$

We can write the equation in complex form as

$$F = C_1\left[J_0\left(\omega r/c\right) + iY_0\left(\omega r/c\right)\right] + C_2\left[J_0\left(\omega r/c\right) - iY_0\left(\omega r/c\right)\right]$$

i.e. $F = C_1 H_0^{(1)}\left(\omega r/_c\right) + C_2 H_0^{(2)}\left(\omega r/_c\right)$ (6.46)

where $A = C_1 + C_2, B = i(C_1 - C_2)$ and $H_0^{(1)}, H_0^{(2)}$ are Hankel functions defined by

$$H_0^{(1)} = J_0\left(\omega r/_c\right) + iY_0\left(\omega r/_c\right) \text{ and } H_0^{(2)} = J_0 = \left(\omega r/_c\right) - iY_0\left(\omega r/_c\right) \quad (6.47)$$

Thus the solution of one-dimensional wave equation is

$$u(r,t) = \left[C_1 H_0^{(1)}\left(\omega r/_c\right) + C_2 H_0^{(2)}\left(\omega r/_c\right)\right]e^{i\omega t}$$

The asymptotic expansions of $H_0^{(1)}$ and $H_0^{(2)}$ are given by

$H_0^{(1)}(x) = \sqrt{\frac{2}{\pi x}}e^{i(x-\pi/_4)}$, $H_0^{(2)}(x) = \sqrt{\frac{2}{\pi x}}e^{-i(x-\pi/_4)}$ for large x and hence the solution (6.48)

reduces to

$$u(r,t) = \sqrt{\frac{2c}{\pi\omega}}\left[C_1 e^{-i\pi/_4}\frac{\exp\{i(\omega/_c)(r+ct)\}}{\sqrt{r}} + C_2 e^{i\pi/_4}\frac{\exp\{i(\omega/_c)(r-ct)\}}{\sqrt{r}}\right] \quad (6.49)$$

for large values of r.

(c) Spherical polar coordinates.

The one-dimensional wave equation in spherical polar coordinates is given by

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) = \frac{1}{C^2}\frac{\partial^2 u}{\partial t^2}, r > 0 \quad (6.50)$$

For periodic solution in time, we assume $u(r,t) = F(r)e^{i\omega t}$. Then the equation (6.50) reduces to

$$\frac{d^2F}{dr^2} + \frac{2}{r}\frac{dF}{dr} + \frac{\omega^2}{c^2}F = 0.$$

Subsituting $F = \left(\frac{\omega}{c}r\right)^{-\frac{1}{2}}G(r)$, this equation gives

$$\frac{d^2G}{dr^2}+\frac{1}{r}\frac{dG}{dr}+\left\{\left(\frac{\omega}{c}\right)^2-\left(\frac{1}{2r}\right)^2\right\}G(r)=0$$

whose solution is $G(r)=A'J_{\frac{1}{2}}\left(\frac{\omega}{c}r\right)+B'J_{-\frac{1}{2}}\left(\frac{\omega}{c}r\right)$

Thus $F(r)=\frac{A}{\sqrt{r}}{}_{\frac{1}{2}}\left(\frac{\omega}{c}r\right)+\frac{B}{\sqrt{r}}{}_{-\frac{1}{2}}\left(\frac{\omega}{2}r\right)$

where $A=\sqrt{\frac{c}{\omega}}A'$, $B=\sqrt{\frac{c}{\omega}}B'$. Noting that (see Study material PG (MT) 03 : Gr.A. Pages 157–158)

$$J_{\frac{1}{2}}(x)=\sqrt{\frac{2}{\pi x}}\sin x, J_{-\frac{1}{2}}(x)=\sqrt{\frac{2}{\pi x}}\cos x$$

we have $F(r)=\sqrt{\frac{2c}{\pi\omega}}.\frac{1}{r}\left[A\sin\left(\omega r/_c\right)+B\cos\left(\omega r/_c\right)\right]$

or in complex form $F(r)=\frac{1}{r}\left[C_1e^{i\omega r/_c}+C_2e^{-i\omega r/_c}\right]$

Thus the required solution of the wave equation is

$$u(r,t)=\frac{1}{r}\left[C_1\exp\left\{i\left(\omega/_c\right)(r+ct)\right\}+C_2\exp\left\{-i\left(\omega/_c\right)(r-ct)\right\}\right] \qquad (6.51)$$

V. Uniquences of the solution

Consider the wave equation for a forced vibrating string of length L described by

$$\frac{\partial^2u}{\partial t^2}-c^2\frac{\partial^2u}{\partial x^2}=F(x,t),\ 0<x<L, t<0$$

where $c^2(x)=T_0/_{\rho(x)}$ subject to the initial conditions

$$u(x,0)=\eta(x),\ \frac{\partial u}{\partial t}(x,0)=v(x), 0\le x\le L$$

and boundary conditions

$$u(o,t)=f(t), u(L,t)=g(t).$$

If possible, we suppose that the above initial boundary value problem satisfying the given conditions for $t<T$ (T being fixed) has two solutions u_1 and u_2. We put $U=u_1-u_2$. Then it readily follows that

$$\frac{\partial^2 U}{\partial t^2}-c^2\frac{\partial^2 U}{\partial x^2}=0,\ 0<x<L, t>0 \tag{6.52}$$

subject to the initial conditions

$$U(x,o)=0, \frac{\partial U(x,o)}{\partial t}=0, 0\le x\le L$$

and boundary conditions

$$U(o,t)=0,\ U(L,t)=0.$$

Now multiplying both sides of (6.52) by $\rho\frac{\partial u}{\partial t}$ and then integrating the resulting equation with respect to x from o to L and with respect to t from o to T (> 0), we get

$$\frac{1}{2}\int_0^l\left\{\rho(x)\left[\frac{\partial U(x,T)}{\partial t}\right]^2+\tau_0\left[\frac{\partial U(x,T)}{\partial x}\right]^2\right\}dx$$

$$-\frac{1}{2}\int_0^L\left\{\rho(x)\left[\frac{\partial U(x,o)}{\partial t}\right]^2+\tau_0\left[\frac{\partial U(x,o)}{\partial x}\right]^2\right\}dx$$

$$-\int_0^T\tau_0\frac{\partial U(L,t)}{\partial x}\frac{\partial U(L,t)}{\partial t}dt+\int_0^T\tau_0\frac{\partial U(o,t)}{\partial x}\frac{\partial U(o,t)}{\partial t}dt=0 \tag{6.53}$$

Now the first two integrals represent the difference in the total energy (vide Example 6.3, Equation 6.4) at times T nd O. The last two integrals represent the work done by the y-components of the tensile force at the ends of the string. Clearly

$$\frac{\partial U(o,t)}{\partial t}=\frac{\partial U(L,t)}{\partial t}=\frac{\partial U(x,o)}{\partial x}=0.$$

Hence the equation (6.53) reduces to

$$\frac{1}{2}\int_0^L\left\{\rho\left[\frac{\partial U(x,T)}{\partial t}\right]^2+\tau_0\left[\frac{\partial U(x,T)}{\partial x}\right]^2\right\}dx=0 \tag{6.54}$$

This means that if the string has no energy at time $t = 0$, the energy remains to be zero if no work is done on string. Let $\rho(x)>0$ and $\tau_0>0$. Then the integral can never be negative so that the integral from o to L is positive. This contradicts equation (6.54). Hence if we assume the integrand to be continuous, then it must be indentically zero, so that for any $t_1<T$, we have

$$\frac{\partial U(x,T)}{\partial t}=0$$

Hence $\frac{\partial U(x,t_1)}{\partial t}=0$ for $o<t_1<T,\ o<x<L$ implying that $U(x,t_1)=$ constant. But $U(x,o)=0$. Thus we have $U(x,t_1)=0$ for $t_1\le T$. Since t_1 is arbitrary, it follows that $U(x,t)=0$ i.e. $u_1(x,t)=u_2(x,t)$ for any t. Thus the given equation has a unique solution.

§ 6.4. Two-dimensional Wave Equation.

The two-dimensional wave equation is given by

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \tag{6.55}$$

I. Separation of variables method.

(a) Cartesian coordinates.

To solve the equation (6.55) we put

$$u(x,y,t)=X(x)Y(y)T(t)$$

Substituting this in (6.55) we get

$$\frac{1}{X}\frac{d^2X}{dx^2}+\frac{1}{Y}\frac{d^2Y}{dy^2}=\frac{1}{c^2T}\frac{d^2T}{dt^2}=-\lambda^2 \text{ (say), a separation constnat}$$

so that $\frac{1}{X}\frac{d^2X}{dx^2}+\lambda^2=-\frac{1}{Y}\frac{d^2Y}{dy^2}=\mu^2$ (say), a separation constant.

Thus we have three ordinary differential equations

$$\frac{d^2X}{dx^2}+p^2X=0,\ \frac{d^2Y}{dy^2}+q^2Y=0 \text{ and } \frac{d^2T}{dt}+r^2c^2T=0 \tag{6.56}$$

where $p^2=\lambda^2-\mu^2, q^2=\mu^2, r^2=p^2+q^2=\lambda^2$. Solutions of the equations (6.56) are

$X=C_1\cos px+C_2\sin px,\ Y=C_3\cos qy+C_4\sin qy, T=C_5\cos(rct)+C_6\sin(rct)$.

Hence the solution of the equation (6.55) is

$$u(x,y,t)=(C_1\cos px+C_2\sin px)(C_3\cos qy+C_4\sin qy)\{C_5\cos(rct)+C_6\sin(rct)\} \tag{6.57}$$

To illustrate the method, we consider the vibrations of rectangular memberane stretched between the lines $x=0,\ x=a,\ y=0,\ y=b$. Then the boundary conditions are

$u=0$ when $x=0,\ a$ and $y=0,b$.

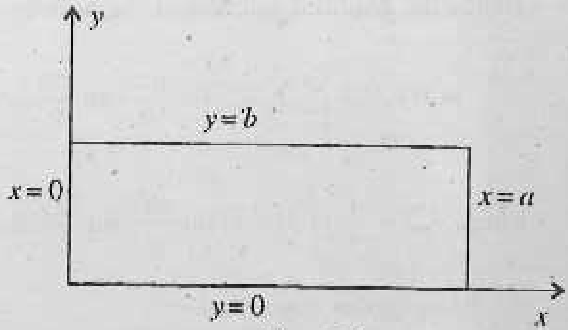


Fig. 6.5

Applying the condition $u=0$ at $x=0$, we have from (6.57), $C_1=0$. The condition $u=0$ at $x=a$ leads to sin $pa=0 \Rightarrow p=m\pi/a, m\in I$. Also using the conditions $u=0$ at $y=0$ and $y=b$ give the results $C_3=0$ and $q=n\pi/b, n\in I$. Hence the solution (6.57) reduces to

$$u(x,y,t)=\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}(A\cos\omega t+B\sin\omega t)$$

where $A=C_2C_4C_5$, $B=C_2C_4C_6$ and $\omega=\pi c\sqrt{\frac{m^2}{a^2}+\frac{\eta^2}{b^2}}$. Using the superposition principle, we therefore have

$$u(x,y,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}(A_{mn}\cos\omega_{mn}t+B_{mn}\sin\omega_{mn}t)$$

Now suppose that the memberane starts from rest with initial position $u = f(x,y)$. Then applying the condition $\frac{\partial u}{\partial t}(x,o) = 0$ we get $B_{mn} = 0$. Also the condition $u = f(x,y)$ at $t = 0$ gives

$$f(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} \sin\frac{m\pi x}{a}.\sin\frac{n\pi y}{b} \tag{6.58}$$

which is a double Fouier series. Multiplying both sides of (6.58) by sin $\frac{m\pi x}{a}.\sin\frac{n\pi y}{b}$ and then integrating from $x = 0$ to a and from o to b, we get

$$\int_{b}^{a}\int_{0}^{b} f(x,y)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}dx\,dy = \frac{ab}{4}A_{mn}.$$

Hence the required solution of the problem is

$$u(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\cos\omega_{mn}t$$

where $A_{mn} = \frac{4}{ab}\int_{0}^{a}\int_{0}^{b} f(x,y)\sin\frac{m\pi y}{a}\sin\frac{n\pi y}{b}dx\,dy$

(b) Plane polar coordinates.

The solve this equation by separation of variable method, we put

$u(r,\theta,t) = R(r)\Theta(\theta)T(t)$

so that the equation (6.59) gives

$$\frac{1}{R}\frac{d^2R}{dr^2} + \frac{1}{rR}.\frac{dR}{dr} + \frac{1}{r^2\Theta}\frac{d^2\Theta}{d\theta^2} = \frac{1}{c^2T}\frac{d^2T}{dt^2} = -\lambda^2, \text{ (say), a separation constant.}$$

Also from this, we get

$$\frac{r^2}{R}\frac{d^2R}{dr^2} + \frac{r}{R}\frac{dR}{dr} + \lambda^2r^2 = -\frac{1}{\Theta}\frac{d^2\Theta}{.d\theta^2} = \mu^2, \text{ (say), a separation constant.}$$

Thus we have three ordinary differential equations to determine R, Θ and T :

$$\frac{d^2R}{dr^2}+\frac{1}{r}\frac{dR}{dr}+\left(\lambda^2-\frac{\mu^2}{r^2}\right)R=0,$$

$$\frac{d^2\Theta}{d\theta^2}+\mu^2\Theta=0,$$

and $$\frac{d^2T}{dt^2}+\lambda^2c^2T=0$$

The solutions of these equations are

$$R=C_1J_\mu(\lambda r)+C_2Y_\mu(\lambda r), \Theta=C_3\cos\mu\theta+C_4\sin\mu\theta,$$

$$T=C_4\cos(\lambda ct)+C_6\sin(\lambda ct).$$

where J_μ and Y_μ are Bessel functions of first and second kind of order μ.

Hence the solution of the equation (6.59) is

$$u(r,\theta,t)=\{C_1J_\mu(\lambda r)+C_2Y_\mu(\lambda r)\}(C_3\cos\mu\theta+C_4\sin\mu\theta)\{C_5\cos(\lambda ct)+C_6\sin(\lambda ct)\} \quad (6.60)$$

To illustrate the method, we consider the vibration of a circular membrane $(0\le r\le a, \theta\le 2\pi)$ whose boundary $r=a$ is held fixed so that $u(a,\theta,t)=0, t\ge 0$ and that initial conditions are

$$u(r,\theta,0)=f(r,\theta), \frac{\partial u(r,\theta,0)}{\partial t}=g(r,\theta) \quad (6.61)$$

Now, since the deflection is a single-valued periodic function of θ of period 2π, μ must be an integer, say $\mu=m$. Also since $Y_\mu(\lambda r)$ is unbounded as $r\to 0$, we must have to take $C_2=0$. Again the boundary condition $u(a,\theta,t)=0$ implies that

$$J_m(\lambda a)=0 \quad (6.62)$$

which has an infinite number of positive roots. Their representation requires two indices, the first one indicating the order n of the Bessel function and the second one representing the solution. We, therefore, suppose that $\lambda_{mn}(m=0,1,2.....;r=1,2,3,....)$ are the roots of the equation (6.61). Then using the principle of superposition, the solution (6.60) can be written as

$$u(r,\theta,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}J_m(\lambda_{mn}r)\big[(a_{mn}\cos m\theta+b_{mn}\sin m\theta)\cos\omega_{mn}t$$

$$+(c_{mn}\cos m\theta+d_{mn}\sin m\theta)\sin\omega_{mn}t\big] \tag{6.63}$$

where $\omega_{mn}=c\lambda_{mn}$

We determine the constants a_{mn}, b_{mn} etc. by the use of the prescribed initial conditions (6.61) as

$$f(r,\theta)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}J_m(\lambda_{mn}r)[an\cos m\theta+b_{mn}\sin m\theta]$$

$$g(r,\theta)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}J_m(\lambda_{mn}r)[c_{mn}\cos m\theta+d_{mn}\sin m\theta]\omega_{mn},$$

so that

$$a_{mn}=\frac{2}{\pi a^2\left[J_m'(\lambda_{mn})\right]^2}\int_0^{2\pi}\int_0^{a}f(r,\theta)J_m(\lambda_{mn}r)\cos m\theta.rdrd\theta$$

$$b_{mn}=\frac{2}{\pi a^2\left[J_m'(\lambda_{mn})\right]^2}\int_0^{2\pi}\int_0^{a}f(r,\theta)J_{mn}(\lambda_{mn}r)\sin m\theta.rdrd\theta,$$

$$c_{mn}=\frac{2}{\pi a^2\omega_{mn}\left[J_m'(\lambda_{mn})\right]^2}\int_0^{2\pi}\int_0^{a}g(r,\theta)J_{mn}(\lambda_{mn}r)\cos m\theta.rdrd\theta,$$

$$d_{mn}=\frac{2}{\pi a^2\omega_{mn}\left[J_m'(\lambda_{mn})\right]^2}\int_0^{2\pi}\int_0^{a}g(r,\theta)J_{mn}(\lambda_{mn}r)\sin m\theta.rdrd\theta$$

Hence the solution for the vibration of a circular membrace is given by (6.63) in which the constnats a_{mn}, b_{mn}, c_{mn} and d_{mn} are given by (6.64).

(c) Spherical polar coordinates.

If the wave function u is axially symmetric, then the wave equation of motion in spherical polar coordinates is given by

$$\frac{\partial^2 u}{\partial r^2}+\frac{2}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right)=\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \tag{6.65}$$

Adopting the method of separation of variables, let

$$u(r,\theta,t)=R(r)\Theta(\theta)T(t)$$

so that the equation (6.65) gives

$$\frac{1}{R}\left(\frac{d^2R}{dr^2}+\frac{2}{r}\frac{dR}{dr}\right)+\frac{1}{\Theta}\left[\frac{1}{r^2\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)\right]=\frac{1}{c^2T}\frac{d^2T}{dt^2}$$

$=-\lambda^2$, (say), a separation constant

so that

$$\frac{r^2}{R}\left[\frac{d^2R}{dr^2}+\frac{2}{r}\frac{dR}{dr}\right]+\lambda^2r^2=-\frac{1}{\Theta}\left[\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)\right]$$

$=n(n+1)$, say, a separation constant,

where n is a positive integer. Thus we have three ordinary differential equations of determine R, Θ and T :

$$\frac{d^2R}{dr^2}+\frac{2}{r}\frac{dR}{dr}+\left\{\lambda^2-\frac{n(n+1)}{r^2}\right\}R=0 \tag{6.66a}$$

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)+n(n+1)\Theta=0 \tag{6.66b}$$

and $$\frac{d^2T}{dt^2}+\lambda^2c^2T=0 \tag{6.66c}$$

To solve (6.66a), we put $R=r^{-\frac{1}{2}}S(r)$ to reduce it into the form

$$\frac{d^2s}{dr^2}+\frac{1}{r}\frac{ds}{dr}+\left\{\lambda^2-\frac{\left(n+\frac{1}{2}\right)^2}{r^2}\right\}S=0$$

which is Bessel's equation of order $n+\frac{1}{2}$ (vide Study Material : PG (MT) 03 : Group A, Sec. 7.8) and its solution is

$$S(r)=C_1J_{n+\frac{1}{2}}(\lambda r)+C_2Y_{n+\frac{1}{2}}(\lambda r)$$

so that $R(r)=r^{\frac{1}{2}}\left[C_1J_{n+\frac{1}{2}}(\lambda r)+C_2Y_{n+\frac{1}{2}}(\lambda r)\right]$

Again putting $\cos\theta=\mu$ so that $\frac{d}{d\theta}=-\sin\theta\frac{d}{d\mu}$ the equation (6.66b) reduces to

$$\left(1-\mu^2\right)\frac{d^2\Theta}{d\mu^2}-2\mu\frac{d\Theta}{d\mu}+n(n+1)\Theta=0$$

which is the well known Legendre's equation of degree n and has the solution

$$\Theta(\theta)=C_3P_n(\mu)+C_4Q_n(\mu)$$

where P_n and Q_n are Legender's functions of the first and second kind respectively (vide Study Materical PG (MT) 03 : Group A, Sec. 7. 10). Also the solution of the equation (6.66c) is

$$T(t)=C_5\cos\omega t+C_6\sin\omega t$$

where $\omega=\lambda c$

Hence by the principle of superposition, the solution of the equation (6.65) is givey by

$$u(r,\theta,t)=r^{-\frac{1}{2}}\sum_{r=0}^{\infty}\left[C_1J_{n+\frac{1}{2}}(\lambda r)+C_2Y_{n+\frac{1}{2}}(\lambda r)\right]\left[C_3P_n(\cos\theta)+C_4Q_n(\cos\theta)\right].$$

$$\left(C_5\cos\omega t+C_6\sin\omega t\right) \tag{6.67}$$

II. Eigen function method

Let us consider a region S in the xy-plane bounded by a closed curve Γ and let $R = S \cup \Gamma$
Consider the problem described by

$$\frac{\partial^2 u}{\partial t^2} - c^2\nabla^2 u = F(x,y,t), x,y \in R, t \geq 0 \tag{6.68}$$

subject to the initial conditions

$$u(x,y,0) = \eta(x,y), \frac{\partial u}{\partial t}(x,y,0) = v(x,y) \text{ in } R$$

and any one of the boundary conditions

(i) $u = 0$ Γ (Dirichlet condition)

(ii) $\frac{\partial u}{\partial n} = 0$ on Γ (Neumann condition)

(iii) $u + \frac{\partial u}{\partial n} = 0$ on Γ (Robin/Mixed condition)

Now at first we consider the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2\nabla^2 u$$

and assume a solution in the form

$$u(x,y,t) = \phi(x,y)T(t)$$

so that the above equation gives

$$\frac{1}{c^2 T}\frac{d^2 T}{dt^2} = \frac{1}{\phi}\nabla^2\phi = -\lambda, \text{ a separation constant.}$$

implying $\nabla^2\phi + \lambda\phi = 0$ and $\frac{d^2T}{dt^2} + \lambda c^2 T = 0$. The first equation is called Helmholtz equation and $\phi = 0$ is its trivial solution. A nontrival solution of this equation exists only for certain values of $\{\lambda_n\}$, called *eigenvalues* and the corresponding solution $\{\phi_n\}$ are the *eigenfunctions*. Corresponding to each eigenvalue λ_n, there exists at least one real-valued and twice coninuously differentiable eigenfunction ϕ_n such that

$$\nabla^2\phi_n + \lambda_n\phi_n = 0 \text{ in S}$$

and $\phi_n = 0$ on Γ.

The sequence of eigenfunctions $\{\phi_n\}$ satisfies the orthogonality property

$$\iint_s \phi_n\phi_m ds = 0 \text{ for all } n \neq m.$$

(c.f. Study Material : PG (MT) 03 : Group A, Pages 78–81).

Now each continuously differentiable functions is S which vanishes on Γ, can have a Fourier series expansion relative to the orthogonal set $\{\phi_n\}$. Hence we can take the solution of the proposed problem in the form

$$u(x,y,t) = \sum_{n=1}^{\infty} C_n(t)\phi_n(x,y)$$

where $c_n(t)$ is to be found out. Substituting this in (6.68). we get

$$\sum_{n=1}^{\infty}\left[\ddot{C}_n(t)\phi_n(x,y) - c^2 C_n(t)\nabla^2\phi_n(x,y)\right] = F(x,y,t)$$

But $\nabla^2\phi_n = -\lambda_n\phi_n$ and, therefour,

$$\sum_{n=1}^{\infty}\left[\ddot{C}_n(t) + \omega_n^2 C_n(t)\right]\phi_n = F(x,y,t) \tag{6.70}$$

where $\omega_n^2 = c^2\lambda_n^2 (n = 1,2,3....)$.

Multiplying both sides of (6.70) by ϕ_m and integrating over the region S and interchanging the order of integration and summation, we obtain

$$\sum_{n=1}^{\infty}\left[\ddot{C}_n(t) + \omega_n^2 C_n(t)\right]\iint_s \phi_n(x,y)\phi_m(x,y)ds = \iint_s F(x,y,t)\phi_m(x,y)ds$$

Using the orthogonality property, we get

$$\ddot{C}_n(t) + \omega_n^2 C_n(t) = F_n(t) \tag{6.71}$$

where

$$F_n(t)=\frac{1}{\|\phi_n\|^2}\iint_s F(x,y,t)\phi_{n},(x,y)ds \text{ and } \|\phi_n\|^2=\iint_s|\phi_n|^2 ds \tag{6.72}$$

The series (6.69) gives with the help of intial conditions

$$\sum_{n=1}^{\infty}C_n(0)\phi_n(x,y)=\eta(x,y) \text{ and } \sum_{n=1}^{\infty}C_n(0)\phi_n(x,y)=v(x,y)$$

Multiplying both sides of these equations by $\phi_m(x,y)$ and integrating over S and then using the orthogonal property, we get

$$C_n(0)=\frac{1}{\|\phi_n\|^2}\iint_s \eta(x,y)\phi_n(x,y)ds.\ \ddot{C}_n(0)=\frac{1}{\|\phi_n\|^2}\iint_s v(x,y)\phi_n(x,y)ds \tag{6.73}$$

Using the method of variation of parameters, the general solution of (6.71) is given by

$$C_n(t)=A_n\cos\omega_n t+B_n\sin\omega_n t+\frac{1}{\omega_n}\int_0^1 F_n(\xi)\sin\omega_n(t-\xi)d\xi.$$

From (6.73) we obtain

$$A_n=\frac{1}{\|\phi_n\|^2}\iint_s \eta(x,y)\phi_n(x,y)ds,\ B_n=\frac{1}{\omega_n\|\phi_n\|^2}\iint_s f(x,y)\phi_n(x,y)ds \tag{6.74}$$

$(n=1,2,3....)$

Hence, the formal series solution of the given problem is

$$u(x,y,t)=\sum_{n=1}^{\infty}\Big[(A_n\cos\omega_n t+B_n\sin\omega_n t)\phi_n(x,y)$$

$$+\frac{1}{\omega_n}\left\{\int_0^1 F_n(\xi)\sin\omega_n(t-\xi)d\xi\right\}\phi_n(x,y)\Big] \tag{6.75}$$

§ 6.5. Three-dimensional Wave Equation.

We consider the wave equation.

$$\nabla^2 u=\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \tag{6.76}$$

and intend to solve this in different system of coordinates by the separation of variable method.

(a) Cartesian coordinates (x, y, z)

Let $u(x,y,z,t) = X(x)Y(y)Z(z)T(t)$ (6.77)

Substituting this is (6.76) in Cartesian form given by

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \tag{6.78}$$

we derive

$$\frac{1}{X}\frac{d^2X}{dx^2}+\frac{1}{Y}\frac{d^2Y}{dy^2}+\frac{1}{Z}\frac{d^2Z}{dz^2}=\frac{1}{c^2T}\frac{d^2T}{dt^2}=-\frac{\omega^2}{c^2},\ \text{(say), a separation constant} \tag{6.79}$$

so that

$$\frac{1}{X}\frac{d^2X}{dx^2}+\frac{1}{Y}\frac{d^2Y}{dy^2}=-\frac{1}{Z}\frac{d^2Z}{dz^2}-\frac{\omega^2}{c^2}=n^2,\ \text{(say), a separation constnat} \tag{6.80}$$

and $$\frac{1}{X}\frac{d^2X}{dx^2}=-\frac{1}{Y}\frac{d^2Y}{dy^2}-n^2=-p^2,\ \text{(say), a separation constant} \tag{6.81}$$

From (6.79) to (6.81) we get the following four ordinary differential equations to find X,Y,Z and T :

$$\frac{d^2X}{dx^2}+p^2X=0,\ \frac{d^2Y}{dy^2}+q^2Y=0,\ \frac{d^2Z}{dz^2}+r^2Z=0\ \text{ and }\ \frac{d^2T}{dt^2}+\omega^2T=0$$

where $q^2=n^2-p^2$, $r^2=\frac{\omega^2}{c^2}-n^2$ so that $p^2+q^2+r^2=\frac{\omega^2}{c^2}$. The solutions of the above equations are easy to derive. We have, therefore, the complete solution of (6.78) by using (6.77) as :

$$u(x,y,z,t)=(C_1\cos px+C_2\sin px)(C_3\cos qy+C_4\sin qy)$$

$$(C_5\cos rz+C_6\sin rz)(C_7\cos\omega t+C_8\sin\omega t) \tag{6.82}$$

It may be noted that the solution (6.82) is periodic of period $2\pi/\omega$.

(b) Cylindrical coordinates (r,θ,z)

The wave equaion in cylindrical coordinates is

$$\frac{\partial^2 u}{dr^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}+\frac{\partial^2 u}{\partial z^2}=\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}. \tag{6.83}$$

Let $u(r,\theta,z,t)=R(r)\Theta(\theta)Z(z)T(t)$ (6.84)

Substituting this in (6.83) we get

$$\frac{1}{R}\left(\frac{d^2R}{dr^2}+\frac{1}{r}\frac{dR}{dr}\right)+\frac{1}{r^2\Theta}\frac{d^2\Theta}{d\theta^2}+\frac{1}{Z}\frac{d^2Z}{dz^2}=\frac{1}{c^2T}\frac{d^2T}{dt^2}=-\frac{\omega^2}{c^2},$$ (say), a separation constant

so that $$\frac{1}{R}\left(\frac{d^2R}{dr^2}+\frac{1}{r}\frac{dR}{dr}\right)+\frac{1}{r^2\Theta}\frac{d^2\Theta}{d\theta^2}=-\frac{1}{Z}\frac{d^2Z}{dz^2}-\frac{\omega^2}{c^2}=-n^2$$ (say), a separation constant

and $$\frac{r^2}{R}\left(\frac{d^2R}{dr^2}+\frac{1}{r}\frac{dR}{dr}\right)+n^2r^2=-\frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2}=m^2$$ (say), a separation constant (6.87)

Thus, from (6.85) to (6.87) we derive the following four ordinary differential equations to find R,Θ,Z and T:

$$\frac{d^2R}{dr^2}+\frac{1}{r}\frac{dR}{dr}+\left(n^2-\frac{m^2}{r^2}\right)R=0,\ \frac{d^2\Theta}{d\theta^2}+m^2\Theta=0,\ \frac{d^2Z}{dz^2}+q^2Z=0,\ \frac{d^2T}{dt^2}+\omega^2T=0$$

where $q^2=\frac{\omega^2}{c^2}-n^2$ and m is assumed to be a positive integer. The solutions of these equations are easy to derive. Hence the general solution of the equation (6.83) is obtained with the help of (6.84) as

$$u(r,\theta,z,t)=\{C_1J_m(nr)+C_2Y_m(nr)\}(C_3\cos m\theta)(C_5\cos qz+C_6\sin qz)$$

$$(C_7\cos\omega t+C_8\sin\omega t) \tag{6.88}$$

The solution is periodic of period $2\pi/\omega$.

(c) Spherical polar coordinates (r_2, θ, ϕ)

In this case the wave equation (6.76) is given by

$$\frac{\partial^2 u}{\partial r^2}+\frac{2}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2}=\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \tag{6.89}$$

To solve this equation by separation of variable technique, we put

$$u(r,\theta,\phi,t)=R(r)\Theta(\theta)\Phi(\phi)T(t)$$

in equation (6.89) and obtain

$$\frac{1}{R}\left(\frac{d^2R}{dr^2}+\frac{2}{r}\frac{dR}{dr}\right)+\frac{1}{\Theta}\left[\frac{1}{r^2\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)\right]$$

$$+\frac{1}{\Phi r^2\sin^2\theta}\frac{d^2\Phi}{d\phi^2}=\frac{1}{c^2T}\frac{d^2T}{dt^2}=-\frac{\omega^2}{c^2}, \text{ (say)},$$

a separation constant (6.91)

so that $\frac{r^2\sin^2\theta}{R}\left(\frac{d^2R}{dr^2}+\frac{2}{r}\frac{dR}{dr}+\frac{\omega^2}{c^2}R\right)+\frac{\sin\theta}{\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)=-\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2}=m^2$, (say),

a separation constant (6.92)

and $\frac{r^2}{R}\left(\frac{d^2R}{dr^2}+\frac{2}{r}\frac{dR}{dr}+\frac{\omega^2}{c^2}R\right)=-\frac{I}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\frac{d\Theta}{d\theta}\right)+\frac{m^2}{\sin^2\Theta}$

$=n(n+1)$, (say), a separation constant. (6.93)

where n is a positive integer. Equations (6.91) to (6.93) produce the following four ordinary differential equations to find out $R, \Theta\ \Phi$ and T:

$$\frac{d^2R}{dr^2}+\frac{2}{r}\frac{dR}{dr}+\left\{\lambda^2-\frac{n(n+1)}{r^2}\right\}R=0 \tag{6.94a}$$

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)+\left\{n(n+1)-\frac{m^2}{\sin^2\theta}\right\}\Theta=0 \tag{6.94b}$$

$$\frac{d^2\Phi}{d\phi^2}+m^2\Phi=0,\ \frac{d^2T}{dt^2}+\omega^2T=0 \tag{6.94c,d}$$

where $\lambda=\omega/c$.

The methode of solution of the equation (6.94a) has been discussed in Section 6.4 II *Case* (c) and is given by

$$R(r)=r^{-\frac{1}{2}}\left[C_1J_{n+\frac{1}{2}}(\lambda r)+C_2Y_{n+\frac{1}{2}}(\lambda r)\right]$$

To solve the equation (6.94b) we put $\cos\theta=\mu$, so that $\frac{d}{d\theta}=-\sin\theta\frac{d}{d\mu}$, to reduce it into the form

$$\frac{d}{d\mu}\left[\left(1-\mu^2\right)\frac{d\Theta}{d\mu}\right]+\left[n(n+1)-\frac{m^2}{1-\mu^2}\right]\Theta=0$$

This is well-known *legendre's associated equation* having two independent solutions $P_n^m(\mu)$ and $Q_n^m(\mu)$, called *associated Legendre's functions* respectively of the first and second kind.

Thus,

$$\Theta(\theta)=C_3P_n^m(\mu)+C_4Q_n^m(\mu)$$

The solutions of the equations (6.94c) and (6.94d) are obvious and are given respenctively by

$$\Phi(\phi)=C_5\cos m\phi+C_6\sin m\phi,\quad T(t)=C_7\cos\omega t+C_8\sin\omega t$$

Hence the solution of the equation (6.89) is given with the help of (6.90) as

$$u(r,\theta,\phi,t)=r^{-\frac{1}{2}}\left[C_1J_{n+\frac{1}{2}}(\lambda r)+C_2Y_{n+\frac{1}{2}}(\lambda r)\right]\left[C_3P_n^m(\cos\theta)+C_4Q_n^m(\cos\theta)\right]$$

$$(C_5\cos m\phi+C_6\sin m\phi)(C_7\cos\omega t+C_8\sin\omega t) \tag{6.95}$$

This solution is also periodic of period $2\pi/\omega$

§ 6.6. Duhamel's Principle for Wave Equation

Let (x,y,z) be any point in a three-dimensional Euclidean space R^3 and $v=v(x,y,z,r,\lambda)$ satisfies for fixed λ the partial differential equation

$$\frac{\partial^2 v}{\partial t^2} - c^2\nabla^2 v = 0, (x,y,z) \in R^3$$

with initial conditions

$$v(x,y,z,0,\lambda) = 0, \quad \frac{\partial v}{\partial t}(x,y,z,0,\lambda) = F(x,y,z,\lambda)$$

where $F(x,y,z,\lambda)$ is a continuous funtion of x,y,z defined in R^3. If the function $u(x,y,z,t)$ is defined such that

$$u(x,y,z,t) = \int_0^1 v(x,y,z,t-\lambda,\lambda)d\lambda,$$

then $u(x,y,z,t)$ satisfies the equation

$$\frac{\partial^2 u}{\partial t^2} - c^2\nabla^2 u = F(x,y,z,t), (x,y,z) \in R^3, t > 0, \tag{6.96}$$

the initial conditions being

$$u(x,y,z,0) = \frac{\partial u}{\partial t}(x,y,z,0) = 0$$

Proof : Let us assume the solution of the equation (6.96) in the form

$$u(x,y,z,t) = \int_0^1 v(x,y,z,t-\lambda,\lambda)d\lambda \tag{6.97}$$

where $v(x,y,z,t-\lambda,\lambda)$ is a one-parameter family solution of

$$\frac{\partial^2 v}{\partial t^2} - c^2\nabla^2 u = 0, \forall\lambda \tag{6,98}$$

Also we assume that at $t = \lambda, v(x,y,z,0\lambda) = 0, \forall\lambda$ (6.99)

Now differentiating both sides of (6.97) with respect to t under the integral sign and using Leibnitz rule we have

$$\frac{\partial u}{\partial t} = v(x,y,z,0,t) + \int_0^1 \frac{\partial v}{\partial t}(x,y,z,t-\lambda,\lambda)d\lambda$$

i.e., $\frac{\partial u}{\partial t} = \int_0^1 \frac{\partial v}{\partial t}(x,y,z,t-\lambda,\lambda)d\lambda$ (By using (6,99))

Differentiating this again with respec to t, we have

$$G(r, r') = \frac{1}{|r_1 - r'|} = \frac{1}{|\rho - r'|}$$

i.e. $\frac{\partial^2 u}{\partial t^2} = \frac{\partial v}{\partial t}(x, y, z, o, t) + \int_0^t c^2 \nabla^2 v d\lambda$ (by using (6.98))

$= \frac{\partial v}{\partial t}(x, y, z, o, t) + c^2 \nabla^2 u$ (by using (6.67))

so that $\frac{\partial^2 u}{\partial t^2} - c^2 \nabla^2 u = \frac{\partial v}{\partial t}(x, y, z, o, t)$ (6.100)

Comparing equations (6.96) and (6.1000), we obtain $\frac{\partial v}{\partial t}(x, y, z, o, t) = F(x, y, z, t)$

Therefore, it v satisfies the equation

$$\frac{\partial^2 v}{\partial t^2} - c^2 \nabla^2 v = 0$$

with the conditions $v(x, y, z, 0, \lambda) = 0$ and $\frac{\partial v}{\partial t}(x, y, z, o, \lambda) = F(x, y, z, \lambda)$ at $t = \lambda$ the $u(x, y, z, t)$ defined by (6.97) satisfies the nonhomogeneous equation (6.96) and the homogeneous initial condtions. The function $v(x, y, z, t)$ is called the *pulse function* or the *force function.*

Exercise

1. By separating the variables, show that one-dimensional wave equation $\frac{\partial^2 u}{dx^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$ has solution of the form $A exp\ (\pm inx \pm ict)$, where A and n are constants. Hence show that functions of the form

$$u(x, t) = \sum_r \left(A_r \cos\frac{r\pi ct}{a} + B_r \sin\frac{r\pi ct}{a} \right) \sin\frac{r\pi x}{a}$$

where A_r's and B_r's are constants, satisfy the wave equation and the boundary conditions $u(o, t) = 0, u(a, t) = 0, \forall t$

2. Find the solution of the radio equation $\frac{\partial^2 u}{dx^2} = LC\frac{\partial^2 u}{\partial t^2}$ appropriage to the case when a periodic e.m.f. A cos pt is applied at the end $x = 0$ of the line, A being constant.

[Ans. $u(x,t) = Acso(pt - p\sqrt{LC}x)$]

3. A tightly stretched string with fixed end points $x = 0$ and $x = l$ is initially in a position given by

i) $u = u_0 \sin^3(\pi x/l), 0 \le x \le l$

ii) $u = 10\sin^3(\pi x/l), 0 \le x \le l$

iii) $u = u_0 \sin 2(\pi x/l), 0 \le x \le l$

iv) $u = \mu x(l - x), 0 \le x \le l$

v) $u = \mu_0 \sin^2(\pi x/l), 0 \le x \le l$

vi) $u = f(x)$ where $f(x) = \begin{cases} \frac{2\lambda x}{l} 0 < x < l/2 \\ \frac{2\lambda}{l}(l - x)\frac{1}{2} < x < l \end{cases}$

and then released. Find the displacement of any points x of the string at any time $t > 0$.

[Ans. *i)* $u(x,t) = \frac{1}{4}u_0\left[3\sin(\pi x/l)\cos(\pi ct/l) - \sin(3\pi x/l)\cos(3\pi ct/l)\right]$

ii) $u(x,t) = 10\cos(\pi ct/l)\sin(\pi x/l)$

iii) $u(x,t) = u_0 \sin(2\pi x/l)\cos(2\pi ct/l)$

iv) $u(x,t) = \frac{8\mu l^2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}\sin\frac{(2n-1)\pi x}{l}\cos\frac{(2n-1)\pi ct}{l}$

v) $u(s,t) = \frac{u_0 l}{12c\pi}\left[9\sin(\pi x/l)\cos(\pi ct/l) - \sin(3\pi x/l)\cos(3\pi ct/l)\right]$

vi) $u(x,t)=\frac{8\lambda}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\left(n\pi/2\right)\cos\left(n\pi ct/l\right)\sin\left(n\pi x/l\right)]$

4. Use D'alembert's solution to find the deflection of vibrating string of unit length having fixed ends with zero initial velocity and initial deflection

i) $f(x)=a(x-x^3)$

ii) $f(x)=a(x^2-x^3)$

Find the deflection $u(x,t)$ at any time t

[Ans. i) $u(x,t)=ax(1-x^2-3c^2t^2)$

ii) $u(x,t)=a(x^2+c^2t^2-x^3-3xc^2t^2)$]

5. A rectangular membrance with fastened edges makes free transverse vibrations. Explain how a formal series solution can be found.

[Ans. For the membrane $x=0,a$, $y=0,b$ and initial conditions $u(x,y,0)=\eta(x,y)$, $\frac{\partial u}{\partial t}(x,y,0)=v(x,y)$

$$u(x,y,t)=\sum\sum\left[A_{mn}\cos(rct)+B_{mn}(rct)\right]\sin\left(m\pi x/a\right)\sin\left(m\pi y/b\right)$$

where $$A_{mn}=\frac{4}{ab}\int_0^a\int_0^b\eta(x,y)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}dxdy$$

$$B_{mn}=\frac{4}{abcr}\int_0^a\int_0^b v(x,y)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}dxdy],\quad r^2=\pi^2\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)$$

6. Solve the initial value problem described by given that

7. Solve the initial boundary value problem described by $\frac{\partial^2 u}{dt^2}=c^2\frac{\partial^2 u}{\partial x^2}, x>0, t>0$ given that $u(x,0)=0, \frac{\partial u}{\partial t}(x,0)=0$, $x>0$ and $u(o,t)=\sin t$, $t>0$

[Ans. $u(x,t)= \begin{cases} 0, x<ct \\ \sin\left(t-x/c\right), x>ct \end{cases}$]

8. Solve the initial value problem described by $\frac{\partial^2 u}{dt^2}-c^2\frac{\partial^2 u}{\partial x^2}=xe^t$ given that $u(x,0)=\sin x$, $\frac{\partial u}{\partial t}(x,0)=0$.. [Ans. $u(x,t)=\sin x\cos ct+(e^t-1)(xt+x)-xte^t$]

§ 6.6. Summary

The hyperbolic differential equation, its occurrence and different types of solutions are considered. Special attention has been given to vibrating strings and membranes. The Duhamel's principle for nonhomogeneous form of the equation is also noted.

UNIT 7 ❑ GREEN'S FUNCTION

§ 7.1 Introduction

We known that the solution of a non-homogeneous ordinary differential equation of the form $Lu(x) = f(x)$, where L is a Sturm-Liouville operator, subject to some boundary conditions at the end points $x = a, b$, of a given interval, can be obtained in an integral form

$$u(x) = \int_a^b G(x,\xi) f(\xi) d\xi$$

provided the function $G(x,\xi)$ is chosen in such a way that

$$LG(x,\xi) = \delta(x-\xi)$$

$\delta(x-\xi)$ being the Dirac d-function. The kernel $G(x,\xi)$ of the above integral operator is called Green's function of the differntial operator. [Vide Study Material PG(MT) 03 : Group A, Pages 74-75 and 82-93]

The above concept can be extended to partial differential equations also. To explain the idea, we consider the equation.

$$L[u(\mathbf{X})] = f(\mathbf{X}) \tag{7.1}$$

where L is some linear prtial differential operator in three independent variable x, y, z and **X** is a vector in three-dimensional space. Then the *green's function* G(X, α) satisfies the equation

$$L[G(X:\alpha)] = \delta(X-\alpha) \tag{7.2}$$

which, or expansion, becomes

$$L[G(x,y,z;\xi,\eta,\zeta)] = \delta(x-\xi)\,\delta(y-\eta)\delta(z-\zeta) \tag{7.3}$$

Here the expression $\delta(X-\alpha)$ is the generalisation of the concept of delta function in three dimensional space R^3. The function $G(X;\alpha)$ represents the effect at the point X due to a source or delta furction input applied at α. The equation (7.2) can easily be interpreted in heat conduction or electrostaics as follows : $G(X;\alpha)$ represents the temperature (or the electrostatic potential) at the point X in R^3 due to a unit source (or unit charge) at α

Now multiplying both sides by (7.2) or $f(\alpha)$ and integrating over the volume V with respect to α, we get

$$L[\int_V G(X;\alpha)\,f(\alpha)\,dv] = \int_V f(\alpha)\delta(X-\alpha)\,dv = f(X)$$

Comparing this with (7.1) we have

$$u(X) = \int G(x;G)\,f(\alpha)dv$$

which is the solution of the equation (7.1). Thus a function $u(x, y, z, \xi, \eta, \zeta)$ is a fundamental solution of the equation $\nabla^2 u = 0$ say if is a solution of the non-homogeneous equation

$$\nabla^2 u = \delta(x, y, z, \xi, \eta, \zeta)$$

This idea can easily be extended to dimensions also.

Let us start with the fundamental solution for a three-dimensional potential problem satisfying the equation

$$\nabla^2 u = \delta(X) \tag{7.5}$$

where u can be interpreted, for example, as the electrostatic potential. We shall consider that solution which depends on the source distance $r=|X|$ only : thus for r <o, u(r) satisfies the equation

$$\nabla^2 u = \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{du}{dr}\right) = 0$$

Integration leads to $u = \frac{A}{r} + B$. Since the petential vanished at infinity, we have $B = 0$ that so $u = A/r$ Integrating (7.5) over a small sphere $\sum$ of radius $\in$ we get

$$\int_\Sigma \text{div grad}\,u\,dv = 1 \quad \text{i.e.} \quad \int_\sigma \frac{du}{dr}\Big|_{r=\in} ds = 0$$

[By divergence theorem & is the surface of $\sum$]

$$\text{or ,} \quad -A\int_{\sigma}\frac{1}{r^2}\Big|_{\epsilon} ds = -\frac{A}{\epsilon^2}(4\pi\epsilon^2) = 1 \qquad \text{i.e.} \quad A = -\frac{1}{4\pi}$$

Thus the fundamental or singular solution $\nabla^2 u = 0$ is $u = -\frac{1}{4\pi r}$

In the case of plane polar coordinates, the fundamental solution of $\nabla^2 u = 0$ i.e. of $\frac{1}{r}\frac{d}{dr}(r, u) = 0$ is obtained, by integration, in the form $u = A \ln r + B$. The constant B remains arbitrary and can be set equal to zero for convenience and the constant A is obtained as above as $A = 1/2\pi$. Thus the fundamental solution is $u = \frac{1}{2\pi}\ln r$

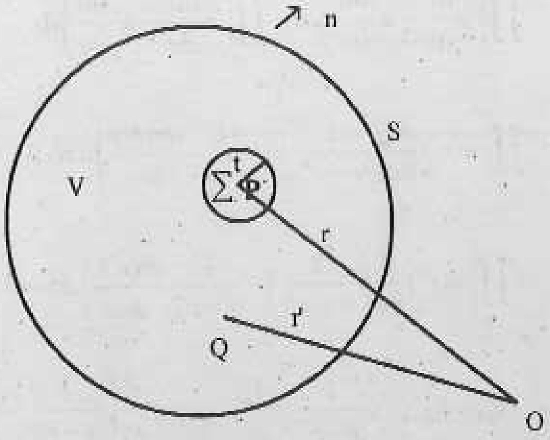


Fig.-7.1

§ 7.2 Green's Function for Laplace's Equation

We now proceed to use Green's function method to find the solution of Laplace's equation. For this, we first introduce Green's function for the equation and study its basic properties. To begin with, we define Green's function for the Dirichlet problem, i.e. we find u such that $\nabla^2 u = 0$ inside some finitely bounded region V enclosed by S, a sufficiently smooth surface, when $u = f$ is prescribed on the boundary S.

We shall find $u(P)$ where $P \in V$. Let $OP = r$ and $\sum$ be a sphere with centre at P and radius ϵ. Let Q be another point in $V' = V' - \Sigma$ or on the boundary S' of $V - \sum$ such that OQ= r and

$$u' = \frac{1}{|r - r'|} \qquad 7.6$$

If u and u' are twice countinously differentiable function in V and have first order derivatives on S then by Green's theorem in the region V', we have (Vide Unit-4, 4.4., Equation, 4.3)

$$\iiint_{V'}(u\nabla^2 u' - u'\nabla^2 u)dv = \iint_{S'}\left(u\frac{\partial u'}{\partial n} - u'\frac{\partial u}{\partial n}\right)ds, \quad (7.7)$$

where n is the unit vector normal to dS drawn outwards from S' and $\frac{\partial}{\partial n}$ denoted differnetiation in that direction, S' being the boundary fo the region V'

Now since $\nabla^2 u = \nabla^2 u' = 0$ within $V - \Sigma$ we have by applying (7.7) to the region V'

$$\iint_{V'}\left(u\frac{\partial}{\partial n} - u'\frac{\partial u}{\partial n}\right)ds + \iint_{S'}\left(u\frac{\partial u'}{\partial n} - u'\frac{\partial u}{\partial n}\right)ds, \quad (7.7)]$$

or $$\iint_{S}\left[u(r')\frac{\partial}{\partial n}\left(\frac{1}{|r-r'|}\right) - \frac{1}{|r-r'|}\frac{dr(r')}{dn}\right]ds = 0$$

$$+\iint_{\sigma}\left[u(r')\frac{\partial}{\partial n}\left(\frac{1}{|r-r'|}\right) - \frac{1}{|r-r'|}\frac{dr(r')}{dn}\right]d\sigma = 0 \quad (7.8)$$

If Q lies on σ, we have $\frac{1}{|r-r'|} = \frac{1}{\in}$ and $\frac{\partial}{\partial n}\left(\frac{1}{|r-r'|}\right) = \frac{1}{\in^2}$ and the elementary surface $d\sigma = \in^2 \sin\theta\, d\theta\, d\phi$ Also on σ,

$$u(r') = u(r) + r.\nabla u$$

or $$u(r') = u(r) + \left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}\right)$$

$$= u(r) + \in\left(\sin\theta\cos\phi\frac{\partial u}{dx} + \sin\theta\sin\phi\frac{\partial u}{\partial y} + \cos\theta\frac{\partial u}{\partial z}\right)$$

$$= u(r) + 0(\in) \text{ on } \sigma$$

so that $$\frac{\partial u(r')}{\partial n} = \frac{\partial u(r)}{\partial n} + 0(\in)$$

Now $$\iint_\sigma u(r')\frac{\partial}{\partial n}\left(\frac{1}{|r-r'|}\right)\partial\sigma=\iint_\sigma[u(r)+0(\in)]\frac{1}{\in^2}\in^2\sin\theta d\theta d\phi$$

$$=u(r)\iint_\sigma \sin\theta d\theta d\phi+o(\in)$$

$$=u(r)\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\sin\theta d\theta d\phi=o(\in)$$

$$=4\pi u(r)+0(\in) \tag{7.9}$$

and $$\iint_\sigma \frac{1}{|r-r'|}\frac{\partial u(r')}{\partial n}\partial\sigma=\frac{1}{\in}\iint_\sigma\left[\frac{\partial u(r)}{\partial n}+0(\in)\right]\in^2\sin\theta d\theta d\phi=0(\in)$$

substitution (7.9) and (7.10) in (7.8) and proceeding to the limit as $\in\to 0$ we get

$$u(r)=\frac{1}{4\pi}\iint_s\left[\frac{1}{|r-r'|}\frac{\partial u(r')}{\partial n}-u(r')\frac{\partial}{\partial n}\left(\frac{1}{|r-r'|}\right)\right]ds \tag{7.11}$$

Thus if u and $\frac{\partial u}{\partial n}$ are known on the boundary S or a region V, we can find the value of u at my interior point of V

A similar result holds in the case of exterior Dirchlet problem. Here we take the region Ω occuring in equation (7.7) to be bounded by S, a small sphere σ sourrounding P and S' a sphere with centre at the origion O and large radius R (cf. Fig. 7.2). Then taking the directions of the normals as indicated in the figure 7.2 and proceedings as above, we get

$$4\pi u(r)+0(\in)+\iint_s\left[\frac{1}{|r-r'|}\frac{\partial u(r')}{\partial n}-u(r')\frac{\partial}{\partial n}\left(\frac{1}{|r-r'|}\right)\right]ds$$

$$\iint_{S'}\left\{\frac{1}{R}\frac{\partial u(r')}{\partial n}+\frac{u(r')}{R^2}\right\}ds'=0$$

Letting $\in\to 0$ as $R\to\infty$ we see that the solution (7.11) is also valie for the exterior Dirichlet problem, provided Ru and R^2u remain finite as $R\to\infty$ Now it seems from eqution (7.11) that to obtain a solution of Dirichlet problem we need to know the values of both u and $\frac{\partial u}{\partial n}$. But we show by introducing the concept of Green's function that this is not so. Let us define Green's function $G(r,r')$ by the equation

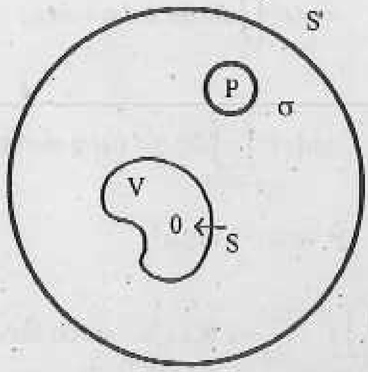


Fig.-7.2

$$G(r,r') = H(r,r') + \frac{1}{|r-r'|} \tag{7.12}$$

where the function $H(r,r')$ satisfies the relations

$$\left(\frac{\partial^2}{\partial x'^2} + \frac{\partial^2}{\partial y'^2} + \frac{\partial^2}{\partial z'^2}\right) H(r,r') = 0 \, in \, V \tag{7.13}$$

$$\text{and} \quad H(r,r') \; H(r,r') + \frac{1}{|r-r'|} = 0 \quad \text{on S} \tag{7.14}$$

in which $r' = (x', y', z')$. Then proceeding as in the derivation of equation (7.11), we can show that

$$u(r) = \frac{1}{4\pi} \iint_s \left[G(r,r') \frac{\partial u(r')}{\partial n} - u(r') \frac{\partial G(r,r')}{\partial n} \right] ds \tag{7.14}$$

Thus if follows that if we can find a function $G(r,r')$ satisfying equations (7.12), (7.13) and (7.14), then the solution of the Dirichlet problem is given by the relation

$$u(r) = \frac{1}{4\pi} \iint_S u(r') \frac{\partial G(r,r')}{\partial n} ds \tag{7.16}$$

Hence the solution of Dirichlet problem is reduced to that of the determination of the Green's function $G(r, r')$

Example 7.1 : For a sphere with centre at the origin and radius a show that $\nabla^2\left(\frac{1}{r}\right) = -r\pi\delta(r)$, where $\delta(r)$ is the Dirac-delta function.

Sulution : We have

$$\nabla\nabla\left(\frac{1}{r}\right) = \nabla^2\left(\frac{1}{r}\right). Let\ \nabla\left(\frac{1}{r}\right) = e_r\frac{\partial}{r}\left(\frac{1}{\partial r}\right) + e_r\frac{\partial}{r}\left(\frac{1}{\partial r}\right) + e_\theta\frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{1}{r}\right) + e_\phi\frac{1}{r}\sin\theta\frac{\partial}{\partial\phi}\left(\frac{1}{r}\right)$$ in spherical polar coordinates

Now $\iiint_V \nabla.\nabla\left(\frac{1}{r}\right)dv = \iint_S \nabla\left(\frac{1}{r}\right).e_r dS$ (by divergence theorem)

$$= \iint_S \frac{\partial}{\partial r}\left(\frac{1}{r}\right)dS = -\frac{1}{a^2}4\pi a^2 = -4\pi$$

Hence $\nabla^2\left(\frac{1}{r}\right) = -4\pi\delta(r)$.

***Theorem 7.1** : Show that Green's function has the symmetric property*

proof. We have seen in Example 7.1 that $\nabla^2\left(\frac{1}{r}\right) = -4\pi\delta(r)$.

Let us define the Green's function $G(r, r')$ by

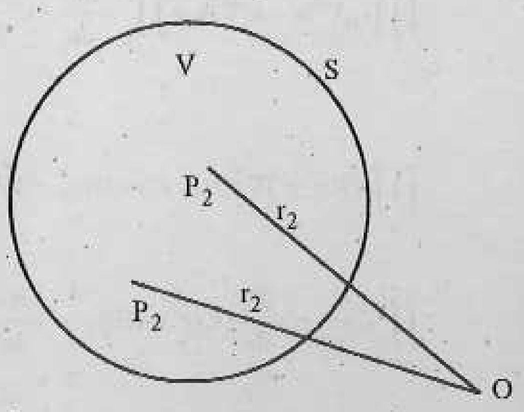


Fig.-7.3

$$G(r,r') = H(r,r') + \frac{1}{|r-r'|}$$

where $H(r,r')$ is harmonic then

$$\nabla^2 G = \nabla^2\left(\frac{1}{|r-r'|}\right) = -4\pi\delta(r-r')$$

We take the position vector of P_1 and P_2 to be r_1 and R_2 and that of a variable point Q to be r′

Assuming $u = G(r_1, r') = \frac{1}{|r_1 - r'|} + H(r_1, r')$

and $u' = G(r_2, r') = \frac{1}{|r_2 - r'|} + H(r_1, r')$

where $G_1(r_1, r) = 0, G(r_2, r') = 0$ on S

and $$\nabla^2 G(r_1, r') = \nabla^2\left(\frac{1}{|r_1 - r'|}\right) = -4\pi\delta(r_1, r')$$

$$\nabla^2 G(r_2, r') = \nabla^2\left(\frac{1}{|r_2 - r'|}\right) = -4\pi\delta(r_1, r')$$

In green's theorem

$$\iiint_V (u\nabla^2 u' - u'\nabla^2 u = \iint_S \left(u\frac{\partial u'}{\partial n} - u'\frac{\partial u}{\partial n}\right) ds$$

we obtain

$$\iiint_B \left[G(r_1, r')\nabla^2 G(r_2, r') - G(r_2, r')\nabla^2 G(r_1 r')\right] dv$$

$$\iint_S \left[G(r_1, r')\frac{\partial G}{\partial n}(r_2, r') - G(r_2, r')\frac{\partial G}{\partial n}(r_1 r')\right] ds$$

implying thereby

$$-4\pi\iiint_V [G(r_1,r')\,\delta(r_2,r') - G(r_1,r')\,\delta(r_1-r')]\,dv = 0$$

Using the property of Dirac function, we have

$$G(r_1,r_2) = G(r_2,r_1) \tag{7.17}$$

Theorm 7.2 *If G be continuous and* $\frac{\partial G}{\partial n}$ *be discontinous at r, then* $\lim_{\epsilon\to 0}\iint_\sigma \frac{\partial G}{\partial n} d\sigma = 1$ *where* $\sum$ *is a smallsphere a radius bounded by the surface* σ

Proof. Since $\nabla^2 G = \delta(r-r')$ we have be integrating bothe sides over the sphere $\sum$

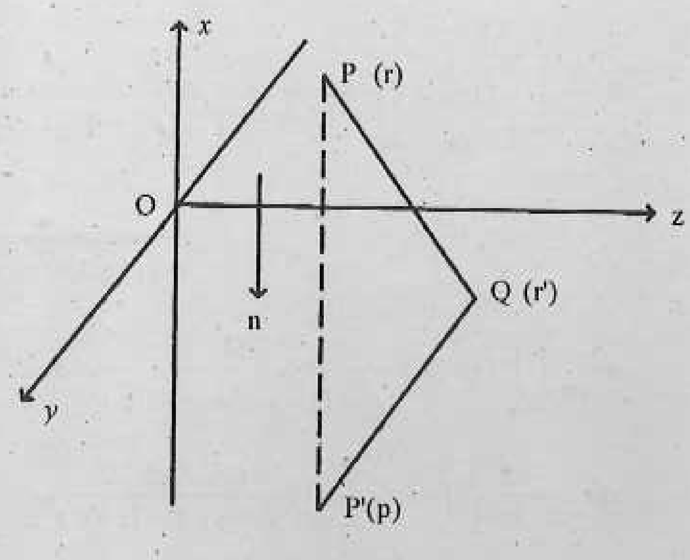
$$\iiint_\Sigma \nabla^2 G.dv = 1 \; i.e. \lim_{\epsilon\to 0}\iiint_\Sigma \nabla^2 G dv = 1 \; i.e. \lim_{\epsilon\to 0}\iint_\sigma \frac{\partial G}{\partial n} ds = 1$$

(by digergene's theorem)

We now consider some cases of solving irichet's problem by means of Green's function

I. Solution of dirichlet problem for a half-space (Semi-infinite space)

Let the half-space be defined by $x \geq 0$ i.e. $0 \leq x < \infty$, $-\infty\, y < \infty$, $-\infty < z < \infty$. Our problem is to find the solution u of the equation $\nabla^2 u = 0$ on $x \geq 0$ and $u = f(y,z)$ on $x = 0$ also $u \to 0$ as $r \to \infty$



Here the Green's function $G(r,r')$ satisfies the following relation

(i) $G(r_1,r') = \dfrac{1}{|r_1 - r'|} + H(r_1,r')$

(ii) $\left(\dfrac{\partial^2}{\partial x'^2} + \dfrac{\partial^2}{\partial y'^2} + \dfrac{\partial^2}{\partial z'^2}\right) H(r,r') = 0$

(iii) $G(r,r') = 0$ on the plane $x = 0$

where $r = (x, y, z)$ is the point P and $r' = (x', y', z')$ is the point Q. Let $P'(\rho)$ be the image point of P(r) in x = 0. If Q lies on x = 0 then, since PQ = P'Q we have using (*i*) and (*iii*)

$$|r - r'| = |\rho - r'| \text{ so that } H(r,r') = -\frac{1}{|\rho - r'|}$$

Thus the Green's function by the method of image is

$$G(r,r') = \frac{1}{|r_1 - r'|} = \frac{1.}{|\rho - r'|} \tag{7.18}$$

Hence the required solution is obtained with the use of (7.16) and (7.18) as

$$u(r) = \frac{1}{4\pi}\iint_S u(\rho')\frac{\partial G(r,r')}{\partial n} ds$$

$$= \frac{1}{4\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(y',z')\left[\frac{\partial}{\partial x'}\left\{\frac{1}{\sqrt{(x-x')^2 + (y-y') + (z-z')}}\right]\right]$$

$$- \frac{1}{\sqrt{(x+x')^2 + (y+y')^2 + (z+z')^2}}\Bigg\}\Bigg]$$

$$= \frac{1}{4\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(y',z')\frac{2x}{\{x^2 + (y-y')^2 + (z+z')^2\}^{3/2}} dy'dz'$$

i.e. $$u(r) = \frac{x}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(y',z')\frac{f(y',y')dy'dz'}{\{x^2 + (y-y')^2 + (z+z')^2\}} \tag{7.19\_}$$

which can be integrated if the nture of the function $f(y',z')$ is given.

II. Solution of dirichlet problem for a circle

Let $P(r,\theta)$ and Q (r',θ') be two points having position vectors r and r' respectively and P' be the inverse point of P with respect to the circle so that $OP.OP' = a^2$ i.e. P' has coordinates $\left(\frac{a^2}{r},\theta\right)$. We construct the Greens function $G(r,r')$ such that

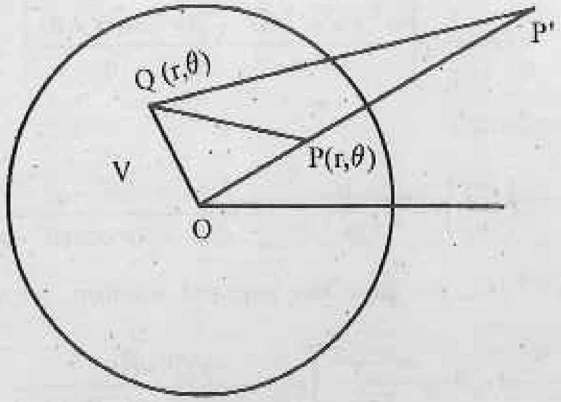


Fig.-7.5

$$G(r_1,r') = \text{In}\frac{1}{|r_1 - r'|} + H(r_1,r')$$

and let $H(r,r') = \text{In}\left(r\frac{P'Q}{a}\right)$

Now, $P'Q^2 = OQ^2 + OP'^2 - 2OQ,OP'\cos(\theta'-\theta)$

$$= r'^2 + \frac{a^4}{r^2} - 2r'.\frac{a^2}{r}\cos(\theta'-\theta)$$

i.e. $r^2 P'Q^2 = r'^2 r' + a'^1 - 2rr^1\cos(\theta'-\theta)$

or $\frac{r^2 P'Q^2}{a^2} = \frac{r^2 r^{12}}{a^2} + a^2 - 2rr'\cos(\theta'-\theta)$

Thus $\quad H(r,r')\ \text{In}\left\{\frac{r'^2}{a^2}r^2+a^2-2rr'\cos(\theta'-\theta)\right\}$

It can be readily verified that s $\nabla^2 H=0$ Also $G=\text{In}\ \frac{r.P'Q}{a.PQ}$ so that on the circle $r=a$ $G=0$

Since $PQ^2=r^2+r'^2-2rr'\cos(\theta'-\theta)$, we have

$$G=\frac{1}{2}\text{In}\left[\frac{a^2+r^2r'^2/a^2-2rr'\cos(\theta'-\theta)}{r^2+r'^2-2rr'(\theta+\theta)}\right]$$

so that on the circle

$$\left(\frac{\partial G}{\partial n}\right)_{r'=a}=\left(\frac{\partial G}{\partial r'}\right)_{r'=a}=\frac{r^2-a^2}{a[a^2-2ar\cos(\theta'-\theta)+r^2]}$$

Hence using (7.16), we have the required solution

$$u(r,\theta)=\frac{a^2-r^2}{2\pi a}\int_0^{2\pi}\frac{f(\theta')d\theta'}{a^2-2ar\cos(\theta'-\theta)+r^2]} \tag{7.20}$$

III. Solution of Dirichlet problem for a Sphere

Here we determine the function $u(r,\theta,\phi)$ sotisfying

$$\nabla^2 u=0,\ 0\le r\le a,\ \ 0\le\theta\le\pi,\ 0\le\phi\le 2\pi$$

subject to $\quad u(a,\theta,\phi)=f(\theta,\phi)$

Green's function for a sphere can be expressed as

$$G(r,r')=\frac{1}{|r-r'|}+H(r,r')$$

Where H satisfies the conditions

$$\left(\frac{\partial^2}{\partial x'^2}+\frac{\partial^2}{\partial y'^2}+\frac{\partial^2}{\partial z'^2}\right)H(r,r')=0 \tag{7.21}$$

and $G(r,r')=0$ (7.22)

on the surface $r=a$ of the sphere.

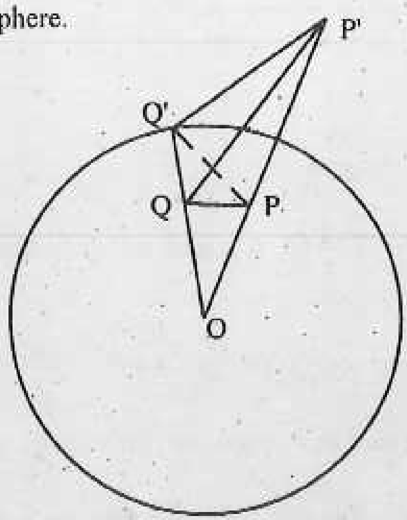


Fig-7.6

Let $P(r,\theta,\phi)$ be a point within the sphere and r' is the inverse point of P' with respect to the sphere so that and $OP=r$ & $OP'=r'$. Let Q' be a variable point on the surface of the sphere. Then from the similar triangles $OQ'P$ and $OQ'P'$ we get

$$\frac{PQ'}{P'Q'}=\frac{r}{a}=\frac{a}{\rho}\Rightarrow PQ'\frac{r}{a}P'Q' \quad \text{where } OQ'=\rho$$

Since this relation is valid for all points on the spherical surface, we have

$$H(r,r')=-\frac{a}{r}\frac{1}{|P'Q'|}=-\frac{a}{r|OP'-OQ'|}=-\frac{a}{r|\rho-r'|}$$

$$=-\frac{a}{r\left|\frac{a^2}{r}\frac{r}{r}-r'\right|}=-\frac{a}{r\left|\frac{a^2}{r^2}r-r'\right|}$$

This form of H satisfies the Laplace equation (7.21). Let $Q(r',\theta',\phi')$ be a variable point inside the sphere. If Q lies on the surface of the sphere, say at Q', then $\frac{PQ}{P'Q}=\frac{r}{a}$. Thus the Green's function for the present problem is

$$G(r,r') = \frac{1}{|r-r'|} - \frac{a/r}{\left|\frac{a^2}{r^2}r - r'\right|}$$

$$= \frac{1}{|PQ|} - \frac{a/r}{|P'Q|} = \frac{1}{R} - \frac{a/r}{R'}$$

where $PQ = R$ and $P'Q = R'$. It is easy to verify that G vavishes on the surface of the sphere.

Now $R^2 = r^2 + r'^2 - rr'\cos\Theta$, $R'^2 = \frac{a^4}{r^2} + r'^2 - \frac{2a^2}{r}r'\cos\Theta$ where

$\cos\Theta = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi - \phi')$. It follows that

$$\frac{\partial G}{\partial n} = \frac{\partial G}{\partial r'} = -\frac{1}{R^2}\frac{\partial R}{\partial r'} + \frac{a/r}{R'^2}\frac{\partial R'}{\partial r'} = -\frac{1}{R^3}\left[R\frac{\partial R}{\partial r'} - \left(\frac{a}{r}\right)\frac{R^3}{R'^3}R'\frac{\partial R'}{\partial r'}\right]$$

$$= -\frac{1}{R^3}\left[R\frac{\partial R}{\partial r'} - \frac{a}{r}\frac{r^3}{a^3}R'\frac{\partial R'}{\partial r'}\right] \quad \left(\because \frac{R}{R'} = \frac{PQ}{P'Q} = \frac{r}{a}\right)$$

so that $$\left(\frac{\partial G}{\partial n}\right)_{r'=a} = -\frac{1}{R^3}\left[(a - r\cos\Theta) - \frac{r^2}{a^2}\left(a - \frac{a^2}{r}\cos\Theta\right)\right]$$

$$= \frac{r^2 - a^2}{aR^3} = \frac{r^2 - a^2}{a\left(r^2 + a^2 - 2ar\cos\Theta\right)^{3/2}}$$

Thus the solution of the interior Dirichlet problem for a sphere is given from (7.16) as

$$u(r,\theta,\phi) = \frac{\left(a^2 - r^2\right)}{4\pi a}\iint \frac{f(\theta',\phi')}{\left(r^2 + a^2 - 2ar\cos\Theta\right)^{3/2}}ds'$$

i.e. $$u(r,\theta,\phi) = \frac{a\left(a^2 - r^2\right)}{4\pi}\int_0^{2\pi}\int_0^{\pi} \frac{f(\theta',\phi')\sin\theta' d\theta' d\phi'}{\left(r^2 + a^2 - 2ar\cos\Theta\right)^{3/2}} \tag{7.23}$$

since $dS' = a^2\sin\theta' d\theta' dp'$. The equation (7.23) is known as *Poisson integral formula.*

§ 7.3 Green's Function for the Diffusion Equation

Let us consider the diffusion equation

$$\frac{\partial T}{\partial t} = \kappa \nabla^2 T \tag{7.25}$$

We solve this equation for $T(r,t)$ in the volume V bounded by the surface S by using Green's function technique, subject to the boundary condition

$$T(r,t) = \theta(r,t), r \in s \tag{7.26}$$

and the inital condition

$$T(r,0) = f(r), r \in V \tag{7.27}$$

We define the Green's function $G(r,r',t-t'), t > r'$ where t' is a parameter, satisfying the following conditions :

(*i*) $\dfrac{\partial G}{\partial t} = \kappa \nabla^2 G$

(*ii*) Boundary condition : $G(r,r',t-t') = 0,\ r' \in S$

(*iii*) The initial condition is $\lim_{t' \to t} G = 0$ for all points in V except at the point where G has the singular solution of the form

$$\frac{1}{8\{\pi k(t-t')\}^{3/2}} \exp\left\{-\frac{|r-r'|^2}{4k(t-t')}\right\}$$

and hence $\dfrac{\partial G}{\partial t'} = -\kappa \nabla^2 G$. The equation (7.25) and the boundary condition (7.26) may, therefore, be rewritten as

$$\frac{\partial T}{\partial t'} = \kappa \nabla^2 T . t' < t$$

$$T(r,t') = \theta(r,t'), r \in S$$

Also $\frac{\partial}{\partial t'}(TG) = G\frac{\partial T}{\partial t'} + T\frac{\partial G}{\partial t'} = \kappa G\nabla^2 T - \kappa T\nabla^2 G$

Let ε be an arbitrary positive constant. Then

$$\int_0^{t-\varepsilon}\left\{\iiint_V \frac{\partial}{\partial t'}(TG)dv'\right\}dt' = \kappa\int_0^{t-\varepsilon}\left\{\iiint_V \frac{\partial}{\partial t'}(G\nabla^2 T - T\nabla^2 G)dv'\right\}dt' \qquad (7.28)$$

Now the L.H.S. $= \int_0^{t-\varepsilon}\left\{\iiint_V \frac{d}{dt}(TG)dt'\right\}dv' = \iiint_V (TG)_0^{t-\varepsilon}dv'$

$$= \iiint_V \left\{(TG)_{t'=t-\varepsilon} - (TH)_{t'=0}\right\}dv'$$

$$= T(r,t)\iiint_V G(r,r',t-t')|_{t'=t-\varepsilon}^{dv'} - \iiint_V G(r,r',t)|f(r')dv'$$

where we have used the initial condition (7.27) and the fact that

$$[T(r',t')]_{t'=t-\varepsilon} = T(r',t-\varepsilon) = T(r,t)$$

Also $\iiint_V G(r,r',t-t')|_{t'=t-\varepsilon}^{dv'} = \iiint_V \frac{1}{8(\pi\kappa\varepsilon)^{\frac{1}{2}}}\exp\left\{\frac{|r-r'|}{4\kappa\varepsilon}\right\}dv' = 1$

as $\varepsilon \to 0$

Again, applying Green's wheorem on the R.H.S. of (7.28) we get

R.H.S $= \kappa\int_0^{t-\varepsilon}\left\{\iiint_{V'}(G\nabla^2 T - T\nabla^2 G)dv'\right\}dt'$

$$= \kappa\int_0^{t-\varepsilon}\left\{\iint_S\left(G\cdot\frac{\partial T}{\partial n} - T\frac{\partial G}{\partial n}\right)ds'\right\}dt'$$

$$= \kappa\int_0^t\left\{-\iint_S T(\ ,t)\frac{\partial G}{\partial n}ds'\right\}dt' \quad (\because G=0, \text{on S and taking } \varepsilon\to 0)$$

Hence the equation (7.28) reduces to

$$T(r,t)=\iiint_V f(r')G(r,r',t)dv'-\kappa\int_0^t\left\{\iint_S T(r',t)\frac{\partial G}{\partial n}\right\}ds' \qquad (7.29)$$

§ 7.4 Green's For the Wave Equation-Helmholtz theorem

Helmholtz Theorem :

Let $u(r)$ be a function of $r=(x,y,z)$ possessing continuous partial derivatives fo first and second order in a region V bounded by a closed surfaces S and satisfies the spatial form of the wave equation $\nabla^2 u+c^2 u=0$ Then

$$\frac{1}{4\pi}\iint_S\left\{\frac{\exp(ic|r-r'|)}{u(r')}\frac{\partial u(r')}{\partial n}-u(r')\frac{\partial}{\partial n}\left(\frac{\exp(ic|r-r'|)}{|r-r'|}\right)\right\}$$

$$=\begin{cases}u(r), r\in V\\ 0, r\notin V\end{cases} \qquad (7.30)$$

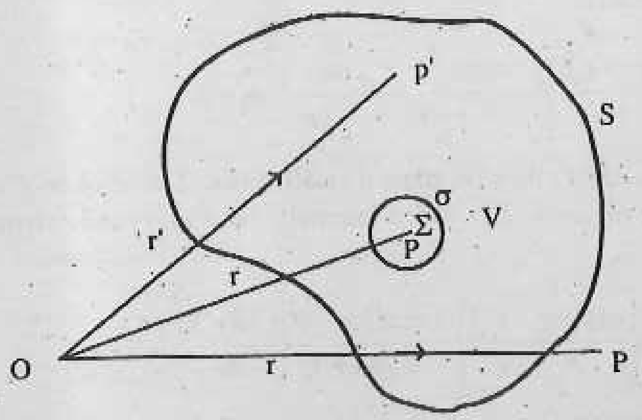


Fig-7.7

Proof. Let u be a solution of the Helmholtz equation

$$\nabla^2 u + c^2 u = 0 \tag{7.31}$$

in the closed region V bounded by the surface S and let all the singularities of u lie outside V.

Consider now the singularity solution

$$u' = \exp(ic|r - r'|)/|r - r'| \tag{7.32}$$

Putting this value of u' in Green's theorem

$$\iiint_V (u\nabla^2 u' - u'\nabla^2 u)dv = \iint_S \left(u\frac{\partial u'}{\partial n} - u'\frac{\partial u}{\partial n}\right)ds \tag{7.33}$$

where n is the outward drawn normal to S, we get the L.H.S. as

$$\iiint_V (u\nabla^2 u' - u'\nabla^2 u)dv = \iiint_V \left[u\nabla^2\{\exp(ic|r - r'|)/|r - r'|\}\right.$$

$$\left. - \{\exp(ic|r - r'|)/|r - r'|\}(-c^2 u)\right]dv$$

$$= \iiint_V \left[-c^2 u\frac{\exp(ic|r - r'|)}{|r - r'|} + c^2 u\frac{\exp(ic|r - r'|)}{|r - r'|}\right]dv$$

$$= 0 \quad (\because |r - r'| \neq 0 \text{ as the point lies outside } V)$$

Hence from (7.33) we have

$$\iint_S \left[u(r')\frac{\partial}{\partial n}\left\{\frac{\exp(ic|r - r'|)}{|r - r'|}\right\} - \frac{\exp(ic|r - r'|)}{|r - r'|}\frac{\partial u(r')}{\partial n}\right]ds = 0 \tag{7.34}$$

If the point r lies inside V, then we draw a small sphere Σ to enclose P and then apply Green's theorem to the region V–Σ bounded internally by σ (say) and externally by S. We, therefore, have

$$\left\{\iint_\sigma + \iint_s\right\}\left[u(r')\frac{\partial}{\partial n}\left\{\frac{\exp(ic|r - r'|)}{r - r'}\right\} - \frac{\exp(ic|r - r'|)}{|r - r'|}\frac{\partial u(r')}{\partial n}\right]ds = 0 \tag{7.35}$$

where $|r - r'| \neq 0$. Noting that on σ

$$\frac{\partial}{\partial n}\left\{\frac{\exp(ic|r - r'|)}{|r - r'|}\right\} = \frac{\partial}{\partial \varepsilon}\left\{\frac{\exp(ic\varepsilon)}{\varepsilon}\right\} = \left(ic - \frac{1}{\varepsilon}\right)\frac{\exp(ic\varepsilon)}{\varepsilon}$$

$$=\left(ic-\frac{1}{|r-r'|}\right)\frac{\exp(ic|r-r'|)}{|r-r'|}$$

we have from (7.35)

$$\iint_S\left[u(r')\frac{\partial}{\partial n}\left\{\frac{\exp(ic|r-r'|)}{|r-r'|}\right\}-\frac{\exp(ic|r-r'|)}{|r-r'|}\frac{\partial u(r')}{\partial n}\right]ds$$

$$=-\iint_\sigma\left\{\left(ic-\frac{1}{|r-r'|}\right)u(r')-\frac{\partial u(r')}{\partial n}\right\}\frac{\exp(ic|r-r'|)}{|r-r'|}ds \qquad (7.36)$$

Using the relations $u(r')=u(r)+o(\varepsilon)$, $\frac{\partial u}{\partial n}=\left(\frac{\partial u}{\partial n}\right)_p+O(\varepsilon)$, $ds=\varepsilon^2\sin\theta d\theta d\phi$,

we have from (7.36)

$$\iint_S\left[u(r')\frac{\partial}{\partial n}\left\{\frac{\exp(ic|r-r'|)}{|r-r'|}\right\}-\frac{\exp(ic|r-r'|)}{|r-r'|}\frac{\partial u(r')}{\partial n}\right]ds$$

$$=\iint_\sigma\left[\left(ic-\frac{1}{|r-r'|}\right)[u(r)+o(\varepsilon)]-\left(\frac{\partial u}{\partial n}\right)_p-o\varepsilon\right]\frac{\exp(ic|r-r'|)}{|r-r'|}\varepsilon^2\sin\theta d\theta d\phi$$

$$=-4\pi u(r)$$

so that we have the result

$$\frac{1}{4\pi}\iint_S\left[\frac{\exp(ic|r-r'|)}{|r-r'|}\frac{\partial u(r')}{\partial n}-u(r')\frac{\partial}{\partial n}\left\{\frac{\exp(ic|r-r'|)}{|r-r'|}\right\}\right]ds$$

$$=\begin{cases}u(r), & r\in v\\ 0, & r\notin v\end{cases}$$

Green's function

We now introduce the Green's function $G(r,r')$ satisfying the conditions

(*i*) $\nabla^2G(r,r')+c^2G(r,r')=0$

and (*ii*) G is is finite and continuous with respect to both the variables r and r'

Replacing $\exp\left(ic|r-r'|/|r-r'|\right)$ by $G(r,r')$ in (7.30), we get

$$u(r)=\frac{1}{4\pi}\iint_S\left[G(r,r')\frac{\partial u(r')}{\partial n}-u(r')\frac{\partial G(r,r')}{\partial n}\right]ds$$

where n is the outward drawn normal to S If $G(r,r')=0$ on S and $G(r,r')$ satisfies the conditions (i) and (ii) above, then

$$u(r)=-\frac{1}{4\pi}\iint_S u(r')\frac{\partial G(r,r')}{\partial n}ds \tag{7.37}$$

Exercises

1. Solve $\nabla^2 u=0$ in the upper half-plane defined by $y\geq 0, -\infty<x<\infty$ using Green's function method, subject to the condition $u=f(x)$ on $y=0$.

$$\left[\text{Ans. } u(x,y)=\frac{y}{\pi}\int_{-\infty}^{\infty}\frac{f(x')dx'}{(x-x')^2+y^2}\right]$$

2. Use the method of images to show that the harmonic Green's function for the half-spare $z\geq 0$ is $G(r,r')=\frac{1}{4\pi}\left(\frac{1}{r}-\frac{1}{r'}\right)$ where $r^2(x-x')^2+(y-y')^2+(z-z')^2$ and $r'^2=(x-x')^2+(y-y')^2+(z+z')^2$

3. Determine the Green's function for the Robin's problem on the quarter infinite place described by $\nabla^2 u=\phi(x,y), x>0, y>0$ subject to the conditions $u=f(y)$ on $x=0$ and $\frac{\partial u}{\partial n}=g(x)$ on $y=0$

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* Sections 7.3 and 7.4 are not included in the syllabi, but are given here for completion of the Unit.

$$\left[\text{Ans. } G(x,y)=\frac{1}{4\pi}\ln\left[\frac{\left\{(x-x')^2+(y-y')\right\}\left\{(x-x')^2+(y+y')^2\right\}}{\left\{(x+x')^2+(y-y')\right\}\left\{(x+x')^2+(y+y')^2\right\}}\right]\right]$$

§ 7.5 Summary

To find analytical solutions of the boundary value problems of Dirichlet type for Laplace's equation, Green's function has been introduced and some basic properties are discussed. Some specific boundary value problems have also been studied. For completion of this unit, the method of solution of diffusion equation and wave equation has also been noted.

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