NETAJI SUBHAS OPEN UNIVERSITY

STUDY MATERIAL

MATHEMATICS
POST GRADUATE

PG (MT) - 02
GROUPS : A & B

Real Analysis, Metric Spaces
&
Complex Analysis

Subject : Mathematics **Post Graduate**

Paper : PG (MT) 02 : Groups : A & B

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PG (MT) – 02
Real Analysis & Metric Spaces
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Complex Analysis

Group–A

Real Analysis & Metric Spaces

Group–B
Complex Analysis

GROUP–A

REAL ANALYSIS & METRIC SPACES

Unit 1 □ Open sets of reals, continuous functions, Functions of bounded Variation

(Algebraic and order properties of real number system R; Supremum and infimum of set of reals; Completeness of R; Interior point and limiting point of a set of reals; Open sets and closed sets in R, Structure of an open set as a Countable Union of disjoint open intervals, Contimity and Uniform Continuity of a real-valued function of a real variable. Monotone functions, points of discontinuity of a monotone function, functions of bounded variation. Representation of a function of bounded variation by monotone functions.)

§ 1.1 Let R denote the set of all real numbers.

Algebraic properties of R : There are two binary operations namely arithmetic (i) addition, denoted by + and (ii) multiplication, denoted by. respectively in R such that (R, +) is a commutative group and (R,) is commutative semi-group with multiplicative identity = 1 ∈ R such that non-zero numbers of R have multiplicable inverses and following distributive Laws hold in (R, +,).

(D) For any three members a b and c in R where $a.(b + c) = a.b + a.c$ and $(a + b).c = a.c + b.c$.

Properties listed above under (i), (ii) and (D) taken all together are field axioms in abstract algebra, and we say that R is a field.

Notations : Let (a) N denote the set of all natural numbers *i.e.*, N = (1, 2, 3, ..., n, n + 1, ...)

(b) Z denote the set of all integers (whole numbers) +ve, –ve and zero *i.e.*, Z = (..., -3, -2, -1, 0, 1, 2, 3, ...)

(c) members of R written as $\frac{a}{b}$ where a, $b \in Z$ with $b \neq 0$ be identified as rational numbers, and Q denote the set of all such rational numbers +ve, –ve and Zero.

The numbers of R/Q are said to be irrational numbers, meaning that they are not ratios of integers. There are many irrational numbers like $\sqrt{2}, -5\sqrt{7}, \log_{10}^{3}, e, \pi$ etc.

§ Order properties of R :

Definition 1.1.1 Given two members $a, b \in$ R, we say $a \leq b$ if and only if $b - a$ is non negative and $\leq$ defines an order relation in R. For example, by Induction, every natural number n satisfies $n > 0$.

Theorem 1.1.1 If $a \in R$ satisfies $0 \leq a < \varepsilon$ for every + ve ε, then $a = 0$.

Proof : Assume the contrary, and let a be +ve. Take $\varepsilon = \frac{1}{3}a$. Then $0 < \varepsilon < a$; So it is not true that $a < \varepsilon$ for **every** + ve ε. The conclusion is $a = 0$.

Remark 1 Product of two + ve numbers is +ve. However positivity of the product if two numbers does not imply that **each factor** is +ve. There if $ab > 0$, then **either** $a > 0$ and $b > 0$ **or** $a < 0$ and $b < 0$.

2. If $ab < 0$, then a and b must possess opposite signs each being non-zero.

Absolute Value : For $a \in$ R, $|a| = \begin{cases} a \text{ if } a > 0 \\ 0 \text{ if } a = 0 \\ -a \text{ if } a < 0 \end{cases}$

Theorem 1.1.2 (Triangle inequality) if $a, b \in$ R, $|a+b| \leq |a| + |b|$.

Proof: Here $-|a| \leq a \leq |a|$ and $-|b| \leq b \leq |b|$. So

$$a + b \leq |a| + |b| \quad \ldots \quad (1)$$

From $a \geq -|a|$ we have $-a \leq |a|$ and similarly $-b \leq |b|$ giving

$$-(a+b) \leq |a| + |b| \quad \ldots \quad (2)$$

From (1) and (2) $\pm (a + b) \leq |a| + |b|$. That is to say $|a+b| \leq |a| + |b|$.

Corollary : For, $a, b \in R$, $||a|-|b|| \le |a-b|$.

Because $\quad |a| = |a-b+b| \le |a-b| + |b|$

or $\quad |a|-|b| \le |a-b| \quad \ldots \quad (1)$

Interchanging a and b we have

$$|b|-|a| \le |b-a| = |a-b| \quad \ldots \quad (2)$$

(1) and (2) give $\pm(|a|-|b|) \le |a-b|$ and hence $||a|-|b|| \le |a-b|$

Definition 1.1.2 For a non-empty subset A of R,

(a) A is said to be bounded above if there is a fixed number $K \in R$ such that $a \le y$ for all $a \in A$, and then K is said to be **an** upper bound for A.

(b) A is said to be bounded below if there is a fixed number $k \in R$ such that $k \le a$ for all $a \in A$, and in this case k is said to be a lower bound for A.

(c) A is said to be bounded if it is bounded above plus bounded below.

(d) A is said to be unbounded if it is **not** bounded.

Definition 1.1.3 (a) if A is bounded above, then the number $M \in R$ is said to be the least upper bound (l.u.b.) or supremum (sup) of A if (i) M is **an** upper bound of A and (ii) given any +ve ε, there is a member $a \in A$ (depending on ε) such that $a > M-\varepsilon$.

(b) If A is bounded below then the number $m \in R$ is said to be **the** greatest lower bound (g.l.b.) or Infimum (Inf) of A if (i) m 's **a** lower bound for A and (ii) given any +ve ε, there is a member $a \in A$ (depending on ε) such that $a < m+\varepsilon$.

Theorem 1.1.3 If A and B are two non-empty subsets of R such that $a \le b$ for all $a \in A$ and for all $b \in B$, then

$$\text{Sup } A \le \text{Inf } B.$$

Proof : Let $b \subset B$, then $a \leq b$ for all $a \in A$. So b is **an** upper bound for A, and hence Sup A $\leq b$.

i. e $\qquad b \geq$ Sup A

This inequality stands for all $b \in B$; thus Sup A is a lower bound for B, and hence Inf B$\geq$SupA, or SupA $\geq$Inf B.

§ 1.2 Intervals in R

Order relation $\leq$ in R invites a natural family of subsets of R called intervals. If and $a,b \in R$ $a < b$, then the subset $\{ x \in R : a < x < b\}$ is called an open interval, denoted by (a, b) with a as the left-hand and point and b as the right-hand end point.

If end points are adjoined to open interval, what results in is the closed interval denoted by $[a, b] = \{x \in R : a \leq x \leq b\}$. Similarly one has left-open and right-open intervals like $(a, b]$ and $[a, b)$.

Each of these intervals is a bounded set of reals and its length = $b - a$.

Sets of the form $\{x \in R : x > a\} = (a,\infty)$ and $\{x \in R : x < a\} = (-\infty, a)$ are called open half-rays or infinite open intervals. Similarly we have infinite closed intervals like $[a, \infty)$ and $(-\infty, a]$. There is a caution. $\pm\infty$ are not numbers *i.e.* $\pm\infty \notin R$; They are symbols of some limits.

Example 1.1.1. If $f : [o,1] \to R$ and $g : [o,1] \to R$ are two functions given by $f(x) = x^2$ and $g(x) = x$ for $0 \leq x < 1$. Then (a) $\sup\limits_{[0,1]} f(x) \leq \sup\limits_{[0,1]} g(x)$

(b) $\sup\limits_{[0,1]} f(x) \leq \inf\limits_{[0,1]} g(x)$ Which one is true? Give reasons.

Solution. Kore $x^2 \leq x \qquad$ in $0 \leq x \leq 1$

$\qquad$ or, $\quad f(x) \leq g(x) \qquad$ in $[0,1]$

Therefore, (a) $\sup_{x\in[0,1]} f(x) \leq \sup_{x\in[0,1]} g(x)$; In fact, L.H.S. = R.HS = 1

Here (b) is not true. Because, $f(x)<g(y)$ for all $x,y\in[o,1]$ for instance, taking $x=1$ and $y=\frac{1}{2}$ we find $f(1) = 1 \not< \frac{1}{2} = g\left(\frac{1}{2}\right)$.

Completeness of R : Every non-empty set of reals that has an upper bound possesses its supremum.

This property of R also called as supremum property is the property of completeness of R. The analogous property for infimum may be obtained as follows :

Suppose B is a non-empty set that is bounded below. Then $A = \{-b : b \in B\}$ is a non-empty set that is bounded above, and by supremum property A has the supremum, say $= u \in R$. It is easy to check that $-u = \text{Inf } B$.

Remark. In view of completeness property R is also called a complete ordered field.

Archimedean property of R : If a is any real number, there is a natural number $n \in N$ such that $n > a$.

An application : If $E = \left\{1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots\right\}$, then Inf E = 0.

Solution : E is a set that is bounded below and Inf E exists; say = u. Also $u \geq 0$ (0 is a lower bound of E). Let $\epsilon > 0$ be arbitrary. Archimedean property shows that there is a natural number $n \in N$ such that $n > \frac{1}{\epsilon}$ or $\frac{1}{n} < \epsilon$, and $0 \leq u < \frac{1}{n} < \epsilon$. As $\epsilon > 0$ is arbitary, we have $u = 0$.

Density of rationals in R : Between only two real numbers there are many rationals.

Corollary If x and y are real numbers with $x < y$, here is an irrational z in between x and y.

Because by Density property of rationals we find a rational r such that

$$\frac{x}{\sqrt{2}} < r < \frac{y}{\sqrt{2}}.$$

So $z = \sqrt{2}r$ is an irrational number with $x < z < y$.

Example 1.1.2 Let $f: [a,b] \to R$ and $g: [a,b] \to R$ be two bounded functions, then

(i) Sup $\{f(x) + g(x): a \le x \le b\} \le \sup \{f(x): a \le x \le b\} + \sup \{g(x): a \le x \le b\}$ and

(ii) Inf $\{f(x): a \le x \le b\}$ + Inf $\{g(x): a \le x \le b\}$

$\le \inf \{f(x) + g(x): a \le x \le b\}$.

Solution : Let M_f = Sup $\{f(x): a \le x \le b\}$ and Let $M_g = \sup \{g(x) : a \le x \le b\}$. Now $f(x) \le M_f$ and $g(x) \le M_g$ for $a \le x \le b$; Then the $f(x) + g(x) \le M_f + M_g$; right-hand side is independent of x in $[a,b]$: So we have sup sup$\{f(x) + g(x)\} \le M_f + M_g$ for all x in $[a, b]$ and therefore (i) follows. Similarly (ii) Holds.

Example 1.1.3 If A and B are two non-empty bounded sets, and $A + B = [a + b : a \in A$ and $b \in B]$. Show that Sup (A + B) = Sup A + Sup B.

and, Inf (A + B) = Inf A + Inf B.

Solution : Since A and B are non-empty bounded sets in R, We have $a \le$ Sup A and $b \le$ Sup B whenever $a \subset$ A and $b \in$ B. So $a + b \le$ Sup B.

Thus sup $\{a + b : a \subset A$ and $b \in A\}$ = Sup $\{A + B\} \le$ Sup A + Sup B. (1)

Given a + ve ε, there is a member $a_0 \subset$ A such that $a_0 >$ SupA $- \frac{\varepsilon}{2}$ (2)

and there is a member $b_0 \in$ B such that $b_0 >$ Sup B $- \frac{\varepsilon}{2}$. Therefore

$a_0 + b_0 >$ Sup (A) + (B) $- \varepsilon$, where $(a_0 + b_0) \in$ A + B. Now coupled with (1) we have Sup (A + B) = SupA + SupB. Similarly one gets Inf (A + B) = Inf A + Inf B.

§ 1.2

We study sets of points on the real line-geometric continuum corresponding to arithmetic continuum of all real numbers. When we say that point x lies on right of the point y we are guided by order of reals to have in mind that $x > y$.

Definition 1.2.1 A point x_0 of a set A of reals is said to be an interior point of A if there is an open interval (a,b) with $x_0 \in (a,b) \subset$ A.

Thus an interior point of A is a member of A.

For Example if A = (0, 1) U $\left\{3, -7, \frac{5}{2}\right\}$, then every element of the open interval (0,1) is an interior point of A, but none of 3, -7 and $\frac{5}{2}$ of A is so. Also a finite subset of reals has no interior point; Also the set Q of all rationals is devoid of any interior point.

Definition 1. 2. 2. A subset G of R is called an open set if all its points are interior points.

For example if G = (–1, + 1) U (2, 3), then G is an open Set without being an open interval; A closed interval $[a, b]$ is not an open set because its end points although being points of the closed interval are not interior points of the closed interval.

Definition 1.2.3 Given a non-empty set E of reals a number $u \in$ R is said to be a limit point of E if *every open interval* containing u shall meet E at a point other than a.

Explanation : If $u \in$ R is a limit point of E$(\neq \Phi)$, then open intervals like $(u - \delta, u + \delta)$, $\delta > 0$ (δ may be as small as one pleases) shall cut E at a point other than u ; in fact, $(u - \delta, u + \delta) \cap$ E contains many members of E other than u. Geometrically, u is not away from E. It is as close to E as one imagines ; but u is not necessarily a member of E. Also if u is *not* a limit point of E, then we find *an open interval* like

$(u-\delta_0, u+\delta_0)_{(\delta_0>0)}$ such that $(u-\delta_0, u+\delta_0)\cap E$ is either empty *or* equals to singleton $\{u\}$

Theorem 1.2.1 A real number $u\,(\in R)$ is a limit point of a non-empty set E if andonly if there is a sequence $\{x_n\}$ of distinct members in E such that $\lim_n x_n = u$.

Proof : Necessary part of the condition is not obvious and we need proving it. Let u be a limit of E. So open interval $(x_0 - 1, x_0 + 1)$ attracts points of E other than u;

Take one such and call it x_1 ; Now take a point $x_2 \in \left(E \cap \left(u - \frac{1}{2}, u + \frac{1}{2}\right)\right)$ other than

u and x_1 : and at nth stage of this process take $x_n \in \left(E \cap \left(u - \frac{1}{n}, u + \frac{1}{n}\right)\right)$ different from u, x_0 ,x_1 ,...,x_{n-1}. Then continuing this process we obtain a desired sequence $\{x_n\}$ in E satisfying $|u - x_n| < \frac{1}{n} \to 0$ as $n \to \infty$. That means $\lim_n x_n = u$.

Corollary : A non-empty finite set of reals has got no limit point. In this connection we may quote following theorem.

Theorem 1.2.2 (Bolzano-Weirstrass Theorem) Every bounded infinite set of reals has a limit point.

Remark : An unbounded infinite set of reals may not have a limit point. For example, the set N of all natural numbers supports this contention.

Definition 1.2.4 A subset G of R is said to be closed if all limit points of G belong to *G*.

For example, the set $\left\{1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots 0\right\}$ is a closed set; because only limit point of the set is zero and that is a member of the set. Also every finite set of reals is a closed set.

Theorem 1.2.3. Every open set of reals is a countable union of disjoint open intervals in R.

Proof : Let G be an open set and $x \in G$. So x is an interior point of G, and attracts an open interval $I = (a,b)$ with $x \in (a,b) \cap G$ Put $A_x = \{a \in R; (a, x] \subset G\}$ and let $\lambda = \text{Inf } A_x$: If $-\infty < \lambda$ then $\lambda \notin G$ by the property of Infimum. λ may be $-\infty$.

In a like manner taking $H_x = \{b \in R ; [x, b) \subset G\}$, and $\mu = \text{Sup } H_x$, we find $\mu \notin G$. and μ may be $+\infty$. Identify this largest open interval $I_x = (\lambda, \mu)$ with x with property $x \in I_x = (\lambda, \mu) \subset G$ in the sense that λ, μ do not belong to G.

Clearly then the family $\{I_x\}_{x \in G}$ of open intervals satisfies $G = \bigcup_{x \in G} I_x$. Now we show that if $u, v \in G$ with $u \neq v$, then either $I_u = I_v$ or $I_u \cap I_v = \phi$.

Suppose $z \in I_u \cap I_v$; If $I_u = (\lambda_u, \mu_u)$, and $I_v = (\lambda_v, \mu_u)$, we have $\lambda_u < z < \mu_u$ and $\lambda_v < z < \mu_u$. Now we show that $\lambda_u = \lambda_v$; otherwise either (i) $\lambda_u < \lambda_v$ or (ii) $\lambda_v < \lambda_u$; In case (i) $\lambda_v \in I_u = (\lambda_u, \mu_u) \subset G$ – a contradiction that end point $\lambda_v \notin G$.Similarly (ii) leads to a similar contradiction. Hence $\lambda_u < \lambda_v$ and in exactly similar manner one checks that $\mu_u = \mu_v$; that is $I_u \cap I_v \neq \phi$ means $I_u = I_v$.

Remaining part of the proof rests on showing that the family $\{I_x\}_{x \in G}$ of non-overlapping open intervals is in fact a countable family of disjoint members. Let an enumeration of the set Q of all rationals be $Q = (r_1, r_2, ..., r_n ...)$. Clearly each I_x attracts member of Q. Choose the member of Q in I_x with smallest index n in the enumeration as stated above. If $n(x)$ is the smallest index n identify I_x with I_{r_n}. In this way, collection of distinct open intervals I_x as $x \in G$ is put in I – I correspondence with a subset of N. So the family is countable, and the proof is complete.

Theorem 1.2.4 A subset G of R is open if and only if its complement (R\G) is closed.

Proof : Let G be an open set in R, and F = R\G, and u be a limit point of F; If $u \notin F$, then $u \notin G$, and u is an interior point of G, because G is open. So there is on an open interval, say $(u - \delta, u + \delta)$, $\delta > 0$ such that $(u-\delta, u+\delta) \subset G$. Clearly, $(u - \delta, u + \delta)$, $\cap F = \phi$ – a contradiction that u is a limit point of F. Thus we show $u \in F$ and F is closed.

Coversely, suppose (R\G) = F is closed, and take $x_0 \in G$, and hence $x_0 \notin F$. So x_0 is *not* a limit point of F because F is closed. Thus one can find an open interval, say, $(x_0 - \delta, x_0 - \delta)$, $\delta > 0$ such that $(x_0 - \delta, x_0 - \delta) \cap F$ is either empty or a singleton $\{x_0\}$. As $x_0 \notin F$, we have $(x_0 - \delta, x_0 + \delta) \cap F = \phi$ showing that $(x_0 - \delta, x_0 + \delta) \subset (R\backslash F) = G$; that means x_0 is an interior point of G. As x_0 is any arbitrary member of G, it follows that G is open.

Remark : The words 'open' and 'closed' may be interchanged without making Theorem 1.2.4 false.

Theorem 1.2.5 Any Union of open sets in R is an open set.

Proof : Let $\{G_\alpha\}_{\alpha\in\Delta}$ be a family of open sets in R and $G = \bigcup_{\alpha\in\Delta} G$, and take $x \in G$. Then $x \in G_\alpha$ for some $\alpha \in \Delta$; As G_α is open, x is an interior point of G_α and there is an open interval, say, $(x - \delta, x + \delta)$, $\delta > 0$ such that $(x - \delta, x + \delta)$, $\subset G_\alpha \subset \bigcup_{\alpha\in\Delta} G_\alpha = G$. Hence x is an interior point of G ; and G is shown as open.

Theorem 1.2.6 If G_1 and G_2 are two open sets, then $G_1 \cap G_2$ is an open set.

Proof : If $G_1 \cap G_2 = \phi$, then we take empty set as both open and closed, $G_1 \cap G_2$ is open. Suppose $u \in (G_1 \cap G_2)$, and then u is an interior point of G_1 and G_2. Thus we find open intervals, say, $(u - \delta_1, u + \delta_1) \subset G_1$, and $(u - \delta_2, u + \delta_2) \subset G_2$, $(\delta_i > 0)$. Taking δ as $0 < \delta < \min(\delta_1 \delta_2)$, it follows that $(u - \delta, u + \delta) \subset (G_1 \cap G_2)$. So u is an interior point of $(G_1 \cap G_2)$. Hence $G_1 \cap G_2$ is an open set.

Remark 1 By induction intersection of a *finite* number of open sets is a open set.

Remark 2 Intersection of an infinite number of open sets may not be an open set in R.

For example, take I_n = *an* open interval = $\left(-\frac{1}{n}, \frac{1}{n}\right)$ for $n = 1, 2, \ldots$; Here each I_n is also an open set and $\bigcap_{n=1}^{\infty} I_n = \{0\}$ = *a* singleton and it is not an open set. In fact, each singleton is a closed set. Further there may be a set in R without being open and closed. The set Q of all rationals in R is neither open nor closed. Also there are sets in R that are both open and closed. In fact, we show later that empty set ϕ and set R itself are only sets in R that are each closed and open (clo-open).

By a straight application of De-Morgan's rule we have Theorems analogues to those of Theorem 1.2.5 and 1.2.6 as under.

Theorem 1.2.7

(a) Intersection of any number of closed sets in R is a closed set.

(b) Union of a finite number of closed sets in R is a closed set.

Remark Arbitrary Union of closed sets may not be a closed set. For example, let us take $F_n = \left[0, \frac{n}{n+1}\right]$ Then each F_n is a closed set (being a closed interval); Here $\bigcup_{n=1}^{\infty} F_n = [0,1)$ which is not a closed set.

Definition 1.2.3 A set E of reals is said to presses Heine-Borel property if every open cover for E has a finite sub-cover.

Note : Notions of open cover and compactness have been given and discussed in some details in Metric spaces (see UNIT 8 §4.1). Taking R as a metric space with usual metric all results in a metric space in relation to compactness shall apply in R.

Theorem 1.2.6 (a) The closed interval $[a,b]$ $(a<b)$ is compact.

Proof : Let $[a,b]$ be not compact ; we seek a contradiction. Then there is an open cover. Say $\zeta = \{G_\alpha\}_{\alpha\in\Delta}$ of $[a,b]$ that admits of no finite sub-cover for $[a,b]$.| Then at least one of the closed sub intervals $\left[a, \frac{a+b}{2}\right]$ and $\left[\frac{a+b}{2}, b\right]$ say = $[a_1,b_2]\left(b_1 - a_1 = \frac{b-a}{2}\right)$ is not covered by a finite sub-family of ζ. We continue this process to construct a decreasing sequence $\{[a_n,b_n]\}$ such that

(i) $[a_1,b_1] \supset [a_2,b_2] \supset ... \supset [a_n,b_n] \supset$ with $b_n - a_n = \frac{b-a}{2n}$ $n = 1, 2,...$ and (ii) None of closed sub-intervals $[a_n,b_n]$ is covered by a finite sub-family of ζ. By Cantors interection Theorem we have $\bigcap_{n-1}^{\infty}[a_n, b_n]$ = a singleton, say, = $\{u\}$. Now $u\in[a,b]$ invites a member, say, G_α of ζ with $u\in G_\alpha$ and hence we have an open interval $(u - \delta, u + \delta), \delta > 0$ so that $(u - \delta, u + \delta), \subset G_\alpha$. Now choose n appropriately so large that $\frac{b-a}{2^{n+1}} < \delta$ and that $u\in[a_n, b_n] \subset (u - \delta, u + \delta), \subset G_\alpha$. That means, for such $n, [a_n,b_n]$ has a finite sub-cover out of ζ,– a contradiction as desired. The proof is complete.

Theorem 1.2.7 A subset E of reals is compact if and only if it is closed and bounded.

Proof : Let E the compact, and take $u\in(R\backslash E)$ and keep u fixed. For each $v\in E$, we have $u \neq v$, we can find two open intervals say $(u - \delta, u + \delta)$ and $(v - \delta, v + \delta), (\delta > 0)$, such that these two open intervals are disjoint.

Now consider the family $\{(v - \delta, v + \delta)\}_{v\in E}$ of all such open intervals $(v - \delta, v + \delta)$; As an open interval is an open set, this family is an open cover for E. By compactness of E there is a finite sub-cover of this family to cover E; say $(v_1 - \delta_1, v_1 + \delta_1)$, $(v_2 - \delta_2, v_2 + \delta_2)....(v_n - \delta_n, v_n + \delta_n)$. That is to say,

$$E \subset \bigcup_{i=1}^{n} (v_i - \delta_i, v_i + \delta_i)$$

Then corresponding open intervals $(u - \delta_i, u + \delta_i)$ with mid point u give an open interval $(u - \bar{\delta}, u + \bar{\delta})$ where $0 < \bar{\delta} < \min[\delta_1, \delta_2, ..., \delta_n]$.

Clearly $(u - \bar{\delta}, u + \bar{\delta}) \subset (u - \delta_i, u + \delta_i)$ for $i = 1, 2, ..., n$ and therefore

$(u - \bar{\delta}, u + \bar{\delta}) \subset (v_i - \delta_i, v_i + \delta_i) = \phi$ because

$(u - \delta_i, u + \delta_i) \cap (v_i - \delta_i, v_i + \delta_i) = \phi$

From (i) we have $\quad (u - \bar{\delta}, u + \bar{\delta}) \cap E = \phi$

or, $\quad (u - \bar{\delta}, u + \bar{\delta}) \cap (R \setminus E)$; thereby u becomes an interior point of (R\E); and (R/E) is open and hence E is closed.

To show that E is bounded, we know that $R = \bigcup_{n=1}^{\infty} (-n, n)$: So $E \subset R = R - \bigcup_{n=1}^{\infty} (-n, n)$. Thus $\{(-n, +n)\}_{n=1,2}$ is an open cover for E; Since E is compact, we have a finite sub-cover, say $(-n_i, + n_1), (-n_2, + n_2), ..., (-n_k, + n_k)$ where we assume $n_1 < n_2 < ... < n_k$. Thus $E = \bigcup_{i=1}^{\infty} (-n_i, n_i) = (-n_k, n_k)$. Hence E is bounded.

Conversely, Let E be a bounded and closed set of reals, and Let $E \subset [a, b]$. Suppose $\{G_\alpha\}_{\alpha \in \Delta}$ is an open cover for E. Since R\E is open, we find that the family $[\{G_\alpha\}_{\alpha \in \Delta} \cup (R \setminus E)]$ is an open cover for $[a, b]$ and by Theorem 1.2.6(a) it is compact. Thus one gets a finite sub-cover say $G_1, G_2, ..., G_n$ and possibly (R/E) for $[a, b]$. Clearly $G_1, G_2, ..., G_n$ forms an open sub-cover for E. Hence E is shown to be compact.

Example 1.2.1 Let A and B be two non-empty bounded sets of reals. Then $A \subset B$ implies $\text{Inf}\, B \le \text{Inf}\, A \le \sup A \le \sup B$.

Solution. Now $\text{Inf}\, B \le b$ for all $b \in B$

So $\text{Inf}\, B \le a$ if $a \in B$ (because $A \subset B$).

This gives $\text{Inf}\, B \le \text{Inf} A$ and rest of inequality chain follows from Definitions. So one has $\text{Inf}\, B \le \text{Inf}\, A \le \sup A \le \sup B$.

Example 1.2.2. For a bounded set A of reals if $T = \{|x-y| : x, y \in A\}$, show that Sup A – Inf A = Sup T.

Solution : Since A is bounded we have the set T as a bounded set.

If $x, y \in A$ we have $x - y \le |x - y| \le \text{Sup } T$

or, $x \le \sup T + y$

If y remains fixed this ineqality shows that $\text{Sup} A \le y + \text{Sup} T$ or, $\text{Sup} A - \text{Sup} T \le y$. Letting y free we get $\text{Sup} A - \text{Sup} T \le \text{Inf} A$

or, $\text{Sup} A - \text{Inf} A \le \text{Sup} T$ (1)

Again $\text{Sup} A \ge x$ for all $x \in A$, and $\text{Inf} A \le y$ for all $y \in A$.

So $\text{Sup} A - \text{Inf} A \ge x - y$ for all $x, y \in A$.

Interchange of x and y does not change L.H.S.; So we have $\sup A - \inf B \ge y - x$ for all $x, y \in A$.

Combining them we have $\text{Sup} A - \text{Inf} A \ge \pm(x - y)$ of all $x, y \in A$

i.e. $\text{Sup} A - \text{Inf} A \ge |x - y|$ of all $x, y \in A$

This inequality gives $\text{Sup} A - \text{Inf} A \ge \text{Sup} T$ (2)

From (1) and (2) we get $\text{Sup} A - \text{Inf} A = \text{Sup} T$.

§ 1.3

Let $[a, b](a < b)$ be a closed interval and $f:[a,b]\to R$ be a function and let $a \le c \le b$.

Definition 1.3.1 *f is* said to be continuous at $x = c$ if corresponding to $\epsilon>0$, there is a = + veδ such that $|f(x) - f(c)|<\epsilon$ whenever $|x-c|<\delta$. *i.e.* $c - \delta<x<c + \delta$.

If c is an end point like a or b, the inequality has to be tailored accordingly.

f is said to be a continuous function over $[a,b]$ if f is continuous at each point of $[a,b]$.

Note : Treatment has been made for properties continuous functions in some details in metric spaces. See UNIT 7, §3,1 R is also a metric space with usual metric and all those results for continuous functions shall apply irrespect of real-valued continuous functions of a real variable.

Definition : 1.3.2 $f:[a,b]\to R$ is said to be uniformly continuous if corresponding to *a + ve* ϵ there is *a + ve* δ such that $|f(x_1) - (x_2)| <\epsilon$ whenever $|x_1 - x_2|<\delta$ as $x_1, x_2 \in[a,b]$.

From the Definition it is a routine exercise to see that if *f* is a uniformlly continuous function, then *f* is a continuous function over $[a ,b]$ but converse is not true. Because $f(x) = x^{-1}$ over open interval (0,1) is a continuous function without being uniformly continuous there.

Every uniformly continuous function over $[a,b]$ sends a cauchy sequence in $[a,b]$ into a similar such sequence.

Proof : Let $\{x_n\}$ be a cauchy sequence in $[a,b]$, let $\epsilon>0$ be given in advance. By uniform continuity there is *a + ve* δ so that $|f(x) - f(u)|<\epsilon$ whenever $|x - u|<\delta$. Since $\{x_n\}$ is cauchy, corresponding to this + *ve* δ we find an index N such that $|x_n-x_m|<\delta$ whenever $n,m \ge N$.

Now by choice of δ, we find $f(x_n) - f(x_m)|<\varepsilon$ whenever $|x_m - x_m| < \delta$ for $n,m \geq N$. That means the sequence $\{f(x_n)\}$ is cauchy.

Remark : The word 'uniformly' can not be removed from Theorem 1.3.1. Because, consider the continuous function $f(x) = \frac{1}{x}$ over open interval (0,1) Here f transforms cauchy sequence $\left\{\frac{1}{n}\right\}$ into a non-cauchy sequence $\{n\}$ in R.

§ 1.4

Definition 1.4.1. A function $f:[a,b] \to R$ is said to be monotone increasing (↑) if $f(x_1) < f(x_2)$ whenever $a \leq x_1 < x_2 \leq b$.

It is said to be strictly monotone increasing if $f(x_1) < f(x_2)$ whenevr $a \leq x_1 \leq x_2 \leq b$.

It is said to be monotone decreasing (↓) if $f(x_1) \geq f(x_2)$ whenever $a \leq x_1 < x_2 \leq b$.

Remark 1 If $f:[a,b] \to R$ is ↑, then $-f$ is ↓, and vice-versa. This permits us to consider only ↑ functions in many problems.

Remark 2 If $f:[a,b] \to R$ is ↑ (or) ↓ then is a bounded function.

Theorem 1.4.1 Let $f:[a,b] \to R$ be an ↑ function, and $a<c<b$; Then

(i) $\lim_{x \to c-} f$ (Left-hand limit of f at c) = Sup $\{f(x): a \leq x < c\}$

(b) $\lim_{x \to c+} f$ (right-hand limit of f at c) = Inf $\{f(x): a < x \leq b\}$.

Proof : (a) If $a \leq x < c$, we have $f(x) \leq f(c)$ because f is ↑.

Now the set $\{f(x): a \leq x < c\}$ has therefore an upper bound = $f(c)$.

So $\sup\{f(x): a \leq x < c\} = L$ (say) exists. Let $\varepsilon > 0$ be given. So we find x_ε with $x_\varepsilon < c$ with $f(x_\varepsilon) > L - \varepsilon$. Now if $0 < \delta_\varepsilon = c - x_\varepsilon$, and $c - \delta_\varepsilon < x < c$ we have $L - \varepsilon < f(x_\varepsilon) \leq f(x) \leq L$.

This gives, $|f(x) - L| < \varepsilon$ whenever $c - \delta_\varepsilon < x < c$. That means $\lim_{x \to c} f(x) = L = \text{Sup}\{f(x): a \le x < c\}$. The proof for (*b*) will be similar.

Corollary : Let $f:[a,b] \to R$ is an ↑ function, and $a < c < b$. Then following statements are equivalent.

(i) f is continuous at $x = c$.

(ii) $\text{Sup}\{f(x): a \le x < c\} = \lim_{x \to c-} f = f(c) = \text{Inf} = \{f(x): c < x \le b\}$

Remark : It is an easy exercise to show that f is continuous at left-hand end point $x = a$ if and only if $f(a) = \text{Inf}\{f(x): a < x \le b\}$; or equivalently if and only if $f(a) = \lim_{x \to c+} f$.

Similar conditions shall apply at right-hand and point $x = b$.

Definition 1.4.2 If $f:[a,b] \to R$ is an ↑ function, and $a < c < b$, then Jump of f at $x = c$, denoted by $w_f(c) = \lim_{x \to c+} f - \lim_{x \to c-} f$.

Explanation : $0 < w_f(c) = \lim_{x \to c+} f - \lim_{x \to c-} f$.

$$= \text{Inf}\{f(x): c < x \le b\} - \text{Sup}\{f(x): a \le x < c\}.$$

If $x = a$, we define $w_f(a) = \lim_{x \to a+} f - f(a)$

and if $x = b$, we define $w_f(b) = f(b) \lim_{x \to b-}$

From the corollary above it is now clear that for an ↑ function f in $[a,b]$ and $a \le c < b$, f is continuous at $x = c$ if and only if $w_f(c) = 0$.

Writing $f(c + 0) = \lim_{x \to c+} f(x)$ and $f(c - 0) = \lim_{x \to c-} f(x)$, we have the jump $w_f(c) = f(c+0) - f(c - 0)$.

Theorm 1.4.3 Let $f:[a,b]\to R$ be an ↑ function,

and $x_0 = a < x_1 < ... < x_n < b = x_{n+1}$, then

$$(f(a+0)-f(a)) + \sum_{k=1}^{n}(f(x_k+0)-(x_k-0)) + (f(b)-f(b-0)) \leq f(b)-f(a) ....(*).$$

Proof : Take $x_k < y_k < x_{k+1}$ $(k = 0,1,2,...,n)$; Then we have

$f(x_k+0)-f(x_k-0) \quad \leq f(y_k)-f(y_{k-1}),\ k = 1,2,....,n,$

$f(a+0)-f(a) \quad \leq f(y_1)-f(a)$, and

$f(b)-f(b-0) \quad \leq f(b)-f(y_n)$. On adding all the above (*) follows.

Theorem 1.4.4 Let $f:[a,b]\to R$ be an ↑ function. For $\alpha > 0$, the set $G_\alpha = \{x \varepsilon [a,b] : \omega_f(x) > \alpha\}$ is a finite set.

Proof: If $x_1, x_2, ..., x_n$ are members of G_α, using (*) we find $n.\alpha \leq f(b)-f(a)$ or $n \leq \frac{1}{\alpha}(f(b)-f(a))$. So n must be finite.

Theorem 1.4.5 Let $f:[a,b]\to R$ be an ↑ function. Then the set of points of discontinues of f is a countable set.

Proof: Put $G_n = \left\{x \varepsilon [a,b] : \omega_f(x) > \frac{1}{n}\right\}$ for any +ve integer n; when the required set of points of discontinuity of f is equal to $\bigcup_{n=1}^{\infty} G_n$. Theorem 1.4.4 says that each G_n is a finite set, and $\bigcup_{n=1}^{\infty} G_n$ is a countable set.

§ 1.5. functions of bounded variation :

Let $f:[a,b]\to R$ be a function and let

P : $x_0 = a < x_1 < x_2 < ... < x_{n-1} < x_n = b$ be a partition of $[a,b]$.

Put $V_p = \sum_{k=0}^{n-1} |f(x_{k+1}) - f(x_k)|$. Clearly $V_p \geq 0$ always.

Definition 1.5.1 l.u.b. or Sup $\{V_p$ for all Partitions $P = [x_0, x_1, ..., x_n]$ of $[a,b]\}$ is called the Total variation of f *over* $[a,b]$, denoted by $\overset{b}{\underset{a}{V}} f$.

If $\overset{b}{\underset{a}{V}} < + \infty$, then f is called a function of bounded variation in $[a,b]$.

Example 1.5.1 A monotone ↑ (or ↓) function over $[a,b]$ is a function of bounded variation.

Solution : Let $P = \{x_0 = a < x_1 < x_2 ... < x_n = b\}$ be a partition of $[a,b]$, and then

$V_p = |f(x_1) - f(x_0)| + |f(x_2) - f(x_1)| + ... + |f(x_n) - f(x_{n-1})|$

$= (f(x_1) - f(x_0)) + (f(x_2) - f(x_1)) + ... + (f(x_n) - f(x_{n-1}))$ in case when f is an ↑ function giving $f(x_i) \geq f(x_{i-1})$ i=1,2,..., n.

$= f(x_n) - f(x_0) = f(b) - f(a)$, which is independent of the partition P of $[a,b]$. So $\overset{b}{\underset{a}{V}} f = f(b) - f(a) < \infty$.

The same will the conclusion in case f is a ↓ function.

Example 1.5.2 Let $f:[a,b]\to R$ be a differentiable function such that $|f'(x)|\leq K$ for some K in $a\leq x\leq b$, then f is a function of bounded variation in $[a, b]$.

Solution : Let $P = \{x_0 = a<x_1<x_2<...<x_n=b\}$ be a partition of $[a,b]$. Then by Mean-value Theorem of Differential calculus we have

$f(x_i)-f(x_{i-1}) = (x_i-x_{i-1})f'(u_i)$ for seme u_i between x_{i-1} and x_i

$(i = 1,2,...,n)$. Thus $|f(x_i)-f(x_{i-1})| = |x_i-x_{i-1}||f'(u_i)|\leq K(x_i-x_{i-1})$ for $i = 1, 2,....n.$

Thus $V_p = \sum_{i=1}^{n}|f(x_i)-f(x_{i-1})| \leq K\sum_{i=1}^{n}(x_i-x_{i-1}) = K(x_n-x_0) = K(b-a)$ which is independent of Partition P of $[a,b]$, and therefore $\overset{b}{\underset{a}{V}}f = \sup\{V_p\}\leq K(b-a) < \infty$. Hence f is a function of bounded variation in $[a,b]$.

Remark : By an argument as in Example 1.5.2 one can show that every Lipscitzian function in $[a,b]$ is of bounded variation in $[a,b]$.

Remark : We have seen in Example 1.5.1 that a monotone function (may he discontinuous) is a function of bounded variation in $[a,b]$. A continuous function may not be a function of bounded variation as shown is Example 1.5.3.

Example 1.5.3. Let $f:[0,1]\to R$ be defined as

$$f(x) = x\sin\frac{\pi}{x}, \quad \text{if} \quad 0<x\leq 1$$
$$= 0 \qquad \text{if} \quad x = 0$$

Show that $\overset{1}{\underset{0}{V}} = +\infty$.

Solution : For a natural number n take the partition P_n of $[0,1]$ as

$\mathrm{P}_n = \left(0 < \frac{2}{2n+1} < \frac{2}{2n+3} < \ldots < \frac{2}{5} < \frac{2}{3} < 1\right)$ Then we have

$$\mathrm{V}_{\mathrm{P}_n} = \left|f\left(\frac{2}{2n+1}\right) - f(0)\right| + \left|f\left(\frac{2}{2n+3}\right) - f\left(\frac{2}{2n+1}\right)\right| + \ldots + \left|f\left(\frac{2}{3}\right) - f\left(\frac{2}{5}\right)\right| +$$

$$\left|f(1) - f\left(\frac{2}{3}\right)\right| = \left(\frac{2}{2n+1} + 0\right) + \ldots + \left(\frac{2}{3} + \frac{2}{5}\right) + \frac{2}{3}$$

$$= 4\left(\frac{1}{3} + \frac{1}{5} + \ldots + \frac{1}{2n+1}\right) \to \infty \text{ as } n \to \infty.$$

So Sup $\{\mathrm{V}_{p_n}\} = +\infty$, although f is continuous in [0,1].

Theorem 1.5.2 If $f:[a,b]\to\mathrm{R}$ is a function of bounded variation, then f is a bounded function.

Proof : Let $\overset{b}{\underset{a}{\mathrm{V}}} < \infty$, and P $=(a,= x_0<x_1=x<x_2=b)$ be a Partition of $[a,b]$.

Then $\mathrm{V}_\mathrm{P} \le \overset{b}{\underset{a}{\mathrm{V}}}(f)$

or $|f(x) - f(a)| + |f(b) - f(x)| \le \overset{b}{\underset{a}{\mathrm{V}}}(f)$

Clearly $|f(x) - f(a)| \le$ L.H.S $< \overset{b}{\underset{a}{\mathrm{V}}}(f)$. So $|f(x)| = |f(x) - f(a) + f(a)|$

$\le |f(x) - f(a)| + |f(a)| \le \overset{b}{\underset{a}{\mathrm{V}}}(f) + |f(a)|$ which is independent of x.

So f is bounded in $[a,b]$.

Example 1.5.4 If $[a,b]\to\mathrm{R}$ is continuous, and $\mathrm{F}(x) = \int_a^x f dt\, ; a \le x \le b$; Show that

$\overset{b}{\underset{a}{V}} F < +\infty$ and estimate $\overset{b}{\underset{a}{V}}(F)$.

Solution : Since f is continuous in $[a,b]$ we know that F is differentiate in $[a,b]$ with $F'(x) - f(x)$ in $a \le x \le b$; and further by continuity of f it follows that F' becomes bounded in $[a,b]$; So F possesses a bounded derivative in $[a,b]$ and hence F is a function of bounded variation in $[a,b]$. Now to obtain value of $\overset{b}{\underset{a}{V}}$ (F) let $P(a = x_0 < x_1 < ... < x_{k+1} < ... < x_n = b)$ be a partition of $[a,b]$. and we have V_P (in respect of F) $= \sum_{k=0}^{n} |F(x_{k+1}) - F(x_k)|$

By Mean-value Theorem of calculus we have $F(x_{k+1}) - F(x_k) = (x_{k+1} - x_k)$ $F'(u_k) = (x_{k+1} - x_k) f(u_k)$ where $x_k < u_k < x_{k+1}$.

Thus $V_P = \sum_{k=0}^{n} |F(x_{k+1}) - F(x_k)| = \sum_{k=0}^{n} (x_{k+1} - x_k) |f(u_k)|$, R.H.S. representing a Reimann sum (Corresponding to P) for $|f|$ over $[a,b]$. Considering all partitions P of $[a,b]$ including the case of norm of P going to zero, above shows that

$$\overset{b}{\underset{a}{V}}(F) = \int_a^b |f| \, dt.$$

Theorem 1.5.3. The sum, difference and Product of two functions of bounded variations are functions of bounded variation.

Proof : Let $f, g : [a,b] \to R$ be two functions or bounded variation in $[a,b]$. The proof for sum and difference $f \pm g$ as functions of bounded variation in $[a,b]$ is easy and left out. Let $p(x) - f(x)g(x)$ in $a \le x \le b$. Take $M = \sup_{a \le x \le b} |f(x)|$ and $K = \sup_{a \le x \le b} |g(x)|$. If

$P = (a = x_0 < x_1 < x_2 < \dots < x_k < x_{k+1} < \dots < x_n = b)$ be a partition of $[a,b]$, we have

$$|p(x_{k+1}) - p(x_k)| \le |f(x_{k+1})g(x_{k+1}) - f(x_k)g(x_{k+1})| + |f(x_k)g(x_{k+1}) - f(x_k)g(x_k)|$$

$$= |g(x_{k+1})||f(x_{k+1}) - f(x_k)| + |f(x_k)||g(x_{k+1}) - g(x_k)| \le |K| f(x_{k+1}) - f(x_k)|$$

$+M\,|g(x_{k+1}) - g(x_k)|$; This inequality leads to

$$\overset{b}{\underset{a}{V}} p \le K \overset{b}{\underset{a}{V}} f + M \overset{b}{\underset{a}{V}} g$$

Theorem 1.5.4 Let $\overset{b}{\underset{a}{V}}(f) < \infty$, $a<c<b$, show that $\overset{b}{\underset{a}{V}}(f) = \overset{c}{\underset{a}{V}}(f) + \overset{b}{\underset{c}{V}}(f)$.

Proof : Let $y_0 = a < y_1 < \dots < y_m = c$ and $z_0 = c < z_1 < \dots < z_n = b$.

and let $V_1 = \sum_{k=0}^{m-1} |f(y_{k+1}) - f(y_k)|$, $V_2 = \sum_{k=0}^{n-1} |f(z_{k+1}) - f(z_k)|$.

Now $y_0 = a < y_1 < \dots y_m = c = z_0 < z_1 < \dots < z_n = b$ becomes a

Partition P of $[a,b]$, and $V_P = V_1 + V_2$. Clearly $V_1 + V_2 = V_P \le \overset{b}{\underset{a}{V}}(f)$

This leads to $\overset{c}{\underset{a}{V}}(f) + \overset{b}{\underset{c}{V}}(f) \le \overset{b}{\underset{a}{V}}(f)$ (1)

Again take a Partition $P(a = x_0 < x_1 < \dots < x_k = c < x_{k+1} < \dots < x_n = b)$ being careful to take $x = c$ as a point of division, denote $c = x_m$.

This partition gives $V_P \sum_{k=0}^{m-1} |f(x_{k+1}) - f(x_k)| + \sum_{k=m}^{n-1} |f(x_{k+1}) - (x_k)|$

or $V_P=V_1(P)+V_2(P)$

$$\leq \overset{c}{\underset{a}{V}}(f)+\overset{b}{\underset{a}{V}}(f).$$

Then R.H.S. being independent of P we have

$$\overset{b}{\underset{a}{V}}(f)\leq \overset{c}{\underset{a}{V}}(f)+\overset{b}{\underset{a}{V}}(f) \qquad (2)$$

From (1) and (2) we have $\overset{b}{\underset{a}{V}}(f)=\overset{c}{\underset{a}{V}}(f)+\overset{b}{\underset{c}{V}}(f)$.

Theorem 1.5.5 (Jordan Theorem) : If $\overset{b}{\underset{a}{V}} f<+\infty$, then $f=$ Difference of two ↑ functions in $[a,b]$.

Proof: If $a<x\leq b$, then $\overset{x}{\underset{a}{V}} f<\infty$, Put $\pi(x)=\overset{x}{\underset{a}{V}}(f)$, and $\pi(a)=0$. We show that $\pi[a,b]\to R$ is an ↑ function.

Theorem 1.5.2 says that $\overset{x}{\underset{a}{V}}(f)+\overset{b}{\underset{x}{V}}(f)=\overset{b}{\underset{a}{V}}(f)$; and hence $\pi(x)$ is an ↑ function in, $[a,b]$. Now put $\gamma(x)=\pi(x)-f(x)$ in $a\leq x<b$

Here if $a<x<y\leq b$ we have $\gamma(y)=\pi(y)-f(y)=\overset{y}{\underset{a}{V}}(f)-f(y)$

$$\overset{x}{\underset{a}{V}}(f)+\overset{y}{\underset{x}{V}}(f)-f(y)=\pi(x)+\overset{y}{\underset{x}{V}}(f)-f(y)$$

Hence $\gamma(y)-\gamma(x)=\pi(x)+\overset{y}{\underset{x}{V}}(f)-f(y)-\pi(x)+f(x)$

$$=\overset{y}{\underset{x}{V}}(f)-(f(y)-f(x))$$

Since $f(y) - f(x) \le \overset{y}{\underset{x}{\mathrm{V}}}(f)$, above gives $\gamma(y) - \gamma(x) \ge 0$

or $\gamma(x) \le \gamma(y)$

So γ is an $\uparrow$ function in $[a,b]$ and we have $f(x) = \pi(x) - \gamma(x)$ for $a \le x \le b$.

Corollary 1 If $\overset{b}{\underset{a}{\mathrm{V}}}(f) < +\infty$, the set points of discontinuity of f is countable.

Corollary 2 If $\overset{b}{\underset{a}{\mathrm{V}}}(f) < +\infty$, then f is Riemann-integrable over $[a,b]$

Unit 2 Lebesgue Measure of Sets, Algebra of Measurable Sets and Measurable Functions, convergence in Measure

(Lebesgue measure of a bounded open set and closed set; Lebesgue Exterior and Interior measure of a bounded set; sub-additive properties; Measurable sets; algebra of measurable sets; measure of limit set of an increasing sequence and decreasing sequence of bounded measurable sets; Measurable functions and their algebra, limit of a sequence of measurable functions; convergence in measure.)

§ 2.1

Lebesgue measure of bounded interval (open or closed or open at one end point and closed at other) is equal to its length. So measure of an open interval (a, b) $(a<b)$ is denoted by $m\,(a,b) = b - a$.

Definition 2.1.1 Measure of a bounded open set G, denoted by m(G) is equal to $\sum_{n=1}^{\infty} m\left(\delta_n\right)$ where $G = \bigcup_{n=1}^{\infty} \delta_n$, and $\{\delta_n\}$ is a countable family of disjoint open intervals.

Explanation : Since G is assumed to be bounded we have $\sum_{n=1}^{\infty} m(\delta_n) < +\infty$ As measure of an open interval is +ve, we have $m(G)>0$ always unless G is empty in which case we put $m(G) = 0$.

Theorem 2.1.1 If a bounded set $G = \sum_{n=1}^{\infty} G_n$, where each G_n is open,

then $m(G) \le \sum_{n=1}^{\infty} m(G_n)$.

Proof : Here G is a bounded open set because each G_n is open and

Let $G= \bigcup_{k=1}^{\infty} \Delta_k$ where $\{\Delta_k\}$ is a countable family of pairwise disjoint open intervals,

and similarly let $G_n = \bigcup_{k=1}^{\infty} \delta_k^{(n)}$, where $\{\delta_k^{(n)}\}$ are pairwise disjoint open intervals for each $n = 1, 2, \ldots$. Now $m(G) = \sum_{k=1}^{\infty} m(\Delta_k)$; Let $\varepsilon>0$ be given arbitrary. There is an index N to satisfy $\sum_{n=N+1}^{\infty} m(\Delta_k) < \varepsilon$

So that we have, $m(G) < \sum_{k=1}^{N} m(\Delta_k) + \varepsilon$ (1)

For each $k = 1,2, \ldots N$ take an open interval $I_k \subset$ closure $I_k = \bar{I}_k \subset \Delta_k$

such that $m(\Delta_k) < m(I_k) + \frac{\varepsilon}{N}$ (2)

Then from (1) we have, $m(G) < \sum_{k=1}^{N} m(I_k) + 2\varepsilon$ (3)

Clearly finite-union $\bigcup_{K=1}^{N} \bar{I}_K$ is a bounded closed set and hence is compact.

Again $\bigcup_{k=1}^{N} \bar{I}_k \subset \bigcup_{n=1}^{\infty} G_n = \bigcup_{n=1}^{\infty}\bigcup_{k=1}^{\infty} \delta_k^{(n)}$. That means $\{\delta_k^{(n)}\}_{k,n=1,2,\ldots}$ forms an open cover for $\bigcup_{k=1}^{N} \bar{I}_k$; by compactness of $\bigcup_{k=1}^{N} \bar{I}_k$. Let $\delta_{k1}^{n1}, \delta_{k2}^{n2}, \ldots, \delta_{kr}^{nr}$ be a finite sub-family of $\{\delta_k^{(n)}\}_{k,n=1,2,\ldots}$ to cover $\bigcup_{k=1}^{N} \bar{I}_k$. Clearly $\bigcup_{k=1}^{N} I_k \subset \bigcup_{k=1}^{N} \bar{I}_k \subset \left(\delta_{k_1}^{(n_1)} \cup \delta_{k_2}^{(n_2)} \cup \ldots \cup \delta_{k_r}^{n_r}\right)$.

Since I_k 's are disjoint, we find, $\sum_{k=1}^{N} m(I_k) = m\left(\bigcup_{k=1}^{N} I_k\right) \leq m\left(\delta_{k_1}^{n_1}\right) + \ldots + m\left(\delta_{k_r}^{n_r}\right)$.

As $\delta_k^{(n)}$ comes from G_n, we have,

$$\sum_{k=1}^{N} m(I_k) \leq \sum_{n=1}^{\infty} m(G_n) \tag{4}$$

Combining (3) and (4) we find $m(G) \leq \sum_{n=1}^{\infty} m(G_n) + 2\varepsilon$.

As $\varepsilon > 0$ is arbitrary letting $\varepsilon \to 0$, we obtain

$$m(G)\sum_{n=1}^{\infty} m(G_n).$$

Definition 2.1.2 Measure of a bounded closed set F is equal to $m(F) = B - A - m(F^c)$, where [A, B] is the smallest closed interval containing F and F^c= complement of F is [A, B].

Explanation: F^c is a bounded open set; because it is a complement of closed set F in [A, B] and we know how to find $m(F^c)$. If F is empty, since empty set is also taken as open set and we have assigned its measure = 0. Further $m(F)$ is always non-negative.

Example 2.1.1 [a, b] (a < b) is closed interval then $m[a,b] = b - a$.

Solution : Here $F = [a,b]$ is a bounded set such that the smallest interval containing F is = $[a,b]$ and hence $F^c = \phi$. So we have $m[a,b] = b - a - m(\phi) = b - a$.

Example 2.1.2 Let $[a_1,b_1], [a_2,b_2],..., [a_n,b_n]$ be a finite number of pairwise disjoint closed intervals, and let $F = \bigcup_{k=1}^{n} [a_k, b_k]$, then $m(F) = \sum_{k=1}^{n} (b_k - a_k)$.

Solution : Take closed intervals in order of increasing values of left end points; then $b_k < a_{k+1}$ $(k=1,2,..., n-1)$. In that case $[a_1,b_n]$ is the smallest closed interval to contain F, and complement

$F^c = (b_1,a_2)\cup(b_2,a_3)\cup...\cup(b_{n-1},a_n)$.

So we have $m(F) = b_n - a_1 - \sum_{k=1}^{n-1} (a_{k+1} - b_k) = \sum_{k=1}^{n} (b_k - a_k)$.

Example 2.1.3 If G is bounded open set and F is a closed subset of G, then $m(F) < m(G)$.

Take a closed interval $[a,b] \supset G$. Then write $[a,b] = G \cup F^c$; So $b - a \leq m(G) + m(F^c)$. Hence $b - a - m(F^c) \leq m(G)$ i.e., $m(F) \leq m(G)$.

Theorem 2.1.2 If G is a bounded open set, then

$m(G) = \text{Sup}\{m(F) : F \text{ is a closed set} \subset G\}$.

Proof : If F is a closed subset of G, we have (See Example 2.1.3)

$$m(F) \leq m(G). \tag{1}$$

Let $G = \bigcup_{k=1}^{\infty} \delta_k$ where δ_k's are mutually disjoint open intervals, and so $m(G) \bigcup_{k=1}^{\infty} m(\delta_k)$.
Let $\varepsilon > 0$ be given arbitrary. We find an index N such that

$$\bigcup_{k=1}^{N} m(\delta_k) > m(G) - \varepsilon / 2 \tag{2}$$

Now choose intervals $[a_k, b_k] \subset \delta_k$ satisfying $b_k - a_k > m(\delta_k) - \frac{\varepsilon}{2N}$, for $k = 1,2,....,N$.

Put $F = \bigcup_{k=1}^{N} [a_k, b_k]$; Then F is a closed subset of G such that,

$$m(F) = \sum_{k=1}^{N} (b_k - a_k) > \sum_{k=1}^{N} m(\delta_k) - \frac{\varepsilon}{2} \tag{3}$$

$> m(G) - \frac{\varepsilon}{2} - \frac{\varepsilon}{2}$ from (2).

$= m(G) - \varepsilon$

This inequality along with (1) shows that $m(G) = \text{Sup}\{m(F): F$ is a closed subset of $G\}$.

Theorem 2.1.3 If F is a bounded closed set,

then $m(F) = \text{Inf}\{m(G): G$ is bounded open set $\supset F\}$.

Proof: Take Δ to be an open interval $\supset F$. So complement $F^c(=\Delta/F)$ is a bounded open set. Given a + ve ε, Theorem 2.22 says that there is a closed subset $H \subset F^c$ such that, $m(H) > m(F^c) - \varepsilon$.

Put $P = (\Delta/H,)$ then P is a bounded open set containing F such that,

$m(P) = m\Delta - m(H) < m(\Delta) - m(F^c) + \varepsilon = m(F) + \varepsilon.$

That means we have completed the proof.

Theorem 2.1.4 If F_1 and F_2 are two bounded closed sets with $F_1 \cap F_2 = \phi$, then

$m(F_1 \cup F_2) = m(F_1) + m(F_2).$

Proof: Let $\varepsilon > 0$ be given arbitrary. Now choose two bounded open sets G_1 and G_2 satisfying $G_i \supset F_i$, and $m(G_i) < m(F_i) + \frac{\varepsilon}{2}$ $(i = 1, 2)$ (See theorem 2.1.3).

Now take $G = G_1 \cup G_2$. Then G is a bounded open set $\supset (F_1 \cup F_2)$ such that

$m(F_1 \cup F_2) \leq m(G) \leq m(G_1) + m(G_2) < m(F_1) + m(F_2) + \varepsilon.$

Since $\varepsilon > 0$ is arbitrary, we have

$$m(F_1 \cup F_2) \leq m(F_1) + m(F_2) \quad (1)$$

Now by property of real number with usual metric we find two disjoint open sets B_1

and B_2 such that $B_i \supset F_i$ ($i = 1, 2$). Again corresponding to a pre-assigned + ve ε, we find open set G such that $G \supset (F_1 \cup F_2)$ and $m(G) < m(F_1 \cup F_2) + \varepsilon$.

Clearly then $B_1 \cap G$ and $B_2 \cap G$ are disjoint bounded open sets containing F_1 and F_2 respectively; and we have,

$$m(F_1) + m(F_2) \leq m(B_1 \cap G) + m(B_2 \cap G) = m(B_1 \cap G) \cup (B_2 \cap G))$$

$$\leq mG < m(F_1 \cup F_2) + \varepsilon \text{ (Since } (B_1 \cap G) \cup (B_2 \cap G) \subset G) \text{ This gives}$$

$m(F_1) + m(F_2) \leq m\ (F_1 \cup F_2)$ as $\varepsilon > 0$ is arbitrary. Combining (1) and (3) we have

$$m\ (F_1 \cup F_2) = m(F_1) + m(F_2). \tag{2}$$

Corollary : If $F_1, F_2, ..., F_n$ are bounded closed sets that are mutually disjoints then $m(F_1 \cup F_2 \cup ... \cup F_n) = \sum_{k=1}^{n} m(F_k)$.

§ 2.2. Let E be a bounded set of reals.

Definition 2.2.1 Lebesgue exterior measure of E denoted by $m*(E)$ is defined as, $m*(E) = \text{Inf}\{m(G) : G \text{ is a bounded open set containing } E(G \supset E)\}$.

So, we always have $0 \leq m*(E) < +\infty$.

Definition 2.2.2 Lebesgue Interior measure of E denoted by $m*(E)$ is defined as $m*(E) = \text{Sup}\{m(F) : F \text{ is a closed subset of } E\}$

So, here again $0 \leq m^*(E) < +\infty$.

Theorem 2.2.1 $m^*(E) \leq m^*(E)$.

Proof : Take G to be a bounded open set containing E. Then for a closed subset of

F of E we have $F \subset E \subset G$ and hence $m(F) \leq m(G)$; also $m_*(E) \leq m(G)$. This being true for any bounded open set $G \supset E$, we find

$$m_*(E) \leq \text{Inf}\{m(G) : G \text{ is a bounded open set}\} = m^*(E).$$

Theorem 2.2.1 If a bounded set $E = \bigcup_{k=1}^{\infty} E_k$, then $m^*(E) \leq \sum_{k=1}^{\infty} m^*(E_k)$.

Proof : If $\sum_{k=1}^{\infty} m^*(E_k)$ diverges, we have finished. Now let $\sum_{k=1}^{\infty} m^*(E_k) < +\infty$. Take $\varepsilon > 0$ arbitrary. We find a bounded open set $G_k \supset E_k$ such that,

$$m(G_k) < m^*(E_k) + \frac{\varepsilon}{2^k} \; (k = 1,2,\ldots,).$$

If Δ is an interval $\supset E$, Then $E \subset \Delta \cap \bigcup_{k=1}^{\infty} G_k$; So we have

$$m^*(E) \leq m\left(\Delta \cap \bigcup_{k=1}^{\infty} G_k\right) = m\left(\bigcup_{k=1}^{\infty} (\Delta \cap G_k)\right) \leq \sum_{k=1}^{\infty} m(\Delta \cap G_k) \leq \sum_{k=1}^{\infty} m(G_k) \leq \sum_{k=1}^{\infty} m^*(E_k) + \varepsilon$$

As $\varepsilon > 0$ is arbitrary we have proved the theorem.

Theorem 2.2.2 If a bounded set $E = \bigcup_{k=1}^{\infty} E_k$ where E_k's are pairwise disjoint,

then $m_*(E) \geq \sum_{k=1}^{\infty} m_*(E_k)$.

Proof : Given an arbitrary + ve ε, we find for a fixed index n, a closed subset $F_k \subset E_k$ satisfying $m(F_k) > m_*(E_k) - \frac{\varepsilon}{n}$ $(k = 1,2,\ldots,n)$. Now F_k 's are pairwise disjoint because E_k's are so. So we have (See Theorem 2.1.4)

$$m_*(E) \geq m\left(\sum_{k=1}^{n} F_k\right) = \sum_{k=1}^{n} m(F_k) > \sum_{k=1}^{n} m_*(E_k) - \varepsilon$$

As $\varepsilon > 0$ is arbitrary we have $m_*(E) \geq \sum_{k=1}^{n} m_*(E_k)$.

Now let n be free, so that $m_*(E) \geq \sum_{k=1}^{n} m_*(E_k)$.

Theorem 2.2.3 Let Δ be an open interval containing E, then

$m^*(E) + m_*(\Delta/E) = m(\Delta)$.

Proof : Let $\varepsilon > 0$ be given in advance. Then we find a closed set $F \subset (\Delta/E)$ with $m(F) > m_*(\Delta/E) - \varepsilon$. Now the set $G = (\Delta/F)$ is a bounded open set containing E such that,

$m^*(E) \leq (G) = m(\Delta) - m(F) - m(\Delta) - m_*(\Delta/E) + \varepsilon$.

As $\varepsilon > 0$ is arbitrary we have $m^*(E) + m^*(\Delta/E) \leq m(\Delta)$ (1)

Again assume $\varepsilon > 0$ is arbitrary, and find a bounded open set G_0, such that $G_0 \supset E$ with $m(G_0) < m^*(E) + \frac{\varepsilon}{3}$.

Let us write Δ (as given) = (A, B) and choose an open interval $(a,b) \subset \Delta$ such that, $A < a < A + \frac{\varepsilon}{3}$ and $B - \frac{\varepsilon}{3} < b < B$. Now put $G = (\Delta \cap G_0) \cup (A,a) \cup (b,B)$

So G is a bounded open set $\supset$ E having property $m(G) < m^*(E) + \varepsilon$.

Further, $\Delta/G = [a,b] \cap (G')$ (Complement of $G = G'$)

and hence (Δ/G) is closed, and since $F \subset (\Delta/E)$ we find,

$m_*(\Delta/E) \geq m(F) = m(\Delta) - m(G) > m(\Delta) - m^*(E) - \varepsilon.$

As $\varepsilon > 0$ is arbitrary we have

$$m_*(\Delta/E) \geq m(\Delta) - m^*(E)$$

or,
$$m^*(E) + m_*(\Delta/E) \geq m(\Delta) \tag{2}$$

From (1) and (2) we obtain $m^*(E) + m_*(\Delta/E) = m(\Delta)$.

§ 2.3 Let E be a bounded set of reals.

Definition 2.3.1 E is said to be Lebesgue measurable if $m_*(E) = m_*(E)$. and this common value is called Lebesgue measure or simply measure of E denoted by $m(E)$.

Theorem 2.3.1 If a bounded set $E = \bigcup_{k=1}^{\infty} E_k$ where each E_k is a measurable set and E_k's are pairwise disjoint, then E is measurable and $m(E) = \sum_{k=1}^{\infty} m(E_k)$.

Proof : The proof rests on following chain of inequalities

$$\sum_{k=1}^{\infty} m(E_k) = \sum_{k=1}^{\infty} m_*(E_k) \leq m_*(E) \leq m^*(E) \leq \sum_{k=1}^{\infty} m^*(E_k) = \sum_{k=1}^{\infty} m(E_k).$$

Remark : The property of Lebesgue measure as in Theorem 2.3.1 is also known (called) as countable additivity property.

Theorem 2.3.2 Union of two bounded measurable sets is a measurable set. (Sets may or may not be disjoint).

Proof : Let E_1 and E_2 be two bounded measurable sets and Let $E = E_1 \cup E_2$. Given *a +ve* ε, we find closed sets F_1, F_2 and bounded open set G_1 and G_2 satisfying

$F_i \subset E_i \subset G_i$ with $m(G_1) < m^*(E_1) + \frac{\varepsilon}{4} = m(E_1) + \frac{\varepsilon}{4}$ and $m(F_1) > m_*(E_1) - \frac{\varepsilon}{4} = m(E_1) - \frac{\varepsilon}{4}$.
Therefore, $m(G_1) - m(F_1) < \varepsilon/2$ and similarly $m(G_2) - m(F_2) < \varepsilon/2$. Now put $G = G_1 \cup G_2$ and $F = F_1 \cup F_2$. Then F is closed and G is a bounded open such that $F \subset E \subset G$ such that,

$$m(F) \leq m_*(E) < m^*(E) \leq m(G) \tag{1}$$

Now G/F is open (we may write $G \cap F'$) and it is bounded. Hence G/F is measurable. Write $G=(G/F) \cup F$, where members on r.h.s are disjoint. So we have,

$$m(G) = m(G/F) + m(F).$$

$$i.e.,\ m(G/F) = m(G) - m(F).$$

In a like manner we have $m(G_k/F_k) = m(G_k) - m(F_k)$, $k = 1,2$.

We now check out $G/F \subset (G_1/F_1) \cup (G_2/F_2)$, as every member is bounded and open we have,

or, $m(G/F) \leq m(G_1/F_1) + m(G_2/F_2) < \varepsilon$

or, $m(G) - m(F) < \varepsilon$ and from (1). We deduce $m^*(E) - m_*(E) < \varepsilon$. As $\varepsilon > 0$ is arbitrary we have $m^*(E) = m^*(E)$.

Corollary 1. Union of finite number of bounded measurable sets is measurable.

Corollary 2. Intersection of a finite number of bounded measurable sets is a measurable set.

Corollary 3. Difference of two bounded measurable sets is a measurable set. Because if E_1 and E_2 are two bounded measurable sets and $E = E_1/E_2$, we write, $E = E_1 \cap E_2'$ where E_2' is complement of E_2 in an open interval containing both

E_1 and E_2

Theorem 2.3.3 If a bounded set $E = \bigcup_{k=1}^{\infty} E_k$ where each E_k is measurable, then E is measurable.

Proof : Here write $B_1 = E_1$

$$B_2 = E_2 \backslash E_1$$

$$B_2 = E_k \backslash (E_1 \cup E_2 \cup \ldots \cup E_{k-1})$$

Clearly, $E = \bigcup_{k=1}^{\infty} B_k$, where B_k's are measurable and are pairwise disjoint, and so Theorem 2.3.1 applies to complete the proof.

Theorem 2.3.4 Intersection of a countable number of bounded measurable sets is a measurable set.

Proof : Let $E = \bigcap_{k=1}^{\infty} B_k$, where each E_k is a bounded measurable set clearly E is a bounded set. If $E = \phi$, we have finished. So let $E \neq \phi$. Now let Δ be an open interval $\supset E$, and put $A_k = \Delta \cap E_k$ $(k = 1,2,\ldots)$.

Then $E = \Delta \cap E = \Delta \cap \left(\bigcap_{k=1}^{\infty} E_k \right) = \bigcap_{k=1}^{\infty} (\Delta \cap E_k) = \bigcap_{k=1}^{\infty} E_k$. Thus $(\Delta \backslash E) = \bigcup_{k=1}^{\infty} (\Delta \setminus A_k)$. So Theorem 2.3.3 applies to show that $(\Delta \backslash E)$ to be measurable and hence E is measurable.

Theorem 2.3.5 Let $\{E_k\}$ be an increasing sequence of measurable sets, such that $\bigcup_{k=1}^{\infty} E_k = E$ is bounded. Then E is measurable and $m(E) = \lim_{n \to \infty} m(E_n)$.

Proof: We write $E = E_1 \cup E_2 \cup \ldots$

$= E_1 \cup (E_2 \backslash E_1) \cup (E_3 \backslash E_2) \cup ....$

Where members on r.h.s. are measurable sets that are mutually disjoint. Then we have E to be measurable, and further,

$$m(E) = m(E_1) + m(E_2 \backslash E_1) + m(E_3 \backslash E_1) + \cdots$$

$$= m(E_1) + (m(E_2) - m(E_1)) + (m(E_3) - (E_2)) + \cdots$$

That is to say, $m(E) = \lim_{n \to \infty} \left\{ m(E_1) + \sum_{k=1}^{n-1} (m(E_{k+1}) - m(E_k)) \right\}$

$$= \lim_{n \to \infty} m(E_n)$$

Theorem 2.3.6 Let $\{E_k\}$ be a decreasing sequence of measurable sets with E_1 bounded and $E = \bigcap_{k=1}^{\infty} E_k$. Then E is measurable and $m(E) = \lim_{n \to \infty} m(E_n)$.

Proof : The proof is done on complementation method and on applying Theorem 2.3.5. Let Δ be an open interval $\supset E_1$. We then have $\{(\Delta \backslash E_n)\}$ as an increasing sequence of measurable sets, and $(\Delta \setminus E) = \left(\Delta \setminus \bigcap_{k=1}^{\infty} E_k \right) = \bigcup_{k=1}^{\infty} (\Delta \setminus E_k)$, with $(\Delta \backslash E_1)$ as bounded. So Theorem 2.3.5 applies here, and we have,

$$m(\Delta \backslash E) = \lim_{n \to \infty} m(\Delta \setminus E_n).$$

That is to say, $m(\Delta) - m(E) = \lim_{n \to \infty} (m(\Delta) - m(E_n))$

$$= m(\Delta) - \lim_{n \to \infty} m(E_n)$$

That is what is wanted.

Example 2.3.1 If E is a bounded measurable set show that there is a subset F⊂E such that F is a countable union of closed sets with $m(F) = m(E)$. (F is called a Borel set like F_σ).

Solution : Corresponding to $\frac{1}{n}$(n = a natural number) we find a closed subset $F_n \subset E$

such that $m(F_n) > m(E) - \frac{1}{n}$.

Put $F = \bigcup_{n=1}^{\infty} F_n$. Then $F \supset F_n$ and hence $m(F) \geq m(F_n) > m(E) - \frac{1}{n}$

Taking $n \to \infty$, we have $m(F) \geq m(E)$ (1)

On the other hand F⊃E gives

$m(F) \leq m(E)$ (2)

(1) and (2) gives $m(F) = m(E)$.

Example 2.3.2 If E is a bounded measurable set show that there is a set T⊃E such that T is a countable intersection of open sets with $m(T) = m(E)$. (T is called a Borel set like F_σ).

Solution : Given a natural number n, we find a bounded open set T_n containing E *i.e.*, $T_n \supset E$ satisfying $m(T_n) < m(E) + \frac{1}{n}$.

Set $T = \bigcap_{n=1}^{\infty} T_n$ Then $T \subset T_n$ gives $m(T) \leq m(T_n) < m(E) + \frac{1}{n}$.

Taking $n \to \infty$, we have,

$m(T) \leq m(E)$ .. (1)

Again $T \supset E$ gives

$m(T) \geq m(E)$ .. (2)

(I) and (2) give

$m(T) = m(E)$

Let E be any unbounded set of reals.

Definition 2.3.2 E is said to be Lebesgue measurable if for **each +ve integer *n*,** the bounded set $E_n = [-n,n] \cap E$ is measurable.

If E is measurable, then $m(E)$ is defined as $m(E) = \lim_{n\to\infty} m(E_n)$.

Explanation : Since all sets E_n are measurable (Assuming E to be measurable), we see $\{E_n\}$ forms an $\uparrow$ sequence of bounded measurable sets; and hence the sequence $\{m(E_n)\}$ is an $\uparrow$ sequence of non-negative reals, and that has a limit finite or $+\infty$. Hence $m(E)$ is either a finite non-negative real or $+\infty$.

Example 2.2.3 Show that the set R of all reals is measurable and find $m(R)$.

Solution : Here $E_n = [-n,+n] \cap R$

$= [-n,n]$ is a closed interval (n being a+ve integer) = a bounded measurable set with $m(E_n) = 2n \to \infty$ as $n \to \infty$. So (i) $R = \bigcup_{n=1}^{\infty} E_n$ and is therefore measurable, and (ii) its measure $m(R) = \lim_{n\to\infty} E_n(\{E_n\})$ is an $\uparrow$ sequence of measureable sets) $= +\infty$.

Remark Every open set of reals is measurable. Because if G is any open set, then for each +ve integer n, put $G_n = [-n,n] \cap G$.

Then $G_n = \bigcap_{k=1}^{\infty}\left\{\left(-n-\frac{1}{k}, n+\frac{1}{k}\right)\cap G\right\}$; all members on r.h.s. are bounded open sets, and are measurable ; So each G_n is measurable. We can in like manner show that every closed set of reals is also measurable.

Example 2.3.4 If $m^*(E) = 0$, then E is measurable and $m(E) = 0$.

Solution : We have $0 \le m_*(E) \le m^*(E) = 0$. That means $m_*(E) = m^*(E)$, showing that E is measurable, and $m(E) = 0$.

Example 2.3.5 Every countable set of reals is measurable and is of measure zero.

Solution : Let $E = (x_1, x_2, x_3, \ldots, x_n, \ldots)$ be a countable set of reals and let $\varepsilon > 0$ be given arbitrary. Take an open Interval I_n with $x_n \in I_n$ and $m(I_n) < \frac{\varepsilon}{2n}$.

Then $I_1 \cup I_2 \cup \ldots \cup I_n \cup \ldots = G$ is an open set $\supset E$.

Therefore $m^*(E) \le m(G) \le \sum_{n=1}^{\infty} m(I_n) \le \varepsilon \sum_{n=1}^{\infty} \frac{1}{2^n} = \varepsilon$. As $\varepsilon > 0$ is arbitrary, we have $m^*(E) = 0$. That means E is measurable and $m(E) = 0$.

Application : The set Q of all rationals is measurable and $m(Q)=0$. Because Q is a countable set.

Example 2.3.6 Let $E = \bigcup_{n=2}^{\infty}\left(n-\frac{1}{n}, n+\frac{1}{n}\right)$; Find $m(E)$.

Solution : Here, if $E_n = \left(n-\frac{1}{n}, n+\frac{1}{n}\right)$, then each E_n is measurable with $m(E_n) = \frac{2}{n}$.

Therefore $E = \bigcup_{n=2}^{\infty} E_n$ is measurable with

$$m(E) = \sum_{n=2}^{\infty} m(E_n) = \sum_{n=2}^{\infty} \frac{2}{n} = 2\sum_{n=2}^{\infty} \frac{1}{n} = 2. \lim_{N\to\infty} \sum_{n=1}^{N} \frac{1}{n} = +\infty.$$

Example 2.3.7 Let $E = \bigcup_{n=1}^{\infty} \left(n - \frac{1}{2^n}, n + \frac{1}{2^n} \right)$, Find $m(E)$.

Solution : Here if $E_n = \left(n - \frac{1}{2^n}, n + \frac{1}{2^n} \right)$ then each E_n is measurable with

$m(E_n) = \frac{2}{2^n} = \frac{1}{2^{n-1}}$ and $m(E) = \sum_{n=1}^{\infty} m(E_n) = 2\sum_{n=1}^{\infty} \frac{1}{2^n} = 2.$

Remark : There are non-measurable sets of reals.

§ 2.4 Let E be a measurable set and f:E→R be a function.

Definition 2.4.1 f:E→R is said to be a measurable function if for every real number α, the set $\{x \in E : f(x) > \alpha\}$ is measurable.

Notations : The set $\{x \in E : f(x) > \alpha\}$ may shortly be put up in the form $E(f > \alpha)$. Similarly, sets $E(f = \alpha)$, $E(f < \alpha)$, $E(f > \alpha)$, $E(f \leq \alpha)$ and $E(a < f < b)$ bear usual meanings.

Example 2.4.1 Every function f: E→R is measurable if E is measurable and $m(E) = 0$.

Here the set $E(f > \alpha)$ is a subset of E and consequently, $E(f > \alpha)$ has measure zero, because $m(E) = 0$.

Definition 2.4.2 Two functions f, g : E→R are said to be equivalent if $m\, E(f \neq g) = 0$. If f and g are equivalent functions over E, we use notation $f \sim g$.

Explanation : If $f \sim g$ over E, the subset A of E where $A = \{x \in E, f(x) \neq g(x)\}$ has measure zero *i.e.,* $f(x) = g(x)$ for $x \in (E \setminus A)$ where $m(E \setminus A) = m(E)$. We say that

f **and** g **agree almost everywhere in E.**

Definition 2.4.3 If a property P holds for all points in E except for points of a subset E_0 of E with $m(E_0) = 0$, it is said that P holds almost everywhere or simple *a.e.*.

For example, it is an easy exercise to see that if f: E→R is a measurable function and $f \sim g$ over E, then g : E→R is also measurable.

Theorem 2.4.1 If f:E→R is measurable function then for every real number α, the set $E(f \geq \alpha)$ is measurable.

Proof : write $E(f > \alpha) = \bigcap_{n=1}^{\infty} E\left(f > \alpha - \frac{1}{n}\right)$, then each member in r.h.s. is measurable, and hence l.h.s. is measurable.

Corollary : For any real α, $E(f = \alpha)$, $E(f \leq \alpha)$ and $E(f < \alpha)$ are measurable.

Remark : If one of sets $E(f \geq \alpha)$, $E(f \leq \alpha)$ and $E(f < \alpha)$ is measurable for every real α, then f is a measurable function.

Example 2.4.2 A bounded set E of reals is measurable if and only if its characteristic function X_E is a measurable function.

Solution : Let $E \subset [a,b]$; Then $X_E(x) = 1$ if $x \in E$

$= 0$ if $x \in ([a,b] \setminus E)$

Suppose, Characteristic function $X_E:[a,b] \to R$ is measurable.

Then $E = \{x \in [a,b]: X_E(x) > 0\}$, which is measurable.

Conversely, if E is measurable, we have for any real α.

The set $\{x\in[a,b]:X_E(x)>\alpha\} = \begin{cases}\phi & \text{if } \alpha\geq 1\\ E & \text{if } 0\leq\alpha<1\\ [a,b] & \text{if } \alpha<0\end{cases}$

where each member on r.h.s. is a measurable set.

So X_E is a measurable function.

Theorem 2.4.2 If f and g : E→R are two measurable functions then $E(f\neq g)$ and $E(f=g)$ are measurable.

Proof : Write $E(f\neq g) = E(f(x)\neq g(x)) = E(f(x)>g(x))\cup E(f(x)<g(x))$.

If we represent all rationals as $r_1, r_2, r_3, \ldots$

Then we have $E(f(x)>g(x)) = \bigcup_{k=1}^{\infty}(E(f(x)>r_k)\cap E(g(x)<r_k))$, since each member on r.h.s. is measurable it follows that l.h.s. is measurable.

Similarly, we show that $E(f(x)<g(x))$ is measurable, and hence $E(f\neq g)$ is measurable; Finally, that $E(f=g)$ is measurable is clear.

Theorem 2.4.3 If f:E→R is a measurable function; then

(i) $f+k$ (k is any real constant) (ii) kf (k any real constant)

(iii) $|f|$ and f^2 and (iv) $\frac{1}{f}$ (when $f(x)\neq 0$ over E) are all measurable functions over E.

Proof : Let α be any real number.

(i) Write $E(f+k>\alpha)$

$= E(f>\alpha-k)$, r.h.s set is measurable because f is so.

(ii) If $k = 0$, (any constant function is measurable), the statement is O.K. for $k \neq 0$,

write $$E(kf > \alpha) \begin{cases} E\left(f > \dfrac{\alpha}{K}\right) & \text{if } K > 0 \\ E\left(f < \dfrac{\alpha}{K}\right) & \text{if } K < 0 \end{cases}$$

Thus kf is measurable.

(iii) $$E(|f| > \alpha) = \begin{cases} E & \text{if } \alpha < 0 \\ E(f > \alpha) \cup E(f < -\alpha) & \text{if } \alpha \geq 0 \end{cases}$$

Hence $|f|$ is measurable.

and $$E(f^2 > \alpha) = \begin{cases} E & \text{if } \alpha < 0 \\ E\left(|f| > \sqrt{a}\right) & \text{if } \alpha \geq 0 \end{cases}$$

So f^2 is measurable.

(iv) for $f(x) \neq 0$, we write,

$$E\left(\frac{1}{f} > \alpha\right) = \begin{cases} E(f > 0) & \text{if } \alpha = 0 \\ E(f > 0) \cap E\left(f < \dfrac{1}{\alpha}\right) & \text{if } \alpha > 0 \\ E(f > 0) \cap \left(E(f < 0) \cap E\left(f < \dfrac{1}{\alpha}\right)\right); & \text{if } \alpha > 0 \end{cases}$$

This way measurability of $\dfrac{1}{f}$ follows.

Example 2.4.3 Every continuous function $f:[a,b] \to R$ is a measurable function.

Solution : Let α be a real number and then (α,∞) is an open set and because f is continuous, $f^{-1}(\alpha,\infty)$ is an open set. That is, the set $E = f^{-1}(\alpha,\infty)$

$=\{x \in [a,b]: f(x) > \alpha\}$ is an

open set and therefore is a measurable set. Hence f is a measurable function.

Example 2,4.4 If F' exists in $[a,b]$ then F' is a measurable function in $[a,b]$.

Solution : Let $F(x) = F(b)$ for $x > b$. So that we write,

$$F'(x) = \lim_{n\to\infty} \frac{F\left(x+\frac{1}{n}\right) - F(x)}{\frac{1}{n}} = \lim_{n\to\infty} n\left(F\left(x+\frac{1}{n}\right) - F(x)\right) = \lim_{n\to\infty} \phi_n(x)$$

where $\phi_n(x) = n\left(F\left(x+\frac{1}{n}\right) - F(x)\right)$ in $a \le x \le b$. By continuity of F each ϕ_n is continuous, and hence is a measurable function in $[a,b]$ Thus F' as point wise limit of $\{\phi_n\}$ is measurable function in $[a,b]$.

Theorem 2.4.4 (Algebra of measurable functions)

If $f, g : E \to R$ are measurable functions, then (a) $f \pm g$ (b) fg and (c) f/g $(g \neq 0)$ are measurable functions.

Proof : (a) for any real a, $g + a$ is measurable. So $E(f > g + a)$ is measurable (see Theorem 2.4.2) ; Now write $E(f - g > a) - E(f > g + a)$ and that settles measurability of ; $f - g$. For $f + g$ write $f + g - f - (-g)$ and result follows.

(b) Write $f(x)g(x) = \frac{1}{4}\{(f(x) + g(x)^2) - (f(x) - g(x))^2\}$

So Theorem 2.4.3 applies and fg is measurable over E.

(c) Write for $g(x)\neq 0$, $\frac{f(x)}{g(x)} = f(x).\frac{1}{g(x)}$ and apply Theorem 2.4.3. The proof is now complete.

Theorem 2.4.5 Let $\{f_n:E\to R\}$ be a sequence of measurable functions and let $f(x) = \lim_{n\to\infty} f_n(x)$ as $x\in E$. Then $f:E\to R$ is measurable.

Proof : Let α be a real number and keep it fixed to consider

$$A_m^{(k)} = E\left(f_k > \alpha + \frac{1}{m}\right), \text{ and } B_m^{(n)} = \bigcap_{k=n}^{\infty} A_m^{(k)}$$

When m and k are natural numbers.

Clearly, abjove sets so constructed are all measurable sets. We show that $E(f>\alpha) = \bigcup_{m,n=1}^{\infty} B_m^{(n)}$.

Let $x_0 \in E(f>\alpha)$; so that $f(x_0) > \alpha$, and then we find a natural number m such that $f(x_0) > \alpha + \frac{1}{m}$. As $\{f_k(x_0)\}$ converges to $f(x_0)$, we can find an n such that for $k \geq n$ we have $f_k(x_k) > \alpha + \frac{1}{m}$.

That means $x_0 \in A_m^{(k)}$ for $k \geq n$; then $x_0 \in B_m^{(n)}$ and hence $x_0 \in \bigcup_{m,n=1}^{\infty} B_m^{(n)}$. Thus

$$E(f > \alpha) \subset \bigcup_{m,n=1}^{\infty} B_m^{(n)} \tag{1}$$

Conversely, take a member $u \in \bigcup_{m,n=1}^{\infty} B_m^{(n)}$ and let $u \in B_m^{(n)}$ for some m and n.

That is to say, $u \in A_m^{(k)}$ for $k \geq n$.

That means, $f_k(u) > \alpha + \frac{1}{m}$ for $k \geq n$.

Passing on limit as $k \to \infty$, we produce $f(u) \geq \alpha + \frac{1}{m}$, Clearly then $f(u) > \alpha$ *i.e.*, $u \in E(f > \alpha)$. So we have shown that,

$$\bigcap_{m,n=1}^{\infty} B_m^{(n)} \subset E(f > \alpha) \qquad (2)$$

Combining (1) and (2) we have $E(f > \alpha) = \bigcap_{m,n=1}^{\infty} B_m^{(n)}$, and the proof is complete.

Definition 2.4.3 A sequence $\{f_n\}$ of measurable functions over E is said to **converge** to f **in measure in E** if for **every** +ve <σ, $\lim_{n\to\infty} mE(|f_n - f| \geq \sigma) = 0$

Theorem 2.4.6 (Lebesgue Theorem on Convergence in Measure).

Let $\{f_n\}$ be a sequence of measurable functions over E and Let $\lim_{n\to\infty} f_n(x) = f(x)$ for each $x \in E$. Then $\{f_n\}$ converges to f in measure in E.

Proof : Take a +ve σ arbitrary, and put for each +ve intiger k,

$E_k(\sigma) = E([|f_k - f| > \sigma)$; and $R_n(\sigma) = \bigcup_{k=n}^{\infty} E_k(\sigma)$ and $H = \bigcap_{n=1}^{\infty} R_n(\sigma)$.

Clearly all sets appearing above are measurable. Since $\{R_n(\sigma)\}$ satisfies

$R_1(\sigma) \supset R_2(\sigma) \supset R_3(\sigma) \supset \cdots$ *i.e.* $\{R_n(\sigma)\}$ is $\downarrow$ and hence $\lim_{n\to\infty} m(R_n(\sigma)) = m(H)$. The proof is aimed to arrive at $H = \phi$. Let $x_0 \in H$. We seek a contradiction. Now $x_0 \in \bigcap_{n=1}^{\infty} R_n(\sigma)$;

So, $x_0 \in R_n(\sigma)$ for $n = 1, 2, \ldots$

i.e, $x_0 \in \bigcap_{k=n}^{\infty} E_n(\sigma)$ for $n = 1, 2, \ldots$.

That is to say, $x_0 \, E_{k_n}(\sigma)$ for k_n for all n = 1,2,.... (1)

Again $x_0 \in E$ gives $\lim_{n\to\infty} f_k(x_0) = f(x_0)$. So there is an index n such that,

$|f_k(x_0) - f(x_0)| < \sigma$ = for $k \geq n$

That means, $x_0 \notin E_k(\sigma)$ for $k \geq n$ (2)

Thus (1) and (2) are contradictory and the proof is complete.

Remark : Converse of Theorem 2.4.6 is not true.

There are examples in support of this Remark. Reader may see Natanson-Vol-I book P96 in this respect.

Theorem 2.47 If $\{f_n\}$ is a sequence of measurable functions converging in measure to f over E, then $\{f_n\}$ converges in measure to every function g equivalent to f in E.

Proof : For +ve σ one has to observe only that

$E(|f_n - g| > \sigma) \subset E(f \neq g) \cup E(|f_n - f| > \sigma)$; since $mE(f \neq g) = 0$

We conclude that $m(E(|f_n - g| \geq \sigma)) \leq m(E(|f_n - f| \geq \sigma))$. Hence proof is complete.

Exercise

Short-answer Type

1. If E and F are two bounded measurable sets with $m(E) = m(F)$. Is it true that E = F? Give reasons.

2. Obtain the measure of the set of all irrationals in $[a,b]$.

3. If G is an open set $\subset [a,b]$, and $m(G)=0$, then $G = \phi$. Examine truth of the statement with reason. Also examine the case if word 'open' is replaced by 'closed'.

4. Every finite set of reals is measurable. Verify it; and determine its measure.

5. Show that every step function in $[a,b]$ is a measurable function.

6. Examine the statement "every measurable function is a continuous function."

7. Show that every non-empty open set has measure +ve.

8. Show that the set of all zeros of $f(x) = \sin\frac{\pi}{x}$, $x \neq 0$ is a measurable set, and obtain its measure.

9. If S and T are bounded sets with $S \supset T$, show that $m^*(S) \geq m^*(T)$.

Broad questions

1. For every bounded set E there are two sets G and H such that (*i*) G is 'F_σ (a countable union of closed sets) and H is G_δ (a countable intersection of open sets) and (ii) $G \subset E \subset H$ and (iii) $m(G) = m_*(E)$ and $m(H) = m^*(E)$. Prove it.

2. If E_1 and E_2 are measurable sets in $[0,1]$ with $m(E_1) = 1$, show that

   $m(E_1 \cap E_2) = m(E_2)$.

3. Show that measure of a bounded closed set F is Inf. of measures of all bounded open sets each containing F.

4. If $E = \bigcup_{k=1}^{\infty} E_k$ where $E_1 \subset E_2 \subset E_3 \subset \ldots$ and E is bounded,

show that $\lim_{n\to\infty} m^*(E_n) = m^*(E)$.

5. If $\{G_n\}$ is a $\uparrow$ sequence if open sets, and $G = \bigcup_{n=1}^{\infty} G_n$, for every $\varepsilon > 0$, show that there is an $n = n(\epsilon)$ satisfying $m(G) < m(G_n) + \varepsilon$.

6. For a bounded set S show that,

$m^*(S) = \text{Inf}\{m(T): S \subset T, \text{ for all bounded measurable sets } T\}$

7. If G and H are bounded measurable sets, show that $(S\backslash T) \cup (T\backslash S)$ is also a bounded measurable set.

8. Prove that a necessary and sufficient condition that a bounded set E is measurable is that for any arbitrary bounded set A

$m^*(A) = m^*(A\cap E) + m^*(A\cap E')$, E' denoting complement of E.

9. Show that (i) every step function over $[a,b]$ is measurable.

(ii) every monotone function over $[a,b]$ is measurable.

10. If $\{f_n\}$ converges in measure to f and $\{g_n\}$ converges in measure to g over a bounded measurable set E, prove that $\{f_n + g_n\}$ converges in measure to $f + g$ over E.

11. If J_1 and J_2 are open intervals of reals, and $f:[a,b]\to R$ is bounded measurable function show that $f^{-1}(J_1 \cup J_2)$ is a measurable set in $[a,b]$.

Unit 3 ☐ Lebesgue Integral and Summable Functions

[Lebesgue Integral (of bounded measurable function). Fundamental properties, like additivity, Law of Mean, Passage to limit under Integral sign. Lebesgue Theorem. Comparison of Riemann and Lebesgue Intergrals, Summable functions with non-negative signs and summable functions with arbitrary signs.)

§ 3.1

By Lebesgue integration it has been made possible to enlarge the class of Riemann integrable (R-integrable) functions in the sense that every R-integrable function is Lebesgue integrable, but there are Lebesgue integrable functions without being R-integrable over an interval $[a,b]$. Let E be a bounded measurable set and $f: E \to R$ be a bounded measurable function.

Let $A < f(x) < B$ whenever $x \in E$ and take a partition

$P(y_0 = A < y_1 < y_2 < \ldots < y_n = B)$ of [A,B] and define

$\ell_k = E(y_k < f < y_{k+1}),\ k = 0,1,\ldots,(n-1).$

Then partition P-sets $\{\ell_k\}$ satisfy the fallowings :—

(3.1.1.) ℓ_k's are pairwise disjoint *i.e.* different ℓ_k's are separated ($\ell_k \cap \ell_{k'} = \phi$ for $k \neq k'$)

(3.1.2) ℓ_k's are measurable subsets of E.

(3.1.3) $m(E) = \sum_{k=0}^{n-1} m(\ell_k)$.

Then two sums s_P and S_P as

$$s_P = y_0 m(\ell_0) + y_1 m(\ell_1) + \cdots + y_{n-1} m(\ell_{n-1}) = \sum_{k=0}^{n-1} y_k m(\ell_k).$$

and $S_p = y_1 m(\ell_0) + y_2 m(\ell_1) + \cdots + y_n m(\ell_{n-1}) = \sum_{k=0}^{n-1} y_k m(\ell_k)$.

are called lower and upper Lebesgue Sums corresponding to partition P of [A,B]. If we designate $\lambda = \max_{0 \le k \le n-1} (y_{k+1} - y_k)$

we obtain, $0 \le S_p - s_p \le \lambda m(E)$. (1)

Theorem 3.1.1 If f:E→R is a bounded measurable function than f is Lebesgue integrable over E.

To prove Theorem 3.1.1 we depend on following Lemmas.

Lemma 3.1.1 Let S_p and S_p be the lower and upper Lebesgue sum in respect of Partition P and let a new point of division be added, then corresponding sums s' and S' satisfy $s_p \le s' < S' < S_p$

Proof : Let new point of division $\bar{y}$ lie between y_i and y_{i+1}

where $P = (y_0 = A < y_1 \cdots < y_i < y_{i+1} < \cdots < y_{n-1} < y_n = B)$.

Thus in new partition $[y_i, y_{i+1})$ is replaced by two half-open intervals $[y_i, \bar{y})$, $[\bar{y}, y_{i+1})$, and the rest of intervals with corresponding auxiliary sets ℓ_k's remain as before. The set ℓ_i is divided into two sets, say,

$\ell'_i = E(y_i < f < \bar{y})$ and $\ell''_i = E(\bar{y} \le f < y_{i+1})$.

Clearly, $e_i = e_i \cup e''_i$ with $\ell'_i \cap \ell''_i = \phi$.

So that $m(\ell_i) = m(\ell'_i) + m(\ell''_i)$. Further New lower sum s' is obtained from old s_p by replacement of $y_i m(\ell_i)$ by $y_i m(\ell'_i) + \bar{y} m(\ell''_i)$. Therefore,

$s' > s_p$.

Other part $S_p \geq S'$ follows through a similar argument.

Hence upon combining these inequalities one gets $s_p < s' < S' \leq S_p$.

Remark : By inviting new point of division in Partition lower sum does not decrease, and upper sum does not increase.

Corollary : None of lower sums is larger than any of upper sum.

Proof : Take any two partitions P_1 and P_2 of the range if *i.e.*, [A,B].

Let lower and upper sums for P_1 and P_2 be S_{p_1} S_{p_2}, and S_{P_1}, Sp_2 respectively.

Construct a partition, say P_3 of [A,B] by taking division points as those of of P_1 *plus* those of P_2. By Lemma 3.1.1.

We find $s_{P_1} < s_{P_3} \leq S_{P3} \leq S_{P2}$ (s_{P3} and S_{P_3} correspond to partition P_3). Thus we have $s_{P_1} \leq S_{P2}$, and this is what was wanted.

Proof of Theorem 3.1.1 Take an upper Sum, say S_{P_0} (corresponding to a partition P_0 of [A,B]. Then corollary above says that for any lower sum s_P we must have

$s_P \leq Sp_0$.

$\therefore$ U = sup{s_P:P is any Partition of [A,B]} $\leq S_{P_0}$

Above when put as $S_{P_0} \geq U$, where we let P_0 free and remembering that r.h.s. is independent of Partition P of [A,B] we find

$$V = \text{Inf}\{S_P\} \geq U.$$

Clearly for any partition P of [A,B] we deduce that

$$s_P \leq U \leq V \leq S_P.$$

Since $Sp - s_p \le \lambda m(E)$, it follows that

$$0 \le V - U \le \lambda m(E).$$

As we may take λ as small as we like it follows that

$$U = V$$

This common value is called the Lebesgue Integral of f over E, and we say f is Lebesgue (L) integrable over E and

write $L\int_E f dx = U(V)$. The proof is complete.

Theorem 3.1.2. (Law of Mean) Let a bounded measurable function $f : E \to R$ satisfy $a \le f(x) \le b$ for all $x \in E$, Then $am(E) < L\int_E f dx < bm(E)$

Proof : Take n as a natural number, and put $A = a - \frac{1}{n}$ and $B = b + \frac{1}{n}$. Clearly then $A < f(x) < B$ for all $x \in E$.

If $P(y_0 - A < y_1 < y_2 < \cdots < y_{n-1} < y_n = B)$ is a Partition of [A,B],

We have $$A\sum_{k=1}^{n-1} m(\ell_k) \le \sum_{k=0}^{n-1} y_k m(\ell_k) \le B\sum_{k=0}^{n-1} m(\ell_k)$$

(symbols having usual meaning) and this inequality-chain gives

$$Am(E) \le s_P \le Bm(E).$$

Proceeding to limit as $\max (y_{k+1} - y_k) = \lambda \to 0$ we arrive at

$$\left(a - \frac{1}{n}\right)m(E) \le L - \int_E f\, dx \le \left(b + \frac{1}{n}\right)m(E).$$

As n is arbitrary taking $n \to \infty$, we get

$$am(\mathrm{E}) \leq \mathrm{L} - \int_{\mathrm{E}} f dx \leq bm(\mathrm{E}).$$

Corollary 1. If f = a constant function, say, = μ

Then $\mathrm{L}\int_{\mathrm{E}} f dx = \mu m(\mathrm{E})$.

Corollary 2. For all bounded measurable functions f over E

where $m(\mathrm{E}) = 0$, $\int_{\mathrm{E}} f dx = 0$

Theorem 3.1.3 (Countable additivity) If $f : \mathrm{E} \to \mathrm{R}$ is a bounded measurable function and

$\mathrm{E} = \bigcup_{k=1}^{\infty} \mathrm{E}_k$ where E_k 's are pairwise disjoint measurable subsets of E,

then $\int_{\mathrm{E}} f dx = \int_{\bigcup_{k=1}^{\infty} \mathrm{E}_k} f dx = \sum_{k=1}^{\infty} \int_{\mathrm{E}_k} f dx.$

Proof : Let $\mathrm{E} = \mathrm{E}' \cup \mathrm{E}''$ with $\mathrm{E}' \cup \mathrm{E}'' = \phi$. and let $\mathrm{A} < f(x) < \mathrm{B}$ for all $x \in \mathrm{E}$.

Suppose $\mathrm{A} = y_0 < y_1 < \cdots < y_{n-1} < y_n = \mathrm{B}$; and put $\ell_k = \mathrm{E}(y_k \leq f < y_{k+1})$;

$\ell'_k = \mathrm{E}'(y_k \leq f < y_{k+1})$ and $\ell''_k = \mathrm{E}''(y_k \leq f < y_{k+1})$;

Then $\ell_k = \ell'_k \cup \ell''_k$ with $\ell'_k \cap \ell''_k = \phi$.

And we have $\sum_{k=0}^{n-1} y_k m(\ell_k) = \sum_{k=0}^{n-1} y_k m(\ell'_k) + \sum_{k=0}^{n-1} y_k m(\ell''_k)$

So proceeding to the limit as $\lambda = \max(y_{k+1} - y_k) \to 0$ we have

$$\int_E f dx = \int_{E'} f dx + \int_{E''} f dx.$$

Now by Induction the result stands O.K. for a finite number of summands.

Finally, take $E = \bigcup_{k=1}^{\infty} E_k$, where $m(E) = \sum_{k=1}^{\infty} m(E_k)$; E_k's. being pairwise disjoint measurable.

So remainder after n terms $\sum_{k=n+1}^{\infty} m(E_k) \to 0$ as $n \to \infty$.

Put $\sum_{k=k+1}^{\infty} E_k = R_n$. As the case for a finite number of summands has been resolved, we are ready to write $\int_E f dx = \sum_{k=1}^{n} \int_{E_k} f dx + \int_{R_n} f dx$.

Now applying Law of Mean (Theorem 3.1.2) we have

$$A.m(R_n) \le \int_{R_n} f dx \le B.m(R_n).$$

As $m(R_n) \to 0$ when $n \to \infty$, we see from above $\lim_{n\to\infty} \int_{R_n} f dx = 0$

From (*) we set $\int_E f dx = \sum_{k=1}^{\infty} \int_{E_k} f dx$.

Corollary If two bounded measurable functions f and g are equivalent over E *i.e.*, if E($f \neq g$) is of measure zero, then $\int_E f dx = \int_E g dx$.

Here if $A = \{x \in E : f(x) \neq g(x)\}$ when $m(A) = 0$

Now by Theorem 3.1.3 $\int_E f dx = \int_A g dx + \int_{(E/A)} f dx$

$$= \int_A f dx + \int_{(E/A)} g dx$$

$$= \int_{(E/A)} g dx \quad \text{as } m(A) = 0$$

$$= \int_A g dx + \int_{(E/A)} g dx \quad \text{since } m(A) = 0$$

$$= \int_{A \cup (E/A)} g dx + \int_E g dx.$$

In particular if f is equivalent to zero function over E, we have $\int_E f dx = 0$. However converse is not true.

Example 3.1.1 Take $f(x) = \begin{cases} 1' & \text{if} \quad x \geq 0 \\ -1 & \text{if} \quad x < 0 \end{cases}$

Then $\int_{[-1,+1]} f dx = \int_{-1}^{0} f dx + \int_{0}^{1} f dx = -1 + 1 = 0.$

But here f is *not equivalent* to zero function in $[-1,1]$.

Theorem 3.1.4 If f:E→R is a non-negative bounded measurable function such that $\int_E f dx = 0$, then f is equivalent to zero function.

Proof : Consider the set E(f>0) which we write as

$$E(f > 0) = \bigcup_{n=1}^{\infty} E\left(f > \frac{1}{n}\right).$$

Suppose f is *not* equivalent to zero function. Then we find n_0

such that $m\left(E\left(f > \frac{1}{n_0}\right)\right) = \alpha > 0.$

Put $A = E\left(f > \frac{1}{n_0}\right)$, $B = (E / A)$; clearly we have $\int_A f dx \geq \frac{1}{n_0}\alpha$, and $\int_B f dx \geq 0$

So $\int_E f dx = \int_A f dx + \int_B f dx \geq \frac{1}{n_0}\alpha$ — a contradiction of hypothesis $\int_E f dx = 0$. Here proof is complete.

Theorem 3.1.5 If f,g:E→R are two bounded measurable functions

Then $\int_E (f + g) dx = \int_E f dx + \int_E g dx.$

Proof : **Here** (f+ g) : E→R is also a bounded measurable function because f and g are so. So $\int_E (f + g) dx$ exists.

Let $a < f(x) < b$ and $c < g(x) < d$ for all. $x \in E$ and take Partitions of $[a, b]$ and of $[c, d]$

as $a = y < y_1 < \cdots < y_{n-1} < y_n = b$ and $c = Y_0 < Y_1 < \cdots < Y_{m-1} < Y_m = d$, and let

$$\ell_k = E(y_k \le f < y_{k+1}), \quad k = 0, 1, \ldots n-1$$

$$E_i = E(Y_i \le g < Y_{i+1}), \quad i = 0, 1, \ldots m-1.$$

Putting $H_{ik} = E_i \cap \ell_k \quad (i = 0, 1, \ldots, (m-1), k = 0, 1, \ldots, n-1)$.

Here $E = \sum_{i,k} H_{ik}$, and H_{ik}'s are pairwise disjoint. So $\int_E (f+g)dx = \sum_{i,k} \int_{H_{ik}} (f+g)dx$

Over $H_{i,k}$ we have $y_k + Y_i \le f(x) + g(x) < y_{k+1} + Y_{i+1}$.

So Law of Mean gives $(y_k + Y_i)m(H_{ik}) \le \int_{H_{ik}} (f+g)dx \le (y_{k+1} + Y_{i+1}) \times m(H_{ik})$

Summing up the inequalities

$$\sum_{i,k} (y_k + Y_i)m(H_{ik}) \le \sum_{i,k} \int_{H_{ik}} (f+g)dx \le \sum_{i,k} (y_{k=1} + Y_{i+1})m(H_{ik}) \quad \ldots \quad (1)$$

Now $\sum_{i,k} y_k m(H_{ik}) = \sum_{k=0}^{n-1} y_k \left(\sum_{i=0}^{N-1} m(H_{ik}) \right)$

$$= \sum_{k=0}^{n-1} y_k m\left(\bigcup_{i=0}^{N-1} H_{ik} \right) = \sum_{k=0}^{n-1} y_k m\left(\bigcup_{i=0}^{N-1} (E_i \cap \ell_k) \right) = \sum_{k=0}^{n-1} y_k m\left(\ell_k \cap \bigcup_{i=0}^{N-1} E_i \right)$$

$$= \sum_{k=0}^{n-1} y_k m(\ell_k \cap E) = \sum_{k=0}^{n-1} y_k m(\ell_k).$$

So first term on L.H.S. represents lower sum s_f, and in a similar way computing rest of terms on both sides we obtain

$s_f + s_g \leq \int_E (f+g)dx \leq S_f + S_g$ where symbols have usual meanings. So passing on limit as max. of lengths of partition sub-intervals $\to 0$ (by increasing number of points of division) we arrive at

$$\int_E f dx + \int_E g dx = \int_E (f+g)dx.$$

Theorem 3.1.6 For a bounded measurable function f over E and λ a constant

$$\int_E \lambda f dx = \lambda \int_E f dx$$

The proof is a routine exercise and left out.

Theorem 3.1.7 If f:E$\to$R is a bounded measurable function then $\left|\int_E f dx\right| \leq \int_E |f| dx$.

Proof : Let P = E($f \geq 0$); then we have $\int_E f dx = \int_P f dx + \int_{E/P} f dx$

$\int_P |f| dx - \int_{E/P} |f| dx$. Also $\int_E |f| dx = \int_p |f| dx + \int_{E/P} |f| dx$; on comparison $\left|\int_E f dx\right| \leq \int_E |f| dx$.

Example 3.1.2 Let $f:[a,b] \to R$ be measurable.

Then f is Lebesgue integrable over $[a,b]$ if and only if $|f|$ is Lebesgue integreble over $[a,b]$.

Solution : Put $f^+ = \max(f,0)$ and $f^- = \max(-f,0)$

(f^+ and f^- are called positive and negative parts of f).

For x fixed in $[a,b]$, if $f(x)>0$, we have $f^{+}(x) = f(x)$ and $f^{-}(x) = 0$.

If $f(x) <0$, then we have $f^{+}(x) = 0$ and $f^{-}(x) = -f(x)$.

If $f(x) = 0$, then $f^{+}(x) = f^{-}(x) = 0$.

Out of these observations we have $f = f^{+} - f^{-}$ and $|f| = f^{+} + f^{-}$ (*)

Now if f is measurable, then both f^{+} and f^{-} are measurable.

If $L-\int_a^b f dx$ exists Theorem 3.1.7 says $|f|$ is L-integrable over $[a,b]$. On the other hand, if $|f|$ is **L-integrable,** since $0 \leq f^{+}(x) \leq |f(x)|$ in $a<x\leq b$, it follows that f^{+} is L-integrable ; Similarly f^{-} is L-integrable over $[a,b]$ and therefore from (*) f is L-integrable over $[a,b]$.

Example 3.1.3. If $f:[a,b] \rightarrow R$ is a bounded measurable function, and $\int_a^b |f| dx = 0$, show that f is equivalent to zero function in $[a,b]$.

Solution : Here $|f|$ is a non-negative bounded measurable function over $[a,b]$ and Theorem 3.1.4 applies and $|f|$ is equivalent to Zero function, and So is f.

3.2. Passage to the limit under sign of Integration :

Let $\{f_n:E \rightarrow R\}$ be a sequence of bounded measurable functions and $\int_0 (x) = \lim_{n\to\infty} f(x)$ for $x \in E$. We want to resolve the question whether $\int_E f_0 dx = \int_E \lim_{n\to\infty} f_n(x)dx = \lim_{n\to\infty} \int_E f_n dx$.

Example 3.2.1 Define $f_n:[0,1]\to R$ as under :

$$f_n(x) = \begin{cases} n & \text{if } 0 < x < \frac{1}{n} \\ 0 & \text{elsewhere in } [0,1]. \end{cases}$$

Then we have each f_n is a step function in [0,1] and is a bounded measurable function over [0,1]. Further $\lim_{n\to\infty} \int_n (x) = 0$ in $0 \le x \le 1$. So $f_0(x)$ means

$f_0(x) = 0$ in $0 \le x \le 1$. Clearly, here $\int_0^1 f_n dx = \int_0^{\frac{1}{n}} f_n dx + \int_{\frac{1}{n}}^1 f_n dx = 1$ for $n = 1,2,\ldots$

and $\int_0^1 f_0 dx = 0$; that is to say, $\lim_{n\to\infty} \int_0^1 f_n dx = 1 \neq 0 = 0 \int_0^1 \lim_{n\to\infty} f(dx)$.

Thus Example 3.2.1 shows that in general passage to limit under integral sign is not valid.

Theorem 3.2.1 Let $\{f_n : E\to R\}$ be a sequence of bounded measurable functions converging in measure to $F:E\to R$ where F is a bounded measurable function. If there is a constant $K > 0$ such that $|f_n(x)| < K$ for all n and for all $x \in E$.

Then $\lim_{n\to\infty} \int_E f_n dx = \int_E F dx$.

(Theorem 3.2.1 is due to Lebesgue and is often named as, Lebesgue Theorem).

Proof : It is possible to obtain a sub-sequence $\{f_{n_k}\}$ of $\{f_n\}$ such that $\lim_k f_n(x) = F(x)$ almost everywhere in E, *i.e.*, $\lim_k f_{nk}(x) = F(x)$ for all $x \in E$ except over a subset of E having zero measure (applying Riesz Theorem) ; and this is why, we obtain from,

$|f_n|(x)| < K$ for all n and for all $x \in E$, the inequality :- $|F(x)| \leq K$ almost everywhere in E.

For a +ve σ put $A_n(\sigma) = E(|f_n - F| \geq \sigma)$, and $B_n(\sigma) = E(|f_n - F| > \sigma)$.

So $$\left|\int_E f_n dx - \int_E F dx\right| \leq \int_E |f_n - F| dx = \int_{A_n(\sigma)} |f_n - F| dx + \int_{B_n(\sigma)} |f_n - F| dx.$$

Now $|f_n(x) - F(x)| \leq |f_n(x)| + |F(x)| < 2K$ almost everywhere in E.

(and hence almost everywhere in $A_n(\sigma)$). So Law of Mean gives us

$$\int_{A_n(\sigma)} |f_n - F| dx \leq 2K.\ m(A_n(\sigma)) \quad \ldots \quad \ldots \quad (1)$$

Since a set of zero measure has no effect on Value of Lebesgue integral, inequality in (1) stands alright. Again by Law of Mean we have

$$\int_{B_n(\sigma)} |f_n - F| dx \leq \sigma.\ m(B_n(\sigma)) \leq \sigma.\ m(E) \quad \ldots \quad (2)$$

From (1) and (2) we find

$$\left|\int_E f_n dx - \int_E F dx\right| \leq 2K.\ m(A_n(\sigma)) + \sigma.\ m(E) \quad \ldots \quad (3)$$

Given an arbitrary +ve $\in$, choose a +ve σ so that $\sigma.\ m(E) \leq \frac{1}{2} \in$.

For this +ve σ convergence in measure says that $m(A_n(\sigma)) \to 0$ as $n \to \infty$;

so we find an index N such that 2K. $m(A_n(\sigma)) < \frac{1}{2} \in$ for $n > N$.

Therefore $\left|\int_E f_n dx - \int_E F dx\right| < \varepsilon$ whenever $n > N$.

Remark : Since convergence in measure is more general than pointwise convergence, Theorem 3.2.1 remains true if one assumes $\lim_{n \to \infty} f_n(x) = F(x)$ almost everywhere in E.

Inequality $\left|\int_n f_n(x)\right| < K$ **for all** n and **for all** $x \in E$ is very strict. Often said $\{f_n\}$ is uniformly bounded over E; and Dominance by +*ve* scalar K has been exploited in the proof. That is why, Theorem 3.2.1 is sometimes designated as a version of **Lebesgue Dominated convergence Theorem.**

§ **3.3.** Let $f:[a,b] \to R$ be a bounded function and $x_0 \varepsilon [a,b]$.

For *a* +*ve* δ let $m_\delta(x_0) = \underset{x_0-\delta<x<x_0+\delta}{\text{Inf}} \{f(x)\}$

and $M_\delta(x_0) = \underset{x_0-\delta<x<x_0+\delta}{\text{sub}} \{f(x)\}$

(If x_0 is one of end points of $[a,b]$ the inequalities have accordingly to be tailored).

So $m_\delta(x_0) \leq f(x_0) < M_\delta(x_0)$; Let $\delta \downarrow 0_+$, Then $m_\delta(x_0) \uparrow$ and $M_\delta(x_0) \downarrow$.

Thus $\lim_{\delta \to 0+} m_\delta(x_0) = m(x_0)$; (say) and $\lim_{\delta \to 0+} M_\delta(x_0) = M(x_0)$ (say) with

$m(x_0) \leq f(x_0) \leq M(x_0)$.

In this way functions $m(x)$ and $M(x)$ called lower and upper Baire functions of f are defined in $[a,b]$ such thit $m(x) \leq M(x)$ is $a \leq x \leq b$.

In elementary analysis we know that f is continuous at x_0 if $m(x_0) = f(x_0) = M(x_0)$. We are ready to establish following theorem.

Theorem 3.3.1 If $\int_a^b f dx$ is Riemann-integrable $\overset{u}{\ldots}\int_a^b f dx$ exists, then $L - \int_a^b f dx$ exits and $L - \int_a^b f dx = R\int_a^b f dx$. The proof rests on supporting Lemma and observations.

Lemma 3.3.1 Let $P_i\left(a = x_0^{(i)} < x_k^{(i)} < \ldots < x_{ni}^{(i)} = b\right)$ be a chain of Partitions ($i = 1,2,\ldots$) of $[a,b]$ with $\lambda_{p_i} = \max\left(x_{k+1}^{(i)} - x_k^{(i)}\right) \to 0$ as $i \to \infty$. Let $m_k^{(i)} = \inf\limits_{\left[x_k^{(i)}, x_{k+1}^{(i)}\right]} \{f(x)\}$ and

$$\phi_i(x) = m_k^{(i)} \qquad \text{for } x \in \left(x_k^{(i)}, x_{k+1}^{(i)}\right)$$

$$0 = \qquad \text{for } x = \text{Division points } x_0^{(i)}, x_1^{(i)} \ldots x_{ni}^{(i)}.$$

Then $\lim\limits_{i \to \infty} \phi_i(x) = m(x)$ for $x \in [a,b] / \left\{x_k^{(i)}\right\}$.

(We may say, $\lim\limits_{i \to \infty} \phi_i = m$ almost everywhere in $[a,b]$).

Proof : Choose an arbitrary i, and fix it. Let $x_{ko}^{(i)} < x < x_{ko+1}^{(i)}$.

For appropriately small $+ve\delta$, $(x - \delta, x + \delta) \subset \left[x_{ko}^{(i)}, x_{ko+1}^{(i)}\right]$

Therefore $m_{ko}^{(i)} \leq m_\delta(x)$

or $\phi_i(x) \leq m_\delta(x)$

Taking $\delta \to 0_+$, we have $\phi_i(x) \leq m(x)$ ... (1)

If $h < m(x)$, choose $+ ve\delta$, such that $m_\delta(x) > h$.

Now take i_0 so large that for $i > i_0$ we have $\left[x_{i0}^{(i)}, x_{k0+i}^{(i)}\right] \subset (x-\delta, x+\delta)$, choice i_0 is O.K. because $\lambda_i \to 0$. So we have $m_{i_0}^{(i)} \geq m_\delta(x) > h$

or, $\phi_i(x) > h$. ... (2)

From (1) and (2) we produce $h < \phi_i(x) \leq m(x)$.

That means $\phi_i(x) \to m(x)$ as $i \to \infty$, and Lemma is proved.

Corallary 1. Functions m and M are measurable.

Because $\lim \phi_i(x) = m(x)$ almost everywhere in $[a, b]$ and because being step function each ϕ_i is measurable. By a similar argument M is also measurable.

Corollary 2. $L - \int_a^b \phi_i dx \to L - \int_a^b m dx$. as $i \to \infty$.

Since f is bounded, if $|f(x) < K,|$ then $|\phi_i(x)| \leq K$ and $|m(x)| \leq K$. Thus these functions are L-integrable. Now apply Lebusgue Theorem on passage to limit under the integral sign.

We observe that $L - \int_a^b \phi_i dx = \sum_{k=0}^{ni-1} \int_{x_k^{(i)}}^{x_{k+1}^{(i)}} \phi_i dx$

$= \sum_{k=0}^{ni-1} m_k^{(i)} \left(x_{k+1}^{(i)} - x_k^{(i)}\right) = S_i$; where S_i is the Lower sum for P_i-th partition and Corollary 2 means that for $i \to \infty$

$s_i \to \text{L}-\int_a^b m\,dx$ ... (3)

Similarly, upper sum S_i approaches $\text{L}-\int_a^b \text{M}\,dx$

i.e., $S_i \to \text{L}-\int_a^b \text{M}\,dx$ ... (4)

Thus f is R-integrable over $[a,b]$ *i.e.*, $\text{R}-\int_a^b f\,dx$ exists if and only if

$S_i - s_i \to 0$ as $i \to \infty$.

That is to say, $\text{R}-\int_a^b f\,dx$ exists if and only if $\text{L}-\int_a^b (\text{M}-m)\,dx = 0$ (from) (3) and (4) As $\text{M} \geq m$ we have M and m are equivalent functions in $[a,b]$*i.e.*, $m(x)=\text{M}(x)$ almost everywhere in $[a,b]$ $[a,b]$*i.e.*, f is continuous almost everywhere in $[a,b]$.

Finally, f being equivalent to m which is measurable, it follows that f is measurable *i.e.*, f is a bounded measurable function over $[a,b]$ and therefore $\text{L}-\int_a^b f\,dx$ exits. From R-integrability of f over $[a,b]$, we have $s_i \to \int_a^b f\,dx$. Because $s_i \to \text{L}-\int_a^b m\,dx$ (see (3)). We at last see that $\text{R}-\int_a^b f\,dx = \text{L}-\int_a^b m\,dx$

Example 3.3.1 Dirichlet's function $\psi : [0,1] \to \text{R}$ given by

$\psi(x)=1$ if x is rational in $[0,1]$

$=0$ if x is irrational in $[0,1]$.

is equivalent to zero function in $[0,1]$. So ψ is a bounded measurable function in $[0,1]$ and $\text{L}-\int_0^1 \psi\,dx$ exists and further $\text{L}-\int_0^1 \psi\,dx = 0$. But we know that ψ is nowhere continuous in $[0,1]$, and $R-\int_0^1 fdx$ does not exist.

3.4. Let $f:E\to R$ be a measurable non-negative function. For a natural number n, Let ${}^n f:E\to R$ be defined as

$${}^n f(x)=f(x) \qquad \text{if } 0\le f(x)\le n$$

$$= n \qquad \text{if } f(x)>n.$$

Thus ${}^n f=\text{Min}(f,n)$.

Example 3.4.1 Let $f(x)=\dfrac{1}{3\sqrt{x}}$ in $0<x\le 1$

$$= 0 \qquad \text{at } x=0.$$

Find ${}^4 f$.

Solution : Here $f(x)<4$ gives $\dfrac{1}{2\sqrt{x}}\le 4$ or $x\ge\dfrac{1}{64}$

Thus $\quad 4f(x)=f(x)=\dfrac{1}{3\sqrt{x}}$ in $\dfrac{1}{64}\le x\le 1$

$$= 4 \qquad \text{if} \qquad 0<x<\frac{1}{64}$$

$$= 0 \qquad \text{if} \qquad x=0$$

Thus given an unbounded non-negative measurable function $f:E\to R$, for each n, n_f is a bounded non-negative measurable function and so $L-\int_E n_f dx$ exists.

Also $\left\{L-\int_E n_f dx\right\}$ in an ↑ sequence of reals and hence it either converges or diverges to = +∞.

Definitions 3.4.1 Let f:E→R, be a non-negative unbounded measurable function. Then $\lim_{n\to\infty} L-\int_E n_f dx$ is called Lebesgue integral of f on E and is denoted as $L-\int_E fdx$.

i.e., $\int_E fdx = \lim_{n\to\infty} L-\int_E n_f dx.$

If $L-\int_E fdx$ is finite, then f is called summable on E.

Example 3.4.2 If $f(x) = \frac{1}{\sqrt[3]{x}}$ in $0 < x \le 1$

$$= 0 \quad \text{at} \quad x = 0$$

Examine $L-\int_0^1 fdx$.

Solution : We see that $L-\int_0^1 fdx = R-\int_0^1 n_f dx$ (because ^nf is bounded and R-integrable.) $= \int_0^{\frac{1}{n^3}} n\, fdx + \int_{\frac{1}{n^3}}^1 {}^n f\, dx = \int_0^{\frac{1}{n^3}} ndx + \int_{\frac{1}{n^3}}^1 \frac{dx}{\sqrt[3]{x}}$

$\frac{1}{n^2} + \left(\frac{3}{2} - \frac{3}{2n^2}\right)$. Hence $\lim_{n\to\infty} L-\int_0^{1n} fdx = \lim_{n\to\infty}\left\{\frac{1}{n^2} + \left(\frac{3}{2} - \frac{3}{2n^2}\right)\right\} = \frac{3}{2}$

Remark : We know that $R-\int_0^1 \frac{1}{\sqrt[3]{x}}$ is an improper Riemann integral. In evaluating the integral in Lebesgue sense *i.e.*, in $L-\int_0^1 \frac{1}{\sqrt[3]{x}}$ there is nothing 'improper'. It is a Lebesgue integral with $L-\int_0^1 \frac{1}{\sqrt[3]{x}} = 3/2$.

In order to define $L-\int_E f dx$ for an unbounded measurable function taking +*ve* and -*ve* signs, we first of all check that such a function may be written as the difference of two non-negative-valued measurable functions.

Given any $f: E \to R$ as unbounded | bounded measurable function over a bounded measurable set E, put

$$f^+ = \max(f, 0)$$

and $f^- = \max(-f, 0)$.

That is to say, as $x \in E$, if $f(x)>0$, put $f^+(x) = f(x)$, and $f^-(x)=0$; and if $f(x)<0$, then put $f^+(x) = 0$ and $f^-(x) = -f(x)$. If $f(x) = 0$, put $f^+(x) = f^-(x) = 0$.

Remark : (i) $f = f^+ - f^-$

(ii) $|f| = f^+ + f^-$

Example 3.4.3 Let $f(x) = x^2 - 1$ in $-2 \le x \le 2$.

Find f^+

Solution : Here $f^+(x) = f(x)$ in $-2 < x \le -1$

$= 0$ in $-1 \le x \le 1$

$= f(x)$ in $\le x \le x \le 2$.

Definition 3.4.2 Let f:E→R be an unbounded measurable function. If both f^+ and f^- are L–intergrable over E, then f is said to be L–integrable over E, and in that case

$$\text{L}-\int_E f dx = \text{L}-\int_E f^+ dx - \text{L}-\int_E f^- dx.$$

Theorem 3.4.1 Let f:E→R be a summable function and g is equivalent to f over E, then $\text{L}-\int_E f dx = \text{L}-\int_E g dx.$

Proof : We have $f(x)$=g(x) almost everywhere in E.

Let each of f and g be non-negative, then for any natural n, $^n f(x) = {}^n g(x)$ almost everywhere in E.

That is to say $^n f$ and $^n g$ are equivalent over E and hence $\text{L}-\int_E^N f dx = \text{L}-\int_E^n g dx$

Passing on limit as $n\to\infty$, we have $\text{L}-\int_E f dx = \text{L}-\int_E g dx.$

Now Let f and g have arbitrary signs. Then $f^+(x) = g^+(x)$ almost everywhere in E.

Therefore $\text{L}-\int_E f^+ dx = \text{L}-\int g^+ dx$. Similarly, $\text{L}-\int f^- dx = \text{L}-\int g^+ dx.$

Therefore, $\text{L}-\int_E f dx = \text{L}\int_E f^+ dx - \text{L}\int_E f^- dx = \text{L}-\int_E f^+ dx - \text{L}\int f^- dx = \text{L}-\int_E g dx.$

Theorem 3.4.2 Let $f:E\to R$ be a summable function. Then for a given $+ve$ ε, there is a $+ve$ δ such that for every measurable subset $e\subset E$ with $m(e)<\delta$, $\left|\int_e f dx\right|<\varepsilon$.

Proof : First Let us suppose f to be non-negative. Then we have $\lim_{n\to\infty}\int_E {}^n f\, dx=\int_E f\, dx$.

Thus given $a{+}ve\,\varepsilon$, we find a natural number N such that $\int_E f dx=\int_E {}^N f dx<\varepsilon/2$

$$0\ \int_E (f-{}^N f)dx<\varepsilon/2 \qquad \ldots \qquad \ldots \qquad (1)$$

Take $a+ve$ $\delta<\dfrac{\varepsilon}{2N}$. Now if e is a measurable subset of E with $m(e)<\delta$,

We have $\int_e {}^N f dx\le\int_e N.dx$ $\quad({}^N f\le N \text{ always})$

$= N.m(e)$

$< N\,\delta<\varepsilon/2 \qquad \ldots \qquad \ldots \qquad (2)$

From (1) and (2) We have $\int_e f dx=\int_e (f-{}^N f)dx+\int_e {}^N f\, dx<\varepsilon/2+\varepsilon/2=\varepsilon$.

For the case when f is of arbitrary signs, we write $f=f^+-f^-$; when each of f^+ and f^- is of non-negative sign over E. So by first part of the proof given $\epsilon>0$, we find $a+ve\delta_1$ such that for every measurable subset e of E with $m(e)<\delta_1$

We have $\int_e f^+ dx<\varepsilon/2 \qquad \ldots \qquad \ldots \qquad (3)$

Simirlarly, there is a $+ve\delta_2$ such that for every measurable subset e of E with $m(e)<\delta_2$ we have

$$\int_e f^- dx < \in /2 \qquad (4)$$

If $0 < \delta < \min(\delta_1 \delta_2)$, for every measurable subset e of E with $m(e) < \delta$ we have

$$\left|\int_e f dx\right| \leq \int_e |f|\, dx = \int_e \left(f^+ + f^-\right) dx = \int_e f^+ dx + \int_e f^- dx < \varepsilon \text{ from (3) and (4).}$$

The proof is now complete.

Exercise

Short Answer Type

1. If $f(x) = \log_e \frac{1}{x}$ in $0 < x \leq 1$, find ^2f.

2. If $f(x) = \frac{1}{2} + \sin x$ in $0 \leq x < 2\pi$,

   Find (*i*) f^+ and (ii) f^-.

3. If f:E→R is L–integrable over a bounded measurable set E with m (E) = 0, show that $\int_E f dx = 0$.

4. If X is the characteristic function of the set of all irrationals in [0,1] evaluate L–$\int_0^1 X dt$.

5. If E is a measurable subset of $[a,b]$ show that

   $\int_E f dx = m(E)$ when $f \equiv 1$.

6. If $f(x) = x^2 - 1$ in $-2 \le x \le 2$

Find f^{-}.

Broad Question

1. If $:[a,b] \to R$ is bounded and L–integrable, and E_1 and E_2 are measurable subsets $\subset [a,b]$, show that

$$\int_{E_1} f dx + \int_{E_2} f dx = \int_{E_1 \cup E_2} f dx + \int_{E_1 \cap E_2} f dx.$$

2. If $f:[a,b] \to R$ is a bounded measurable function and $L-\int_a^b f^2 dx = 0$ show that f is equivalent to zero function in $[a,b]$.

3. If $f:[a,b] \to R$ is a bounded measurable function such that $f(x) \ge 0$ almost everywhere in $[a,b]$, and if $L-\int_a^b f dx = 0$, Show that $f(x)=0$ almost everywhere in $[a,b]$.

4 Let $f:(0,1] \to R$ be taken as

$$f(x) = \begin{cases} n & \text{for } \frac{2n+1}{2n(n+1)} < x \le \frac{1}{n} \\ -n & \text{for } \frac{1}{n+1} < x \le \frac{2n+1}{2n(n+1)} \end{cases} \qquad (n = 1, 2, 3, \ldots)$$

Verify that f is not summable over (0,1].

$$\left(\text{Hint}: \int_0^1 |f| dx = \sum_{n=1}^{\infty} \int_{\frac{1}{n+1}}^{\frac{1}{n}} |f| dx = \sum_{n=1}^{\infty} \frac{1}{n+1} = +\infty \right)$$

5. If $f(x) = \frac{1}{x^p}$ in $0 < x \leq 1$; for $p < 1$, show that $L-\int_0^1 f dx = \frac{1}{p+1}$.

6. If $f:[a,b] \to R$ is bounded and L-integrable over $[a,b]$ and $F(x) = \int_a^x f dt$ in $a \leq x \leq b$, Show that F is continuous in $[a,b]$.

7. If $f:[a,b] \to R$ is summable and if $F(x) = \int_a^x f dt$ in $a \leq x \leq b$, verify that $F'(x) = f(x)$ almost everywhere in $[a,b]$.

Unit 4 ☐ Riemann Sticeltjas Integral, Fourier Series

(Riemann-Stieltjes Integral (R.S. integral), Existence of R.S. integral, Fundamentals properties including additive property, Itergration by parts, Law of Mean, Passage to the limit under R.S. integral sign. Fourier series for a function, Riemann-Lebesgue, Dirichlet's Integral, Convergence of Fourier Series).

3.1 Stieltijes Integral is an important generalisation of Reimann Integral.

Let f, g:$[a,b]\to$R be two bounded functions over a closed interval $[a,b]$ and g be monotonically ↑.

If $P(a=x_0<x_1<x_2<\cdots x_r<x_{r+1}<\cdots<x_n=b)$ be a partition of $[a,b]$,

Let $U(P,f,g)=M_1(g(x_1)-g(x_0))+M_2(g(x_2)-g(x_1))+\cdots+M_r(g(x_r)-g(x_{r-1}))+\cdots+M_n(g(x_n=b)-g(x_{n-1}))$.

$$=\sum_{r=1}^{n} M_r(g(x_r)-g(x_{r-1})) \qquad g(x_r)\geq g(x_{r-1})) \text{ because g is} \uparrow$$

and similarly, $L(P,f,g)=\sum_{r=1}^{n} m_r(g(x_r)-g(x_{r-1}))$, where

$$M_r=\sup_{x_{r-1}\leq x\leq x_r}\{f(x)\} \text{ and } m_r=\operatorname{Inf}_{x_{r-1}\leq x\leq x_r}\{f(x)\},\ r=1,2,\ldots,n.$$

Sums $U(P,f,g)$ and $L(P,f,g)$ are respectively called an upper and a lower Riemann–Stieltjes (R–S) sum of f with respect to (w.r.t)g corresponding to partition P of $[a,b]$.

If $M=\sup_{a\leq x\leq b}\{f(x)\}$ and $m=\operatorname{Inf}_{a\leq x\leq b}\{f(x)\}$, we have

$$m(g(b)-g(a))\leq L(P,f,g)\leq U(P,f,g)\leq M(g(b)-g(a)).$$

This chain of inequality shows that lower and upper R–S Sums are always bounded. We are in a position to define the Upper and Lower R–S integral of f w.r.t in $[a,b]$ as Inf $\{U(P, f, g)\} = \int_a^b fdg$ and Sup $\{L(P, f, g)\} = \int_{-a}^b fdg$

Where Inf and Sup are taken over all partitions P of $[a,b]$ with $\max\limits_{1\le r\le n}(x_r - x_{r-1}) \to 0$.

Definition 4.1.1 When **upper and lower R–S integrals** are equal *i.e.*, when $\int_a^{-b} fdg = \int_a^{-b} fdg$, this common value is called the **Riemann-Stieltjes integral** (on Simply **Stieltjes integral.** In that case R–S integral is written as $\int_a^b fdg$.

Remark : When $g(x)=x$ in $a<x\le b$, then Riemann-integral R–$\int_a^b fdx$ becomes a special case R-S integral $\int_a^b fdg$.

Properties of R-S Sums

I. For any partition P of $[a,b]$

$$L(P, f, g) \le U(P, f, g)$$

II. If P_1 is a given Partition of $[a,b]$ and P_2 is another with more division points *i.e.*, $P_2 = P_1$ plus added division points (P_2 called a refinement of P_1), then

$$U(P_2, f, g) \le U(P_1, f, g) \text{ and } L(P_2, f, g) \ge L(P_1, f, g)$$

That is to say, upper R.S. Sums ↓ and Lower R-S Sums↑.

III. If P_1 and P_2 are any two partitions of $[a,b]$ (independent of each other),

$$L(P_1, f, g) \le U(P_2, f, g).$$

Thus any lower R-S Sum does not exceed any upper R-S Sum. And this at once gives rise

$$\underline{\int}_a^b f dg \leq \overline{\int}_a^b f dg$$

Theorem 4.1.1 f is R-S integrable w.r.t g in $[a,b]$ iff corresponding to any +veε, there is a partition P of $[a,b]$ such that

$$U(P, f, g) - L(P, f, g) < \varepsilon. \qquad (1)$$

Proof : For every partition P of $[a,b]$ we have

$$L(P, f, g) \leq \underline{\int}_a^b f dg \leq \overline{\int}_a^b f dg \leq U(P, f, g)$$

In case (1) holds for some P we deduce above chain of inequality

$$0 \leq \overline{\int}_a^b f dg - \underline{\int}_a^b f dg \leq \varepsilon.$$

Since $\varepsilon > 0$ is arbitary, we have $\underline{\int}_a^b f dg = \overline{\int}_a^b f dg$

So f is R-S integrable w.r.t g in $[a,b]$.

Conversely, Let f be R-S integrable w.r.t g in $[a,b]$, and let $\varepsilon > 0$ be given. So we find Partitions P_1 and P_2 of [a,b] such that.

$$U(P_1, f, g) - R\text{-}S\int_a^b f dg < \frac{\varepsilon}{2}$$

and $R\text{-}S\int_a^b f dg (P_2, f, g) < \varepsilon/2$

by an appeal to Inf. and Sup. property. Consider the refinement $P_1 = P_1 \cup P_2$ as a Partition of $[a,b]$ with added division points than either of P_1 and P_2; wherein following chain holds.

$$U(P, f, g) \le U(P_1, f, g) < \int_a^b f dg + \frac{\varepsilon}{2} < L(P_2, f, g) + \varepsilon \le L(P, f, g) + \varepsilon.$$

This gives $U(P, f, g) - L(P, f, g) < \varepsilon$.

Following is a list of fundamental properties of R-S integral

(a) $\int_a^b (f_1 + f_2) dg = \int_a^b f_1 dg + \int_a^b f_2 dg$

(b) $\int_a^b f d(g_1 + g_2) = \int_a^b f dg_1 + \int_a^b f dg_2$

(c) For any constants λ and μ $\int_a^b (\lambda f) d(\mu g) = \lambda\mu \int_a^b f dg$ **(If r.hs. members exist, so does l.h.s. → in (a), (b) and (c).**

(d) **If a<c<b and all integrals exist, then**

$$\int_a^b f dg = \int_a^c f dg + \int_c^b f dg.$$

Remark : If $\int_a^b f dg$ exists. it is a routine exercise to show that both integrals $\int_a^c f dg$ and $\int_c^b f dg$ exist $(a<c<b)$. However, converse is not true.

Example 4.1.1 Let f,g:[-1,1]→R be taken as

$f(x)$=0 in $-1\le x\le 0$ and $g(x)$=0 in $-1\le x\le 0$

= 1 in $0<x\le 1$ = 1 in $0<x\le 1$.

Then $\int_{-1}^{b} fdg$ and $\int_{0}^{1} fdg$ exist ; but $\int_{-1}^{1} fdg$ does not exist.

Solution : It is easy to check that $\int_{-1}^{1} fdg$ exist ; In fact any R-S Sum is equal to zero, and so is the case with $\int_{0}^{1} fdg$. In respect of $\int_{-1}^{1} fdg$, take a partition P of [−1,+1] taking care to exclude 0 as a point division in P, then corresponding\

$$\text{R-S Sum} = \sum_{k=0}^{n-1} M_k(g(x_{k+1})-g(x_k)) \ (P(-1=x_0<x_1<\cdots<x_i<x_{i+1}<\cdots<x_n=1)).$$

shall be either 0 or 1. So $\int_{-1}^{1} fdg$ does not exist.

Example 4.1.2 Show that $\int_{0}^{3} x^2 d[x]=14$, where $[x]$ is the largest integer not larger than x.

Solution : Write $\int_{0}^{3} = \int_{0}^{1} + \int_{1}^{2} + \int_{2}^{3}$

each integral on r.h.s. exist and to evaluate $\int_{0}^{1} x^2 d[x]$

We take any Partition $(0=x_0<x_1<x_2<\cdots<x_{n-1}<x_n=1)$ of [0,1] wherein R-S sunm $\sum_{r=0}^{n} M_r\,(g(x_r)-g(x_{r-1}))$ $(g(x)=[x])$. becomes = $1^2(1-0)=1$. In fact all sums coincide

and $\int_0^1 x^2 d[x]=1$ In a similar argument $\int_1^2 x^2 d[x]=2^2(2-1)=4$ and $\int_2^3 x^2 d[x]=3^2(3-2)=9$.

Hence required integral $\int_0^3 x^2 d[x]=1+4+9=14$.

Theorem 4.1.2 The integral $\int_a^b f dg$ exists if f is continuous in $[a,b]$ and g is of bounded variation in $[a,b]$.

Proof : We know that a function of bounded varition in $[a,b]$ is represented as the difference of two ↑ functions in $[a,b]$. We may therefore assume g to be an ↑ function in $[a,b]$. Let P($a=x_0<x_1<x_2<\cdots<x_n=b$) be a Partition of $[a,b]$ and corresponding R-S lower and upper sums be $s=\sum_{k=0}^{n-1} m_k(g(x_{k+1})-g(x_k))$

and $S=\sum_{k=0}^{n-1} M_k(g(x_{k+1})-g(x_k))$. Then $s \le S$.

It is a routine exercise to check that lower sums do not decrease and upper Sums do no increase upon addition of new points of division in P. Also none of sums s surpasses any of sums S.

Put I=Sup$\{s\}$, sup being taken in respect of all partitions including those with $\max_{1\le r\le n}(x_r - x_{r-1}) \to 0$

Now $s \le I \le S$.

Let $\varepsilon>0$ be given arbitrary, Since f is continuous in $[a,b]$, there is $a+ve$ δ such that $|f(u)-f(v)|<\varepsilon$ whenever $|u-v|<\delta$.

If $\underset{1\le r\le n}{\text{Max}}(x_r - x_{r-1})$ is taken smaller than δ, we find $M_k - m_k < \varepsilon$ for $k = 0,1,\ldots,(n-1)$.

That means, $S - s < \varepsilon(g(b) - g(a))$; So Theorem 4.1.1 says that $\int_a^b f dg$ exists.

Theorem 4.1.3 (Integration by parts)

If f and g are ↑ in $[a,b]$ and f is continuous in $[a,b]$,

then $\int_a^b f dg = f(b)g(b) - f(a)g(a) - \int_a^b g df$

Proof: For a partition $P(a=x_0<x_1<\ldots<x_n=b)$ of $[a,b]$; take t_i with $x_{i-1} \le t_i \le x_i$ and $t_0 = a$, $t_{n+1} = b$, then $Q\ (t_0, t_i \ldots, t_{n+1})$ is another partition of $[a,b]$.

Now let $S(P, f, g) = \sum_{i=1}^{n} f(t_i)(g(x_i) - (x_{i-1}))$

$= f(b)g(b) - f(a)g(a) - \sum_{i=1}^{i+1} g(x_{i-1})(f(t_i) - f(t_{i-1}))$

$= f(b)g(b) - f(a)g(a) - S(Q, g, f)$ for $t_{i-1} \le x_{i-1} \le t_i$

Thus if max $(x_r - x_{r-1})$ (in respect of P) →0, then $\max(t_{i+1} - t_i)$ (in respect of Q)→0. Now we know under $\max(x_r - x_{r-1})$(in respect of P)→0 $S(P, f, g) \to \int_a^b f dg$. In consequence, $S(Q, g, f) \to \int_a^b g df$ under $\max(t_{i+1} - t_i)$ (in respect of Q) →0. Proceeding to above limit we obtain. $\int_a^b f(b)g(b) = f(b) - f(a)g(a) - \int_a^b g df$.

Theorem 4.1.4 (Mean-Value Theorem). If $f : [a,b] \to R$ is continuous and $g:[a,b] \to R$ is ↑ Then there is a point c between a and b such that

$\int_a^b f dg = f(c)(g(b)-g(a))$.

Proof: Put M= $\sup_{a\le x\le b}\{f(x)\}$, $m=\operatorname{Inf}_{a\le x\le b}\{f(x)\}$; thus we have $m\le f(x)\le M$ in $[a, b]$

and $m\int_a^b 1.dg \le \int_a^b f dg \le M\int_a^b 1.dg$ or $m(g(b)-g(a)) \le \int_a^b f dg \le M(g(b)-g(a))$.

So there is a number μ between m and M such that $\int_a^b f dg = \mu(g(b)-g(a))$.

By intermediate value property of continuous function f over $[a,b]$, we get one point, say, c between a and b such that $f(c)=\mu$. So we have $\int_a^b f dg = f(c)(g(b)-g(a))$.

4.2. Passage to limit under R-S integral sign.

Theorem 4.2.1 If $f:[a,b]\to R$ is continuous and $g:[a,b]\to R$ is of bounded variation then

$$\left|\int_a^b f dg\right| \le M \overset{b}{\underset{a}{V}}(g). \text{ where } \sup_{a\le x\le b}|f(x)|.$$

Proof : For any Partition $(a=x_0<x_1<...<x_r<x_{r+1}<...<x_n=b)$ and $x_r<\xi_r\le x_{r+1}$

we have $$\left|\sum_{k=0}^{n-1} f(\xi_k)(g(x_{k+1})-g(x_k))\right| \le M\sum_{k=0}^{n-1}|g(x_{k+1})-g(x_k)|$$

$$\le M \overset{b}{\underset{a}{V}}(g)$$

This inequality gives rise in the process of limit as $\max_{1\le r\le n}(x_r - x_{r-1})$ (in respect of Partition) $\to 0$ the inequality as desired namely $\left|\int_a^b f dg\right| \le M \overset{b}{\underset{a}{V}}(g)$.

Theorem 4.2.2. Let $\{f_n:(a,b)\to R\}$ be a sequence of continuous functions converging uniformly to f *in* $[a,b]$ and $g:[a,b]\to R$ be a function of finite variation, then

$$\lim_{n\to\infty}\int_a^b f_n dg=\int_a^b \lim_{n\to\infty} f_n dg=\int_a^b fdg$$

Proof : Put $M_n = \max_{a\leq x\leq b} |f_n(x)-f(x)|$; By uniform convergence of $\{f_n\}$ the limit function f is continuous in $[a,b]$. So M_n exit as a finite non-negative real. Now,

$$\left|\int_a^b f_n dg-\int_a^b fdg\right|=\left|\int_a^b (f_n-f)dg\right|\leq M_n \overset{b}{\underset{a}{V}}(f)$$

(See Theorem 4.1.1). Now apply unif-limit $f_n=f$ to check that $M_n\to 0$ as $n\to\infty$. Hence we get as wanted; namely, $\lim_{n\to\infty}\int_a^b f_n dg=\int_a^b fdg$.

Example 4.2.1 Apply Integration by parts formula *to* show that $\int_0^3 x\, d([x]-x)=\frac{3}{2}$ when $[x]$ denotes the largest integer not larger than x.

Solution : here $f(x)=x$ in $[0,3]$ is continuous and is of bounded variation in $[0,3]$ (infact it is $\uparrow$ is $[0,3]$). And $g(x) = [x]-x$ is of bounded variation in $[0,3]$. So we apply Integration by parts formula to have

$$\int_0^3 x\, d([x]-x)=\{x[x]-x\}_0^3-\int_0^3([x]-x)dx$$
$$=0-\left(\int_0^1+\int_1^2+\int_2^3\right)$$
$$=-\int_0^1(-x)dx-\int_1^2(1-x)dx-\int_2^3(2-x)dx$$
$$=\frac{1}{2}-1+\left(\frac{2^2}{2}-\frac{1}{2}\right)-2.1+\left(\frac{3^2}{2}-\frac{2^2}{2}\right)$$
$$=\frac{3}{2}.$$

Example 4.2.2 Let $f:[a,b]\to R$ be taken as

$$f(x) = 1 \text{ at } x=\xi \in [a,b]$$

$$=0 \text{ at } x \neq \xi \text{ in } [a,b].$$

If g is monotone ↑ in $[a,b]$ with ξ as a point of continuity,

Show that $\int_a^b f dg = 0$

Solution :- Let $P(a = x_0 < x_1 < \ldots < x_r < \xi < x_{r+1} < \ldots < x_a = b)$ be a Partition of $[a, b]$. By continuity of g at ξ, given a +ve ε, these is a +ve δ such that

$|g(x) - g(\xi)| < \varepsilon$ whenever $|x - \xi| < \delta$.

Now $|g(x_r) - g(x_{r-1})| = g(x_r) - g(\xi) + g(\xi) - g(x_{r-1})|$

$\leq |g(x_r) - g(\xi)| + |g(\xi) - g(x_{r-1})|$

$< \varepsilon + \varepsilon = 2\epsilon$ whenever $\max\limits_{1 \leq r \leq n} (x_r - x_{r-1}) < \delta$

in respect of Partition P.

Accordingly, $S(P, f, g)$

$= \sum_{r=1}^{n} f(t_r)(g(x_r) - g(x_{r-1}))$ with $x_{r-1} \leq t_r \leq x_r$

$= 0$ if $t_r = \xi$.

$= g(x_r) - g(x_{r-1})$. when $t_r = \xi$.

This gives $|S(P, f, g)| < 2\varepsilon$ whenever norm of Partition

$$P = \max_{1 \le r \le n} (x_r - x_{r-1}) < \delta.$$

Proceeding to limit we arrive J $\int_a^b f \, dg = 0.$

§ 4.3

Let : $f : [-\pi, \pi] \to R$ be a bounded R-integrable function with period $= 2\pi$. Then a Trigonometric series of the form $\frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$, where

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos nt \, dt; \; n = 0, 1, 2, \ldots$$

and $b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin nt \, dt; \; n = 1, 2, \ldots$

is said to be the Fourier series generated by f in $[-\pi, \pi]$.

The constant Coefficients a_0, a_n and b_n are known as Fourier Coefficients for f. Generally we write

$$f \sim \frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

Remark : There is a caution. Symbol $\sim$ is not = (equal to).

It is too early to ascertain whether infinite series $\frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$ of variable terms is at all convergent or not ; and if convergent at some x, whether sum $= f(x)$.

Example 4.3.1 Let $f(x)=0$ in $-\pi \le x < 0$

$= 1$ in $0 \le x \le \pi$.

Obtain the fourier series for f.

Solution : Here $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx \, dx$

$= \frac{1}{\pi}\int_0^{\pi} \cos x \, dx$; and hence $a_o = 1$, and $a_n = 0$ for $n = 1,2,...$; Also

$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx \, dx = \frac{1}{\pi}\int_0^{\pi} \sin nx \, dx = \frac{1 - \cos n\pi}{n\pi}$.

Thus $b_n = \frac{2}{n\pi}(n = 1,3,5...)$ and $b_n = 0(n = 2,4,6,...)$

So Fourier series generated by f is given as $f \sim \frac{1}{2} + \frac{2}{\pi}\left(\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + ...\right)$.

Here we see at $x = 0$, the r.h.s. series has the sum $= \frac{1}{2}$ which is **not** equal to $f(o) = 1$ (given).

Some preliminaries :- Let $f : [-\pi, \pi] \rightarrow R$ be a bounded integrable function.

(1) f is called **an even function if** $f(-x) = f(x)$ in $-\pi \le x < \pi$.

For example, $f(x) = \cos x$ is an even function.

(2) f is called **an odd function if** $f(-x) = -f(x)$ in $-\pi \leq x \leq \pi$.

For example, $f(x) = \sin x$ is an odd function.

(3) For an even function f, $\int_{-\pi}^{\pi} f dx = 2\int_{0}^{\pi} f dx$.

(4) For an odd function f $\int_{-\pi}^{\pi} f dx = 0$

(5) If $\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$,

Then (i) $b_1 = b_2 \ldots = 0$ **in case f is an even function.**

(ii) $a_0 = a_1 = a_2 = \ldots 0$ **in case f is an odd function.**

Theorem 4.5.1 (Riemann -Lebesgue theorem)

If $f:[a,b] \to R$ is bounded and R-integrable then

(i) $\lim_{n\to\infty} \int_a^b f(x) \sin nx \, dx = 0$

(ii) $\lim_{n\to\infty} \int_a^b f(x) \cos nx \, dx = 0$

Proof : (i) Since f is R-integrable in $[a,b]$, given $\epsilon > 0$, there is a Partition P of $[a,b]$ such that $U(p,f) - L(P,f) < \frac{\varepsilon}{2}$ (symbols have usual meanings). Let

$P = (a = x_0 < x_1 < x_2 < \ldots < x_{r-1} < x_r < \ldots < x_n = b)$.

Now $\int_a^b f(x)\sin nx\,dx = \int_{x_0}^{x_1} f(x)\sin nx\,dx + \int_{x_1}^{x_2} f(x)\sin nx\,dx + \ldots + \int_{x_{n-1}}^{x_n = b} f(x)\sin nx\,dx$

$$= \sum_{r=1}^{n} f(x_{r-1})\int_{x_{r-1}}^{x_r} \sin nx + \sum_{r=1}^{n}\int_a^b (f(x) - f(x_{r-1}))\sin nx\,dx$$

Therefore

$$\left|\int_a^b f(x)\sin nx\,dx\right| \le \sum_{r=1}^{n} |f(x_{r-1})|\left|\int_{x_{r-1}}^{x_r} \sin nx\,dx\right| + \sum_{r=1}^{n}\left|\int_{x_{r-1}}^{x_r} (f(x) = -f(x_{r-1}))\sin nx\,dx\right|$$

Now $x \in [x_{r-1}, x_r]$ implies $|f(x) - f(x_{r+1})| \le M_r - m_r$.

Therefore $|(f(x) - f(x_{r-1}))\sin nx| \le M_r\ m_r$, and

$$\left|\int_{x_{r-1}}^{x_r} \sin nx\,dx\right| \le \frac{1}{n}\{|\cos nx_r| + |\cos nx_{r-1}|\} \le \frac{2}{n}.$$

Therefore, $\left|\int_a^b f(x)\sin nx\,dx\right| \le \frac{2}{n}\sum_{r=1}^{n} |f(x_{r-1})| + \sum_{r=1}^{n}(M_r - m_r)(x_r - x_{r-1})$

$< \frac{2}{n}\sum_{r=1}^{n} |f(x_{r-1})| + \varepsilon/2$ (from above argument).

Now with this partition P, we find an index N such that

$< \frac{2}{n}\sum_{r=1}^{n} |f(x_{r-1})| + \varepsilon/2$ *for* $n \ge N$.

Therefore, $\left|\int_a^b f(x)\sin nx\,dx\right| < \varepsilon/2 + \varepsilon/2 = \varepsilon$ for $n \ge N$.

That means, $\lim_{n\to\infty}\int_a^b f(x)\sin nx\,dx=0$, and similarly one obtains

$$\lim_{n\to\infty}\int_a^b f\ \cos nx\cdot dx=0.$$

Important properties :

I Let $f:[-\pi,\pi]\to R$ be bounded and R-integrable and f be periodic with period

$=2\pi$. Then $\int_{-\pi}^{\pi} f(t)dt=\int_{-\pi}^{\pi} f(t+a)dt$ for any real a.

Proof : Here $\quad \int_{-\pi}^{\pi} f(t+\mathrm{a})dt=\int_{-\pi+\mathrm{a}}^{\pi+\mathrm{a}} f(u)du$

$$=\int_{-\pi+\mathrm{a}}^{-\pi} f(t)dt+\int_{-\pi}^{\pi} f(t)dt+\int_{\pi}^{\pi+\mathrm{a}} f(t)dt$$

$$=\int_{-\pi+\mathrm{a}}^{-\pi} f(u+2\pi)du \text{ (by periodicity of } f)+\int_{-\pi}^{\pi} f(t)dt+\int_{\pi}^{\pi+\mathrm{a}} f(t)dt$$

$$=\int_{-\pi+\mathrm{a}+2\pi}^{-\pi+2\pi} f(t)dt+\int_{-\pi}^{\pi} f(t)dt+\int_{\pi}^{\pi+\mathrm{a}} f(t)dt$$

$$=\int_{\pi+a}^{\pi} f(t)dt+\int_{-\pi}^{\pi} f(t)dt+\int_{\pi}^{\pi+\mathrm{a}} f(t)dt=\int_{-\pi}^{\pi} f(t)dt.$$

II. If $f:[-\pi,\pi]\to R$ be a bounded and R-integrable and f is periodic with period $=2\pi$; and if f is $\uparrow$ in (a,α); $0<\alpha<a$, then $\lim_{n\to\infty}\int_0^a f(t)\frac{\sin nt}{t}dt.\frac{\pi}{2}f(0+)$.

Proof: Since f is $\uparrow$ in a right-interval $(0,\alpha)(0<\alpha<a)$, we know that $f(0+)$ exits. **We assume f(0+) = f(0).**

Now take $g(t) = f(t) - f(0+)$ in $0<h<\alpha$. By second Mean-Value Theorem we find h_1

h_1 between 0 and h such that

$$\int_0^h f(t)\frac{\sin nt}{t}\,dt = g(0)\int_0^{h1}\frac{\sin nt}{t}dt + g(h)\int_{h1}^{h}\frac{\sin nt}{t}dt.$$

$$= g(h)\int_{nh1}^{nh2}\frac{\sin u}{u}dt.$$

Now $\int_0^\infty \frac{\sin t}{t}dt$ is convergent, and there is a +ve K. such that

$$\left|\int_0^y \frac{\sin t}{t}\right| < \text{K for large } y. \qquad \text{..} \qquad (1)$$

Here taking n sufficiently large, we have

$$\left|\int_{nh1}^{nh}\frac{\sin t}{t}\right| = \left|\int_0^{nh}\frac{\sin t}{t}dt - \int_0^{nh1}\frac{\sin t}{t}dt\right| < 2\text{K} \qquad \text{..} \qquad (2)$$

As $g(h)\to 0$ as $h\to 0_+$, we find a + $ve\delta$ so that $|g(h)| < \varepsilon$ when $0<h<\delta$.

Thus $\left|\int_0^h g(t)\frac{\sin nt}{t}dt\right| < 2\text{K}\varepsilon$ when $0<h<\delta$ (from (1) (2)). .. (3)

Now $\int_0^a g(t)\frac{\sin nt}{t}dt = \int_0^\delta g(t)\frac{\sin nt}{t} + \int_\delta^a g(t)\frac{\sin nt}{t}dt$

As $\frac{g(t)}{t}$ is integrable in $[\delta, a]$, the last integral $\to 0$ as $n\to\infty$.

Therefore in view of (3) we obtain

$$\lim_{n\to\infty}\int_0^a g(t)\frac{\sin nt}{t}dt = 0$$

or $\lim_{n\to\infty}\int_0^a g(t)\frac{\sin nt}{t}dt = 0$ (4)

Since $\lim_{n\to\infty}\int_0^a \frac{\sin nt}{t}dt = \lim_{n\to\infty}\int_0^{na}\frac{\sin u}{u}du = \int_0^\infty \frac{\sin u}{u}du = \frac{\pi}{2}$,

We obtain from (4), $\lim_{n\to a}\int_0^a f(t)\frac{\sin nt}{t}dt = \lim_{n\to a}\int_0^a f(0+)\frac{\sin nt}{t}dt$

$$= f(0+).\frac{\pi}{2}$$

$$= \frac{\pi}{2}f(0+).$$

(III) Let f be as in II above, and $0 < a < \pi$, then

$\lim_{n\to\infty}\int_0^a f(t)\frac{\sin nt}{\sin t}dt = \frac{\pi}{2}f(0+).$

Proof : Write $f(t)\frac{\sin nt}{\sin t} = f(t)\frac{t}{\sin t}.\frac{\sin nt}{t}$

We know that $\frac{t}{\sin t}$ is +ve and ↑ in $(0,\alpha]$ $\left(\alpha < \frac{\pi}{2}\right)$,

So if f is ↑ in $(0,\alpha)$, We have $f(t).\frac{t}{\sin t}$ as bounded integrable and ↑ in $(0,\alpha]$,

we have $f(t).\frac{t}{\sin t}$ as bounded integrable and ↑ in $(0,\alpha]$.

So, as $t\to 0+$, $f(t)\to f(0+)$ implies $f(t).\frac{t}{\sin t}\to f(0+)$.

Therefore, $\lim_{n\to\infty}\int_0^a f(t)\frac{\sin nt}{\sin t}dt = \lim_{n\to\infty}\int_0^a f(t)\frac{t}{\sin t}.\frac{\sin nt}{t}dt$.

$$= f(0+).\ \frac{\pi}{2}.\text{(sec II)}.$$

Remark : In case f is $\downarrow$, then $-f$ is $\uparrow$ and then

$$\lim_{n\to\infty}\int_0^{\pi}(-1)f(t)\frac{\sin nt}{\sin t}\,dt = (-1).\,f(0+)\frac{\pi}{2}.$$

4.6 Convergence Theorem.

Theorem 4.6.1 Let $f:[-\pi,\pi]\to R$ be bounded and there be a Partition P in $[-\pi,\pi]$ so that f is monotonic in each of finite number of open sub-intervals (due to P) of $[-\pi,\pi]$, the sum of Fourier series generated by f in $[-\pi,\pi]$ is given by

$$\frac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)=\frac{1}{2}\{f(x-0)+f(x+0)\}\text{ if } x\in(-\pi,\pi)$$

$$=\frac{1}{2}\{f(\pi-0)+f(-\pi+0)\}\text{ if } x=\pm\pi.$$

Proof : We assume that f is periodic with period $= 2\pi$; so $f(x+2\pi)=f(x)$ outside $[-\pi,\pi]$. By monotonic character *of f*, we know that $f(x\pm 0)$ exist, for $x\in(-\pi,\pi)$.

Now $S_m=\frac{1}{2}a_0+\sum_{n=1}^{m}(a_n\cos nx+b_n\sin nx)=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(t)dt+\sum_{n=1}^{m}\frac{1}{\pi}\int_{-\pi}^{\pi}f(t)\cos n(t-x)dt.$

$$=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(t)\left\{1+2\sum_{n=1}^{m}\cos n(t-x)\right\}dt$$

$$=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(t+x)\left\{1+2\sum_{n=1}^{m}\cos nt\right\}dt \text{ (using property I).}$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t+x) \frac{\sin\left(m+\frac{1}{2}\right)t}{\sin\frac{t}{2}} dt.$$

$$= \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(2t+x) \frac{\sin(2m+1)t}{\sin t} dt.$$

$$= \frac{1}{\pi} \int_{0}^{\frac{\pi}{2}} f(-2t+x) \frac{\sin(2m+1)t}{\sin t} dt + \frac{1}{\pi} \int_{0}^{\frac{\pi}{2}} f(2t+x) \frac{\sin(2m+1)t}{\sin t} dt \ldots \qquad (*)$$

$$= \frac{1}{\pi} f(x-0).\frac{\pi}{2} + \frac{1}{\pi}.f(x+0).\frac{\pi}{2} \text{ as } m \to \infty \text{ by property III.}$$

$$= \frac{1}{2} \{ f(x-0) + f(x+0) \}$$

Therefore, we have $\frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$

$$= \frac{1}{2} \{ f(x-0) + f(x+0) \} \text{ if } x \in (-\pi, \pi)$$

$$= \frac{1}{2} \{ f(\pi-0) + f(-\pi+0) \} \text{ if } x = \pm\pi.$$

$(f(\pi+0) = f(-\pi+0))$, and $(f(-\pi-0) = f(\pi-0))$.

Remark. From (*) above it is evident that *a* necessary and sufficient condition for convergence of the Fourier series at x is that

$\lim_{n \to \infty} \int_{0}^{\frac{\pi}{2}} \{f(x-2t) + f(x+2t)\} \frac{\sin(2n+1)t}{\sin t} dt$ exits as a finite quantity which shall be the sum of Fourier series at x. The above integral, that is, $\int_{0}^{\frac{\pi}{2}} \{f(x-2t) + f(x+2t)\} \frac{\sin(2n+1)t}{\sin t} dt$ is popularly known as Dirichlets integral.

Remark 2. At a point x where f is continuous, we have

$$\frac{1}{2}(f(x-0)+f(x+0)=f(x)\ (x\neq\pi).$$

So in addition to stipulation as in Theorem 4.6.1 if f is continuous at all points of $(-\pi,\pi)$, its Fourier series converges with its sum $-f(x)$ at all points x. At a point x of discontinuity of f however the sum $=\frac{1}{2}(f(x-0)+f(x+0)$ as shown above.

Theorem 4.6.2 If $f:[-\pi,\pi]$ is bounded and integrable and is ↑ in $(-\alpha,0)$ and in $(0,\alpha)$ for $0<\alpha<\pi$. Then $\frac{1}{2}a_0+\sum_{n=1}^{\infty}a_n=\frac{f(0-)+(0+)}{2}$

Proof : As f is ↑ in $(-\alpha,0)$ and in $(0,\alpha)$ $(0<\alpha<\pi)$, therefore $\lim_{x\to 0+} f(x)$ exists *i.e.*, $f(0+)$ and similarly $f(0-)$ exist finitely.

$$\text{Now } \frac{1}{2}a_0+\sum_{n=1}^{m}a_n=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(t)\mathrm{dt}+\frac{1}{\pi}\int_{-\pi}^{\pi}f(t)\left(\sum_{n=1}^{m}\cos nt\right)dt$$

$$=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(t)\left(1+2\sum_{n=1}^{m}\cos nt\right)dt=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(t).\frac{\sin\left(m+\frac{1}{2}\right)t}{\sin\frac{1}{2}t}dt$$

$$=\frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\mathrm{f}(2t).\frac{\sin(2m+1)t}{\sin t}dt$$

$$=\frac{1}{\pi}\int_{0}^{\frac{\pi}{2}}f(-2t).\frac{\sin(2m+1)t}{\sin t}dt+\frac{1}{\pi}\int_{0}^{\frac{\pi}{2}}f(2t).\frac{\sin(2m+1)t}{\sin t}dt$$

$$\to \frac{1}{\pi}.f(0-)\frac{\pi}{2}+\frac{1}{\pi}f(0+).\frac{\pi}{2} \text{ as } m\to\infty$$

$$= \frac{1}{2}\ (f(0-)+f(0+1)).$$

Example 4.6.1 If $f : [-\pi,\pi]\to R$ is bounded and R-integrable, and $\{a_n\},\{b_n\}$ are Fourier coefficients of the Fourier series corresponding to f, then $\sum_{n=1}^{\infty}(a_n{}^2+b_n{}^2) < +\infty$.

Solution : We have

$$\int_{-\pi}^{\pi}\left\{f(t)-\sum_{n=1}^{m}(a_n\cos nt+b_n\sin nt)\right\}^2 dt \geq 0 \text{ always.}$$

$$\text{L.H.S.} = \int_{-\pi}^{\pi}f^2(t)dt-2\sum_{n=1}^{m}\left\{a_n\int_{-\pi}^{\pi}f(t)\cos nt\ dt\right\}-2\sum_{n=1}^{m}\left\{b_n\int_{-\pi}^{\pi}f(t)\sin nt\ dt\right\}$$

$$+\int_{-\pi}^{\pi}\{\Sigma(a_n\cos nt+b_n\sin nt)\}^2 dt$$

$$=\int_{-\pi}^{\pi}f^2(t)dt-2\pi\sum_{n=1}^{m}a_n^2-2\pi\sum_{n=1}^{m}b_n^2+\pi\sum_{n=1}^{m}a_n^2+\pi\sum_{n=1}^{m}b_n^2$$

$$=\int_{-\pi}^{\pi}f^2(t)dt-\pi\sum_{n=1}^{m}\left(a_n^2+b_n^2\right)$$

(We have used $\int_{-\pi}^{\pi} \sin pt \cos qt\ dt = 0$ if $p\neq q$,

$$\int_{-\pi}^{\pi}\cos pt\ \cos qt\ dt = 0 \text{ if } p\neq q$$

$$= \pi \text{ if } p = q$$

and $\int_{-\pi}^{\pi} \sin pt \sin qt \, dt = 0$ if $p \neq q$

$= \pi$ if $p = q$.)

This gives $\sum_{n=1}^{m}\left(a_n^2 + b_n^2\right) \leq \frac{1}{\pi}\int_{-\pi}^{\pi} f^2(t)dt < +\infty$ ... (*)

R.H.S being independent of m, it follows that $\sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right) < +\infty$

Note : The inequality (*) is known as Bessels inequality.

Corollary : If $\{a_n\}$ and $\{b_n\}$ are Fourier coefficients of a fourier series, then $\lim_{n \to \infty} a_n = 0$ $\lim_{n \to \infty} b_n$ independent of the Fourier series being convergent or not.

Example 4.6.2 Show that $\sum_{n=1}^{\infty}\frac{\cos x}{\sqrt{n}}$ is not Fourier series.

Solution : Here $a_n = \frac{1}{\sqrt{n}}$, and $b_n = 0$ and $\sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right) = \sum_{n=1}^{\infty}\frac{1}{n}$

which is a divergent series. Hence $\sum_{n=1}^{\infty}\frac{\cos nx}{\sqrt{n}}$ is not a Fourier series corresponding to any admissible function.

Example 4.6.3 Obtain the Fourier series of sines and cosines of multiple of x for $f(x)$ given by

$f(x) = x - \pi$ in $-\pi < x < 0$

$- \pi x$ in $0 < x < \pi$.

Hence Verify that $\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots$

Solution : Here if $f \sim \frac{1}{\pi} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$

we have $\frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos nt \, dt = \frac{1}{\pi} \left[\int_{-\pi}^{0} (t - \pi) \cos nt \, dt + \int_0^{\pi} (\pi - t) \cos nt \, dt \right]$

$$= \frac{1}{\pi} \int_0^{\pi} (-t - \pi + \pi - t) \cos nt \, dt = -\frac{2}{\pi} \int_0^{\pi} t \cos nt \cdot dt$$

$$= -\frac{2}{\pi} \left\{ \frac{t \sin nt}{n} + \frac{\cos nt}{n2} \right\}_0^{\pi}$$

$$= \frac{2}{\pi} \cdot \frac{1 - (-1)^n}{n^2}; \; n = 1, 2, 3, \dots$$

$\therefore a_0$ (for $n = 0$) $= -\frac{2}{\pi} \int_0^{\pi} t dt = -\pi$.

Again $b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f \sin nt \, dt = \frac{1}{\pi} \left[\int_{-\pi}^{0} (t - \pi) \sin nt \, dt + \int_0^{\pi} (\pi - t) \sin nt \, dt \right]$

$$= \frac{1}{\pi} \int_0^{\pi} (\pi + t + \pi - t) \sin nt \, dt = 2 . \frac{1 - (-1)^n}{n}$$

Thus Fourier series corresponding to f is

$$-\frac{\pi}{2}+\frac{4}{\pi}\sum_{m=1}^{\infty}\left\{\frac{1}{(2m-1)^2}\cos(2m-1)x+\frac{\pi}{2m-1}.\sin(2m-1)x\right\} \quad \cdots \quad (*)$$

because n being even, $n = 2n$ gives $1-(-1)^{2m}=0$

n being odd, $n = 2m-1$ gives $1-(-1)^n = 2$

We see further that f is monotone in $(-\alpha,0)$ and in $(0,\alpha)$ $0<\alpha<\pi$ and hence $f(0-)$ and $f(0+)$ exist; and Convergence Theorem says that Fourier series (*) has the sum

$=\frac{1}{2}[f(0-)+f(0+)]=\frac{1}{2}(-\pi+\pi)=0$. So putting $x = 0$ in (*) we obtain

$$\frac{-\pi}{2}+\frac{4}{\pi}\sum_{m=1}^{\infty}\frac{1}{(2m-1)^2}=0$$

or $$\sum_{m=1}^{\infty}\frac{1}{(2m-1)^2}=\frac{\pi^2}{8}$$

i.e., $1+\frac{1}{3^2}+\frac{1}{5^2}+\ldots=\frac{\pi^2}{8}$.

Theorem 4.6.3 If the Trigonometric series

$\frac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)$ (a_n, b_n's are scalars) is uniformly convergent in $[-\pi,\pi]$ then it is a Fourier series.

Proof : Let $\frac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)$ be uniformly convergent and let $\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) = f(x)$ in $[-\pi,\pi]$.

Then $\frac{1}{2}a_0 \cos mx + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)\cos mx$ converges uniformly to $f(x)\cos mx$ $(m = 0,1,2,...)$ in $[-\pi,\pi]$. Now term by term integration gives

$$\int_{-\pi}^{\pi} a_0 \cos mx \, dx = \frac{1}{2}\int_{-\pi}^{\pi} a_0 \cos mx \, dx = \pi a_0 \text{ when } m=0$$

giving $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx.$

and $\int_{-\pi}^{\pi} f(x)\cos mx \, dx = \int_{-\pi}^{\pi} a_m\left(\frac{1+\cos 2mx}{2}\right)dx$

$$= \pi a_m \text{ when } m = 1,2,...$$

giving $a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos mx \, dx.$ $(m = 1,2,...)$

Similarly, $\int_{-\pi}^{\pi} f(x)\sin mx \, dx = \int_{-\pi}^{\pi} b_m\left(\frac{1-\cos 2mx}{2} dx\right) = \pi b_m$

giving $b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos mx \, dx.$ $(m = 1,2,...)$

So given Trigonometric series is a Fourier series.

Example 4.6.4 Verify that $\sum_{n=1}^{\infty}\frac{\sin nx}{n^2}$ is a Fourier series

Solution : Here, by M-test we see that the series $\sum_{n=1}^{\infty}\frac{\sin nx}{n^2}$ is uniformly convergent in $[-\pi,\pi]$ since $\left|\frac{\sin nx}{n^2}\right| \leq \frac{1}{n^2}$ for all $x \in[-\pi,\pi]$ and $\sum_{n=1}^{\infty}\frac{1}{n_2}$ is convergent. Therefore given Trigonometric series is a Fourier series.

Fourier series for even and odd functions

Let f be an even function in $[-\pi,\pi]$, then $f(-t) = f(t)$ in $[-\pi,\pi]$.

So here $f(t)\cos nt$ becomes an even functions and $f(t)\sin nt$ becomes an odd function in $[-\pi,\pi]$. So that $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\ \sin\ nt\ dt = 0$ for $n = 1,2,...$

and $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos nt\ \ dt = \frac{2}{\pi}\int_{0}^{\pi} f(t)\cos nt\ dt$, for $n = 1,2,...$

Similarly if f is an odd function in $[-\pi,\pi]$, then $f(t)\cos nt$ becomes an odd function and $f(t)\sin nt$ becomes an even function in $[-\pi,\pi]$; so that

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos nt\ dt = 0,\ n=1,2,...$$

and $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos nt\ dt = \frac{2}{\pi}\int_{0}^{\pi} f(t)\sin nt\ dt, n=1,2,...$

Example 4.6.5 : Obtain Fourier series generated by $f(x) = |x|$ in $|x|$ in $[-\pi,\pi]$. Is the series convergent at $x = 0$? If so, compute the sum at $x = 0$.

Solution : Here $f(x) = -x$ in $-\pi \leq x < 0$

$= x$ in $0 \leq x \leq \pi$.

The function f is monotonic in $(-\pi,0)$ and in $(0,\pi)$. Also it is an even function in $[-\pi,\pi]$; therefore its Fourier series consists of cosine terms only.

Now $a_0 = \frac{2}{\pi}\int_0^\pi x\,dx = \pi$

and $a_n = \frac{2}{\pi}\int_0^\pi x\cos nx\,dx = \frac{2}{\pi n^2}(\cos n\pi - 1)$.

$$= \begin{cases} 0 & \text{if } n \text{ is even} \\ -\frac{4}{\pi n^2} & \text{if } n \text{ is odd} \end{cases}$$

Therefore the Fourier series for f is

$$\frac{\pi}{2} - \frac{4}{\pi}\left(\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ldots\right) \quad \ldots \quad (*)$$

Since f is monotone in $(-\pi, 0)$ and in $(0, \pi)$ we have $f(0-) = 0 = f(0+)$. Thus by convergence Theorem Fourier series for f converges to $\frac{1}{2}[f(0-) + f(0+)] = 0$ at $x = 0$. Hence putting $x = 0$ in (*) we find the sum of the series = 0.

Exercise

Short answer type questions;

1. If f is an add function show that $\int_{-\pi}^{\pi} f dx = 0$

2. Examine if $\sum_{n=1}^{\infty} \frac{\cos nx}{\sqrt{n}}$ is a Fourier series in $[-\pi, \pi]$ with reasons.

3. Examine if $\sum_{n=1}^{\infty}(\sin nx + \cos nx)$ is a Fourier series in $[-\pi,\pi]$ with reasons.

4. Obtain Fourier series for $f(x) = 0$ in $-\pi \leq x \leq \pi$.

5. Show by Definition thatR-S integral $\int_0^2 x^2 d[x]=5$. where $[x]$ denotes the largest integer not larger than x.

6. If f and g are R-S integrable over $[a,b]$ w.r.t α. Then $f \pm g$ are R-S integrable over $[a,b]$ w.r.t α. Prove it.

7. Show that R–S $\int_0^t d[x]=[t]$ for any real t where $[x]$ denotes the largest integer not larger than x.

8. If f is continuous in $[a,b]$, and $g(x) = 0$ at $x=a$ and $g(x) = 1$ when $x>a$, show that R–S $\int_a^b f dg = f(a)(g(b)-g(a))$.

Broad Questions

1. If $f(x) = x$ when x is an irrational

$= 0$ when x is a rational,

and $g(x) = 0$ when $x<0$

$= \frac{1}{2}$ when $x = 0$

$= 1$ when $x > 0$.

Show that f is R-S integrable over $[-1,+1]$ $w.r.t$ g and R-S $\int_{-1}^{1} f dg = 0$.

Is f Riemann integrable over $[-1,+1]$? Give reasons.

2. If f is continuous in $[a,b]$ and g is a function of bounded variation in $[a,b]$, show that the Function $F(x) = \int_a^x f dg$ is a function of bounded variation in $[a,b]$.

3. If $[x]$ is the largest integer not larger than x,

(i) Show that $\int_0^4 x d([x]-x) = 2$.

(ii) $\int_a^1 d(e^{2x}) = \frac{1+e^2}{2}$

4. If f is continuous in $[a,b]$ and $g:[a,b] \to R$ has a g' which is R-integrable over $[a,b]$, prove that R-S$\int_a^b f\, dg$ = R - $\int_a^b f g'\, dx$

5. Find the Fourier series for f in $[-\pi,\pi]$ Given that

$f(x) = x \quad$ in $-\pi<x<0$

$= \pi-x$ in $0<x<\pi$.

Examine sum of the series at $x = 0$ and π with reasons.

6. Show that Fourier series for $f(x) = x + x^2$ in $[-\pi,\pi]$

is $\frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}(-1)^n\left(\frac{\cos nx}{n^2} - \frac{\sin nx}{2n}\right)$

What shall be the sum of the series at $x = + \pi$? Give reasons. Deduce that

$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots$$

7. Obtain the Fourier series for cosK x in $[-\pi,\pi]$ and deduce therefrom

$$\pi \cot K\pi = \frac{1}{K} + \frac{2K}{K^2-1^2} + \frac{2k}{K_2-2^2} + \ldots \quad (K \neq \text{an integer}).$$

Unit 5 ❑ Metric Space, Opensets, Closed Sets and Algebra, Closure, Interior and Boundary of Sets

(**Metric spaces,** examples, open balls, closed balls, open sets, metric topology, neighbourhoods of a point. Hausdorff property, limit point of a set, closed set, algebra of open and closed sets, closure, Interior and **boundary of a set)**

§ 1.1

Let X be a non-empty set. Then Cartesian product of X with itself, denoted by X × X is also a non-empty set consisting of all ordered pairs (x,y) of points as $x,y \in X$. For example if $X = \{a,b,c\}$,

Then $X \times X = \{(a,a),(a,b),(a,c),(b,a),(b,c),(c,a)(c,b),(b,b)$ and $(c,c)\}$. Here ordered pair (a,b) is not equal to ordered pair (b,a) unless $a = b$.

In the Cartesian product X × X two members (x,y) and (x,y) are equal *i.e.,* $(x,y) = (u,v)$ if and only if $x = u$ ***and*** $y = v$.

***Example* 1.1.1** Let R denote the set of all real numbers, then $R \times R = \{(a,b) : a,b \in R\}$; and geometrically R×R represents Euclidean plane R^2 where each point is represented by an ordered pair (a,b) *i,e.,* by a member of R × R and vice-versa.

Definition 1.1.1 A function $d : X \times X \to R$ is called a metric or a distance function over X if it satisfies following conditions, known as metric or distance axioms :

(M.1) $d(x,y) \geq 0$ for all $x,y \in X$, and $d(x,y) = 0$ if and only if $x = y$. (Property of non-negativity),

(M. 2) $d(x,y) = d(y,x)$ for all $x,y \in X$ (Property of symmetry),

and (M. 3) $d(x,z) \leq d(x,y)+(y,z)$ for all x,y and $z \in X$ (property of triangle inequality).

If d is a metric on X, the pair (X, d) is called a metric space. Indefinite article 'a' tells us that X may invite many metrics, and accordingly with same X, one may have many metric spaces (X, d) as d changes. For example, given a metric space (X, d) one may replace d by $\frac{d}{2}, \frac{d}{3}, \cdots$ and obtains metric spaces $\left(X, \frac{1}{k}d\right)$ for natural number k = 2, ...

Example 1.1.2 The set R of all reals is a metric space (R, d) where $d(x,y) = |x-y|$, as $x,y \in$ R.

Solution : Here property of absolute value $|x|$, of real number x gives (M.1) and for (M.2) we have $d(y,x) = |y-x| = |(-1)(x-y)| = |(-1)(x-y)| = |x-y| = d(x,y)$ and triangle-inequality of d is a consequence of similar property of $|x|$ for any real x. So, (R,d) is a metric space.

Example 1.1.3 If X is any non-empty set, there is always a metric d, called the trivial metric over X, so that (X,d) is a metric space where we take,

$$d(x,y) = \begin{cases} 1, \text{if } x \neq y \text{ in X} \\ 0, \text{if } x = y \text{ in X.} \end{cases}$$

Example 1.1.4 The set ¢ of all complex number $z = x + iy$ (x, y are reals and $i^2 = -1$) is a metric space (¢, d) where $d(z,z') = |z-z'|$ as, $z,z' \in$ ¢.

Example 1.1.5 The Cartesian product R×R×...×R (n times) = R^n of the set R of all reals with itself, called the Euclidean n-space is a metric space (R^n,d) where

$$d\left(\underset{\sim}{x}, \underset{\sim}{y}\right) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \ldots + (x_n - y_n)^2} \text{ as } \underset{\sim}{x} = (x_1, x_2, \ldots, x_n)$$

and $\underset{\sim}{y} = (y_1, y_2, \ldots, y_n)$ ordered n-typles of reals are any two members of R^n.

Solution : Here (M.1) is easy to check; remembering that $\sum_{i=0}^{n}(x_i - y_i)^2 = 0$ if and only if $x_i = y_i$ for $i = 1,2,...,n$. Symmetry property (M.2) is clear. To derive triangular inequality (M.3) we employ following important inequality, known as Cauchy-Schwarz inequality (C-S inequality).

C-S inequality : For any two sets $(a_1, a_2, ..., a_n)$ and $(b_1, b_2, ..., b_n)$ of n reals,

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \sum_{i=1}^{n} a_i^2 . \sum_{i=1}^{n} b_i^2 .$$

This C-S inequality gives $\left(\sum_{i=1}^{n}(a_i b_i)^2\right)^{\frac{1}{2}} \leq \left(\sum_{i=1}^{n} a_i^2\right)^{\frac{1}{2}} + \left(\sum_{i=1}^{n} b_i^2\right)^{\frac{1}{2}}$,

known as Minkowski inequality as a special case.

Now let $\underset{\sim}{x} = (x_1, x_2, ..., x_n)$, $\underset{\sim}{y} = (y_1, y_2, ..., y_n)$ and $\underset{\sim}{z} = (z_1, z_2, ..., z_n) \in \mathrm{R}^n$.

Put $a_i = x_i - y_i$, $b_i = y_i - z_i$ $(i = 1,2,..., n)$. We apply Minkowski inequality in a form as above and obtain.

$$\sqrt{\sum_{i=1}^{n}(x_i - z_i)^2} \leq \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2} + \sqrt{\sum_{i=1}^{n}(y_i - z_i)^2} .$$

That means, $d\left(\underset{\sim}{x}, \underset{\sim}{z}\right) \leq d\left(\underset{\sim}{x}, \underset{\sim}{y}\right) + d\left(\underset{\sim}{y}, \underset{\sim}{z}\right)$, and this is (M.3).

Remark : For $p \geq 1$, one can show that (R^n, d_p) are all metric spaces where

$$d_p\left(\underset{\sim}{x}, \underset{\sim}{y}\right) = \left(\sum_{i=1}^{n} |x_i - y_i|^p\right)^{\frac{1}{p}}$$ as $\underset{\sim}{x} = (x_1, x_2, \ldots, x_n)$ and $\underset{\sim}{y} = (y_1, y_2, \ldots, y_n) \in R^n$. For $p = 2$, the corresponding metric for R^n as in **Example 1.1.6** is called the usual or Euclidean Metric for R^n.

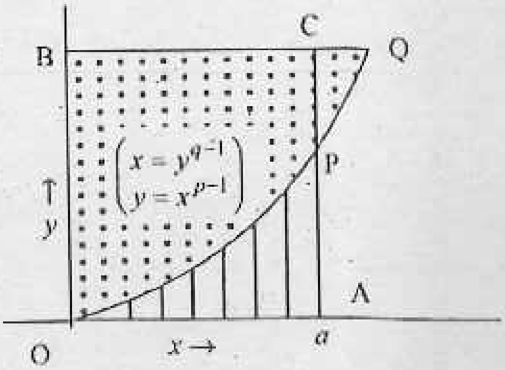
Sequence spaces ℓ_p (p>1), Hilbert space ℓ_2; Holder's and Minkowski's inequality.

Let p be a real number > 1, and take q to satisfy $\frac{1}{p} + \frac{1}{q} = 1$. So, $q > 1$, *i.e.*, $\frac{p+q}{pq} = 1$ or, $(p-1)(q-1) = 1$ or $\frac{1}{p-1} = q-1$. Then p and q are called conjugate indices to each other. Thus $y = x^{p-1}$ implies $x = y^{q-1}$.

The Collection $\ell_p = \left\{\underset{\sim}{\xi} : \underset{\sim}{\xi} = (\xi_1, \xi_2, \ldots)\right\}$, ξ_i are real / complex with $\left(\sum_{i=1}^{\infty} |\xi_i|^p\right)^{\frac{1}{p}} < \infty$ is known as real/complex sequence space with metric $\rho\left(\underset{\sim}{\xi}, \underset{\sim}{\eta}\right) = \left(\sum_{i=1}^{\infty} |\xi_i - y_i|^p\right)^{\frac{1}{p}}$, as $\underset{\sim}{\xi} = (\xi_1, \xi_2, \ldots)$ and $\eta = (\eta_1, \eta_2, \ldots) \in \ell_p$. We show at present that ℓ_p is a metric space. For $p = 2$ sequence space ℓ_2 is known as a Hilbert space. For this purpose we need applying Hölder's and Minkowski's inequality.

Hölder inequality : Let $\underset{\sim}{\xi} = (\xi 1\ \xi 2 \ldots)$ and $\underset{\sim}{\eta} = (\eta 1, \eta 2) \in \ell_p$ be members of ℓ_p and ℓ_q respectively (p and q are conjugate indices). Then

$$\sum_{i=1}^{\infty}|\xi_i\eta_i| \le \left(\sum_{i=1}^{\infty}|\xi_i|^p\right)^{\frac{1}{p}}\left(\sum_{i=1}^{\infty}|\eta_i|^q\right)^{\frac{1}{q}}.$$



Proof : If a and b are two + ve reals, then ab equals to area of a rectangle OACB (See fig), and we see at once that Area OACB ≤ area OAPO (as bounded by $y = x^{p-1}$, x-axis etc.) + area OPQCBO (as bounded by $x = y^{q-1}$, Y-axis etc.)

$$\text{or, } ab \int_0^a x^{p-1}dx + \int_0^b y^{q-1}dy = \frac{a^p}{p} + \frac{b^q}{q} \qquad \ldots \qquad (1)$$

First Let $\sum_{i=1}^{\infty}|\xi_i'|^p = 1 = \sum_{i=1}^{\infty}|\eta_i'|^q$; Taking $|\xi_i'|$ and $|\eta_i'|$ for a and b respectively in (1)

we have $|\xi_i'\eta_i'| \le \frac{1}{p}|\xi_i'|^p + \frac{1}{q}|n_i'|^q$ and hence, $\sum_{i=1}^{\infty}|\xi_i'\eta_i'| \le \frac{1}{p}\sum_{i=1}^{\infty}|\xi_i'|^p + \frac{1}{q}\sum_{i=1}^{\infty}\eta_i'|^q$ (since r.h.s.

series are convergent.) $= \frac{1}{p} + \frac{1}{q} = 1$ by assumption.

$$\text{or, } \sum_{i=1}^{\infty}|\xi_i'\eta_i'| \le 1 \qquad \ldots \qquad (2)$$

Now take $\underset{\sim}{\xi} = (\xi_1, \xi_2, \ldots.) \in \ell_p$ and $\underset{\sim}{\eta} = (\eta_1, \eta_2, \ldots) \in \ell_q$;

and put $\xi_i' = \dfrac{\xi_i}{\left(\sum_{i=1}^{\infty}|\xi_i|^p\right)^{\frac{1}{p}}}$ and

$$\eta_i' = \frac{\eta_i}{\left(\sum_{i=1}^{\infty}|\eta_i|^q\right)^{\frac{1}{q}}}.$$

So that, $\sum_{i=1}^{\infty}|\xi_i'|^p = 1 = \sum_{i=1}^{\infty}|\eta_i'|^q$ and from (2) we have,

$$\sum_{i=1}^{\infty}|\xi_i\eta_i| \le \left(\sum_{i=1}^{\infty}|\xi_i|^p\right)^{\frac{1}{p}} \cdot \left(\sum_{i=1}^{\infty}|\eta_i'|^q\right)^{\frac{1}{q}} \quad \ldots \quad (3)$$

This is Hölder's inequality.

Corollary : Cauchy-Schwarz (C-S) Inequality :

$$\sum_{i=1}^{\infty}|\xi_i\eta_i| \le \sqrt{\sum_{i=1}^{\infty}|\xi_i|^2} \cdot \sqrt{\sum_{i=1}^{\infty}|\eta_i|^2}$$

This follows from (3) by taking $p = 2$ (hence $q = 2$).

Minkowski's inequality : If $\underset{\sim}{\xi} = (\xi_1, \xi_2, \ldots)$ and $\underset{\sim}{\eta} = (\eta_1, \eta_2, \ldots)$ are two members of ℓ_p, then $\left(\sum_{i=1}^{\infty}|\xi_i + \eta_i|^p\right)^{\frac{1}{p}} \le \left(\sum_{i=1}^{\infty}|\xi_i|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{\infty}|\eta_i|^p\right)^{\frac{1}{p}}$.

Proof : $\sum_{i=1}^{\infty}|\xi_i + \eta_i| = \sum_{i=1}^{\infty}|\xi_i + \eta_i||\xi_i + \eta_i|^{p-1}$

$$\le \sum_{i=1}^{\infty}|\xi_i||\xi_i + \eta_i|^{p-1} + \sum_{i=1}^{\infty}|\eta_i||\xi_i + \eta_i|^{p-1}$$

$$\le \left(\sum_{i=1}^{\infty}|\xi_i|^p\right) \cdot \left(\sum_{i=1}^{\infty}|\xi_i + \eta_i|^{(p-1)q}\right)^{\frac{1}{q}} + \left(\sum_{i=1}^{\infty}|\eta_i|^p\right)^{\frac{1}{p}} \cdot \left(\sum_{i=1}^{\infty}|\xi_i + \eta_i|^{(p-1)q}\right)^{\frac{1}{q}}$$

by holder's inequality.

$$=\left(\sum_{i=1}^{\infty}|\xi_i+\eta_i|^{(p-1)q}\right)^{\frac{1}{q}}\left(\left(\sum_{i=1}^{\infty}|\xi_i|^p\right)^{\frac{1}{p}}+\left(\sum_{i=1}^{\infty}|\eta_i|^p\right)^{\frac{1}{p}}\right)$$

$$=\left(\sum_{i=1}^{\infty}|\xi i+\eta_i|^p\right)^{\frac{1}{q}}\left(\left(\sum_{i=1}^{\infty}|\xi_i|^p\right)^{\frac{1}{p}}+\left(\sum_{i=1}^{\infty}|\eta_i|^p\right)^{\frac{1}{p}}\right)$$

$$\text{or, }\left(\sum_{i=1}^{\infty}|\xi_i+\eta_i|^p\right)^{1-\frac{1}{q}}\le\left(\sum_{i=1}^{\infty}|\xi_i|^p\right)^{\frac{1}{p}}+\left(\sum_{i=1}^{\infty}|\eta_i|^p\right)^{\frac{1}{p}}$$

$$\text{or, }\left(\sum_{i=1}^{\infty}|\xi_i+\eta_i|^p\right)^{\frac{1}{p}}\le\left(\sum_{i=1}^{\infty}|\xi_i|^p\right)^{\frac{1}{p}}+\left(\sum_{i=1}^{\infty}|\eta_i|^p\right)^{\frac{1}{p}}.$$

Example 1.1.6. Sequence-space l_p $(p\ge 1)$ is a metric space with respect to metric ρ

where $\rho\left(\underset{\sim}{\xi},\underset{\sim}{\eta}\right)=\left(\sum_{i=1}^{\infty}|\xi_i-\eta_i|^p\right)^{\frac{1}{p}}$, $\underset{\sim}{\xi}=(\xi_1,\xi_2,\ldots)$, $\underset{\sim}{\eta}=(\eta_1,\eta_2,\ldots)\in\ell_p$.

Solution : Here if $\underset{\sim}{\xi},\underset{\sim}{\eta}\in\ell_p$, we have $\rho\left(\underset{\sim}{\xi},\underset{\sim}{\eta}\right)=\left(\sum_{i=1}^{\infty}|\xi_i-\eta_i|^p\right)^{\frac{1}{p}}\ge 0$ always, and

$\rho\left(\underset{\sim}{\xi},\underset{\sim}{\eta}\right)=0$ if and only if $\xi_i\eta_i$ for all i *i.e.*, $\rho\underset{\sim}{\xi},\underset{\sim}{\eta}$. Thus (M.1) axiom is verified. Property of symmetry *i.e.*, (M.2) is also clear. For triangle inequality (M.3) we may assume $\rho>1$; because for $\rho=1$ the inequality is trivially clear. If $\underset{\sim}{\xi}=(\xi_1,\xi_2,\ldots)$,

$\underset{\sim}{\eta} = (\eta_1, \eta_2, \ldots)$ and $\underset{\sim}{\xi} = (\xi_1, \xi_2, \ldots)$ are three members of ℓ_p we have

$$\rho\left(\underset{\sim}{\xi}, \underset{\sim}{\xi}\right) = \left(\sum_{i=1}^{\infty} |\xi_i - \xi_i|^p\right)^{\frac{1}{p}}$$

$$= \left(\sum_{i=1}^{\infty} |\xi_i - \eta_i + \eta_i - \xi_i|^p\right)^{\frac{1}{p}} \leq \left(\sum_{i=1}^{\infty} |\xi_i - \eta_i|^p\right) + \left(\sum_{i=1}^{\infty} |\eta_i - \xi_i|^p\right)^{\frac{1}{p}}$$

by Minkowski inequality

$$= \rho\left(\underset{\sim}{\xi}, \underset{\sim}{\eta}\right) + \rho\left(\underset{\sim}{\eta}, \underset{\sim}{\xi}\right)$$

Example 1.1.7 The collection $C_R\ [a,b]$ of all real-valued continuous functions over a closed interval $[a,b]$ forms a metric space with metric ρ called supmetric where $\rho(f,g) = \sup_{a \le x \le b} |f(x) - g(x)|$, $f, g \in C_R\ [a,b]$.

Solution : We know that sum (and difference) of two real-valued continuous functions over $[a,b]$ is again a continuous function over $[a,b]$, and so is the case in respect of taking $|f|$ for a member $f \in C_R\ [a,b]$. Also by property of a continuous function over $[a,b]$ we have, $\rho(f,g)$ exixts as a finite ≥ 0 number and further $\rho(f,g) = 0$ if and only if $f = g$ in $C_R\ [a,b]$. Because, say $\rho(f,g)=0$.

So, $\sup_{a \le x \le b} |f(x) - g(x)| = 0$; giving $f(x) - g(x) = 0$ for all x in $[a,b]$ and hence $f(x) = g(x)$ in $a \le x \le b$ or $f = g$;

and (M.1) is verified. (M.2) is also clear. For (M.3), take f, g and h as any three members of $C_R\ [a,b]$ We have if $a \le x \le b$,

$|f-h|(x) = |f(x) - h(x)| = |f(x) - g(x) + g(x) - h(x)| \le |f(x) - g(x)| + |g(x) - h(x)|$

$\le \sup_{a\le x\le b} |f(x) - g(x)| + \sup_{a\le x\le b} |g(x) - h(x)| = \rho(f,g) + \rho(g,h)$; R.H.S. being independent of x in $[a,b]$ we obtain by taking sup on L.H.S. as $a \le x \le b$,

$\rho(f,h) \le \rho(f,g) + \rho(g,h)$, and verification of M(3) is complete.

Theorem 1.1.1 In metric space (X,d) if $x,y,z \in X$; then $|d(x,z) - d(y,z)| \le d(x,y)$.

Proof : By triangle inequality we have $d(x,z) \le d(x,y) + d(y,z)$,

So, $d(x,z) - d(y,z) \le d(x,y)$ ... (1.1.3)

Interchanging x and y we get,

$d(y,z) - d(x,z) \le d(y,x) = d(x,y)$

or, $-(d(x,z) - d(y,z)) \le d(x,y)$ ... (1.1.4)

Combining (1.1.3) and (1.1.4) we get $|d(x,z) - d(y,z)| \le d(x,y)$.

Theorem 1.1.2 The Certesian product $X = X_1 \times X_2$ where (X_1, d_1) and (X_2, d_2) are metric spaces is a metric space (X,d) where $d(x,y) = d_1(x_1,y_1) + d_2(x_2,y_2)$, where

$x = (x_1, x_2)$; $y = (y_1, y_2)$ are any two members of $X = X_1 \times X_2$.

Proof : (M.1) axiom for d follows from that of d_1 and d_2. Symmetry axiom (M.2) is also clear in respect of d.

For triangle inequality let $x = (x_1,x_2)$, $y = (y_1 y_2)$, $z = (z_1,z_2)$, be any three elements of X; Then we have, $d(x,z) = d_1(x_1,z_1) + d_2(x_2,z_2) \le d_1(x_1 y_1) + d_1(y_1 z_1) + d_2(x_2,y_2) + d(y_2,z_2)$ because metrics d_1 and d_2 satisfy triangle inequality.

or, $d(x,z) \le \{d_1(x_1,y_1) + d_2(x_2,y_2)\} + \{d_1(y_1,z_1) + d_2(y_2,z_2)\}$

$= d(x,y) + d(y,z)$ which is (M.3).

and the proof is complete.

Example 1.1.7 If (X,d) is a metric space, show that $\left(X, \frac{d}{1+d}\right)$ is a metric space.

Solution : Put $\rho(x,y) = \frac{d(x,y)}{1+d(x,y)}$ for $x, y \in X$. Metric axioms (M.I) and (M.2) in

favour of ρ follow from corresponding axioms for d. For triangle inequality for ρ let us consider a function $f : [0,\infty] \to$ Reals where $f(u) = \frac{u}{1+u}$ as $u \geq 0$. Then f is a differentiable function with derivative $f'(u) = \frac{1}{1+u} - \frac{u}{(1+u)^2} = \frac{1}{(1+u)^2} > 0$ for all $u \geq 0$.

That means that f is a monotone increasing function in $[0,\infty]$. Take x, y and z as any three elements in X, then by triangle inequality property of d we have,

$d(x,z) < d(x,y) + d(y,z)$ and since f is increasing,

We have $f(d(x,z)) < f(d(x,y) + d(y,z))$

or, $\frac{d(x,z)}{1+d(x,z)} \leq \frac{d(x,y)+d(y,z)}{1+d(x,y)+d(y,z)}$

$= \frac{d(x,y)}{1+d(x,y)+d(y,z)} + \frac{d(y,z)}{1+d(x,y)+d(y,z)}$

$\leq \frac{d(x,y)}{1+d(x,y)} + \frac{d(y,z)}{1+d(y,z)}$

or, $\rho(x,z) \leq \rho(x,y) + \rho(x,z)$.

Example 1.1.9 Let X be a non-empty set and $\rho : X\times X \to$ Reals be defined as (i) $\rho(x,y) = 0$ if and only if $x = y$; and

(ii) $\rho(x,y) \leq \rho(x,z) + \rho(y,z)$ for **any three** elements x, y and z in X. Show that (X,ρ) is a metric space.

Solution : For (M.1) axiom put $x = y$ in (ii) we get

$0 \leq \rho(x,z) + \rho(x,z) = 2\rho(x,z)$

i.e., $\rho(x,z) \leq 0$ and since x and z are arbitrary elements in X, non-negativity of ρ is clear, and by (*i*) (M.1) axiom is verified. For Symmetry put $x = z$ in (ii) to have $\rho(x,y) \leq 0 + \rho(y,x,)$ and interchanging x and y gives,

$\rho(y,x) < \rho(x,y)$; Thus we get $\rho(x,y) = \rho(y,x)$ which is (M.2). Finally, triangle inequality for ρ follows from given condition (ii) plus axion of Symmetry.

Example 1.1.10 Let p:R×R→R be defined by,

$p(x,y) = \min[1, y-x]$ if $x \leq y$;

$= 0$ if $x>y$.

Examine if p is a metric over the set R of all reals.

Solution : Here p is non-negative and $p(x,x) = 0$ for all reals x; but $p(x,y) = 0$ does

not imply $x = y$; because $p(3,1) = 0$ by definition. Hence (M.1) axiom fails. Thus p is not a metric.

§ **1.2** Let (X,d) be a metric space, and $x_0 \in X$ and r be a +ve real.

Definition 1.2.1 The subset $\{x \in X : d(x_0,x)<r\}$ of X, denoted by $B_r(x_0)$ is called an open ball in X centred at x_0 with radius = r.

Clearly, $B_r(x_0)$ is not empty because $d(x_0,x_0) = 0 < r$ and $x_0 \in B_r(x_0)$.

Example 1.2.1 An open ball $B_r(x_0)$ in R with usual metric d

$= \{x \in R: d(x_0,x) < r\}$

$= \{x \in R: |x - x_0| < r\}$

$= \{x \in R: x_0 - r < x < x_0 + r\}$

= an open interval $(x_0 - r, x_0 + r)$ with mid-point x_0 and length = $2r$.

Similarly, in the metric space q of all complex numbers with usual metric we see that $B_r(z_0)$ looks as an open circular disc with centre at $z_0 \in$ q having radius = r.

Definition 1.2.2 The subset $\{x \in X: d(x_0,x) \leq r\}$ of X is called a closed ball with centre $= x_0$ and with radius $= r$.

The subset $\{x \in X: d(x_0,x) = r\}$ of X is called a sphere centred at x_0 having the radius = r. It is also called boundary (Bdr) of open (or closed) ball $B_r(x_0)$; Note that centre of a sphere is not a member of the sphere.

Example 1.2.2 If X is non-empty and (X,d) is a metric space with the trivial metric d, describe all open balls centred at a point of X.

Solution : Let $x_0 \in$ X and r be a +ve real, then an open ball $B_r(x_0)$ of

X = $\{x \in X: d(x_0, x) < r\}$. Now,

$B_r(x_0) = \{x_0\}$, if $0 < r < 1$

= X, if $r > 1$.

So open balls centred a point of X are either singleton $=\{x_0\}$ or the whole space X.

Definition 1.2.3 Given a non-empty subset of G of (X,d), an element $u \in G$ is called an interior point of G, if there is an open ball $B_r(u)$ $(r > 0)$ such that $B_r(u) \subset G$.

For example, in the sub-set G = ((0,l)U2) of reals with usual metric every member of open interval (0,1) is an interior point of G, but $2 \in G$ is not an interior point of G.

Definition 1.2.4 A subset G in (X,d) is called an open set if every point of G is an interior point of G.

For example, the subset $(-1,1) \cup (2,3)$ of reals with usual metric is an open set without being an open interval in R. In (X,d) every open ball is an open set, and above statement shows converse is not true. However we have following Theorem in this connection.

Theorem 1.2.1 If a subset G of (X,d) is open, then G is a Union of open balls in (X,d).

Proof : Without loss of generality we take G to be a non-empty open set in (X,d) and $u \in G$. Then u being an interior point of G we find an open ball $B_r(u)$ such that $B_r(u) \subset G$. Thus we write $\bigcup_{U \in G} B_r(u) \subset G$. The reverse inclusion $G \subset \bigcup_{U \in G} B_r(u)$ is always true, and on combining we deduce $G = \bigcup_{U \in G} B_r(u)$. The proof is complete.

Theorem 1.2.2 Any union of open sets in *(X,d)* is an open set in *(X,d)*.

Proof : Let $\{G_\alpha\}_{\alpha\in\Delta}$ with Δ as an index set be a family of open sets in (X,d) and

$G = \bigcup_{\alpha\in\Delta} G_\alpha$. Take $u \in G$. Then u is a member of $G_{\alpha 0}$ for some $\alpha_0 \in \Delta$. Since $G_{\alpha 0}$ is open we find an open ball $B_r(u)$ centred at u with $B_r(u) \subset G_{\alpha 0} \subset \bigcup_{\alpha\in\Delta} G_\alpha = G$. This shows that u is an interior point of G. So G is an open set in *(X,d)*.

Theorem 1.2.3 If G_1 and G_2 are open sets in *(X,d)*, then $G_1 \cap G_2$ is an open set in *(X,d)*.

Proof: Let G_1 and G_2 two open sets in *(X,d)* and $G = G_1 \cap G_2$. Suppose $G \neq \phi$ and take $x \in G$. Then $x \in G_1$ and $x \in G_2$; G_1 being open we find an open ball $B_{r_1}(x)$ such that,

$$B_{r_1}(x) \subset G_1 \qquad \ldots \qquad (1.2.1)$$

Similarly, let $B_{r_2}(x) \subset G_2$ and take a +ve $r < \min(r_1, r_2)$. Then the open ball $B_r(x) \subset G_{ri}(x) \subset G_i$ for $i = 1, 2$; and hence $B_r(x) \subset (G_1 \cap G_2) \subset G$;

Thus x is an interior point of G and x being an arbitrary element of G, G is shown as an open set.

Corollary : An intersection of a finite number of open sets is an open set in *(X,d)*.

Example 1.2.3 An intersection of an infinite number of open sets in a metric space may not be an open set.

Solution : For each natural number n take the open interval $I_n = \left(-\frac{1}{n}, \frac{1}{n}\right)$ in real number space R with usual metric. Then each I_n is an open ball, and hence an open set in R. Now $\bigcap_{n=1}^{\infty} I_n = \bigcap_{n=1}^{\infty}\left(-\frac{1}{n}, \frac{1}{n}\right) = \{0\}$ is not an open set in R.

Definition 1.2.5 Given a subset G in *(X,d)* Interior of G, denoted by Int G is the set of all interior points of G.

Remark 1 In a metric space we assume ϕ (empty set) as an open set.

Remark 2 Int G is the largest open subset of G in the souse that if K is an open subset of G, then $K \subset \text{Int } G$.

Remark 3 In a metric space (X,d) if ζ_d denotes the collection of all open subsets of *(X,d)* then we have (i) $\phi, X \in \zeta_d$ (ii) Any union of members of ζ_d is member of ζ_d and (iii) Intersection of a finite number of members of ζ_d is a member of ζ_d. For this reason ζ_d is said form a Topology on X, called the metric Topology corresponding to given metric d on X.

Theorem 1.2.4 In a metric space *(X,d)* given two distinct points u and v in X, there are disjoint open sets G and H containing u and v respectively.

Proof : Let $u, v \in X$ with $u \neq v$. Then $d(u,v) > 0$. Let us take a +ve real $r < \dfrac{d(u,v)}{2}$. Then open Balls $B_r(u)$ and $B_r(v)$ centred at u and v are disjoint; If not, take $x \in (B_r(u) \cap B_r(v))$. Then $x \in B_r(u)$ gives $d(u,x)<r$ and, similarly, $d(v,x)<r$. Then $d(u,v) \leq d(u,x) + d(x,v) < r + r = 2r < d(u,v)$, which is impossible.

Hence $B_r(u) \cap B_r(v) = \phi$.

Remark : This property of a pair of distinct points in a metric space (X,d) attracting a pair of disjoint open balls (and hence open sets) is called Hausdorff property of the space.

Definition 1.2.6 A subset N of a metric space (X,d) is called a neighbourhood of a point $x \in X$ if there is an open set O in X such that $x \in O \subset N$.

For example, in the space of R of reals with usual metric an open interval $\left(x-\frac{1}{n}, x+\frac{1}{n}\right)$ is a nighbourhood of $x \in R$ for each natural number n. There are in general, many neighbourhoods of a point in a metric space. If N_x denotes the collection of all neighbourhoods of $x \in X$ in metric space (X,d), then we have following theorem.

Theorem 1.2.5 A subset G in a metric space in open if and only if G is a neighbourhood of each of its points.

Proof : The proof is easy and left out.

Theorem 1.2.6 In a metric space (X,d) if $x \in X$ then

(a) any member of N_x is always non-empty.

(b) If $N \in N_x$ and $N \subset H$, then $H \in N_x$,

(c) If $N_1, N_2 \in N_x$, Then $N_1 \cap N_2 \in N_x$,

(d) If $N \in N_x$, there is a member $N^* \in N_x$ with $N^* \subset N$

such that $N \in N_u$ for every member $u \in N^*$.

Proof : Proofs for (a)-(c) are easy and left out. For (d) let $N \in N_x$. Then we find an open set O_x containing x such that,

$$O_x \subset N$$

Now put $N^* = O_x$ which is a neighbourhood of each of its points;

Further $N^* \subset N$ gives $N \in N_u$ for every member $u \in N^*$.

§ 1.3

Definition 1.3.1 An element $u \in X$ is called a limit point of a non-empty subset A of metric space (X,d), if every **neighbourhood N of u** meets A at some point other than u. By notation, if $A \cap (N\setminus\{u\}) \neq \phi$.

For example, O is a limit point of set $A = \left(1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots\right)$ in space R of reals with usual metric, but any non-zero real is not a limit point of A.

Definition 1.3.2 A subset F in a metric space (X,d) is called a closed set if A contains all its limit limits.

Example 1.3.1 The set N of all natural numbers in real number space with usual metric is a closed set, but an open interval of reals is not a closed set in metric space of reals with usual metric.

Theorem 1.3.1 A subset F in a metric space (X,d) is closed if and only if its complement $(X\setminus F)$ in X is an open set in (X,d).

Proof : Suppose F is a closed set in (X,d) and $G = (X\setminus F)$; If $x \in G$, then $x \notin F$ and x is not a limit point of F because F is closed. So we find an open ball $B_r(x)$ such that,

$B_r(x) \cap F = \phi.$ or, $B_r(x) \subset (X\setminus F) = G.$

That means, x is an interior point of G. So G is open.

Conversely, assume $(X\setminus F)$ to be open and take u as a limit point of F, and if possible, let $u \notin F$; *i.e.*, $u \in (X\setminus F)$. Since $(X\setminus F)$ is open we find an open ball $B_r(u) \subset (X\setminus F)$, showing $B_r(u) \cap F = \phi$ and hence $(B_r(u) \setminus \{u\}) \cap F = \phi$ – a contradiction that u is limit point of F. The proof is now complete.

Remark : Words 'closed' and 'open' as appeared in Theorem 1.3.1 may be interchanged without making Theorem false.

Theorem 1.3.2 In a metric space *(X,d)* intersection of any number of closed sets is closed.

Proof : Let $\{F_\alpha\}_{\alpha\in\Delta}$ be a family of closed sets in X, and put $F = \bigcap_{\alpha\in\Delta} F_\alpha$ (Δ is an index set). Taking complement in X, we have $X\backslash F = X\backslash \bigcap_{\alpha\in\Delta} F_\alpha$

$$= \bigcap_{\alpha\in\Delta}(X \setminus F_\alpha)$$

by De-Morgan's rule. From Theorem 1.3.1 it follows that each $X\backslash F_\alpha$ is an open set in X and r.h.s. is an open set; Thus X\F becomes an open set, giving F as a closed set in X.

Theorem 1.3.3 In a metric space Union of a finite number of closed sets in a closed set.

Proof : The proof is done using De-Morgan's rule as above.

Remark : In a metric space Union of an infinite number of closed set may not be a closed set. Take closed interval $I_n = \left[0, \frac{n}{n+1}\right]$ for n = 1, 2, 3, ... Then $\bigcup_{n=1}^{\infty} I_n = \bigcup_{n=1}^{\infty}\left[0, \frac{n}{n+1}\right][0,1)$ which is not a closed set in R with usual metric.

Theorem 1.3.4 The only non-empty subset of reals with used metric that is both open and closed is the set R of all reals.

We shall take up this result by employing notion of connectedness of space of reals later on.

Definition 1.3.3 Given a subset A in a metric space *(X,d)* its closure denoted by $\overline{A}$ is defined as $\overline{A} = A \cup A'$, where A' is called the derived set of A = set of all limit points of A.

For example (i) closure of an open interval (a,b) of reals with usual metric is equal to the closed interval $[a,b]$.

(ii) Closure of the set Q of all rationals in space R of all reals with usual metric is equal to R.

Remark : Closure $\overline{A}$ is a closed set.

Theorem 1.3.5 In a metric space (X,d) for any subset A, (i) $\text{Int}\,A = X\backslash\left(\overline{X/A}\right)$, and (ii) $\overline{A} = X\backslash\text{Int}(X\backslash A)$.

Proof : Now $\overline{X/A}$ is closed; so $X\backslash\left(\overline{X/A}\right)$ is open. Further, $X\backslash A \subset \left(\overline{X/A}\right)$, and taking complement, $A \supset X\backslash\left(\overline{X/A}\right)$.

So, $X\backslash\left(\overline{X/A}\right)$ is an open subset of A and therefore,

$$X\backslash\left(\overline{X/A}\right) \subset \text{Int}\,A \quad \ldots \quad (1.3.1)$$

If G is any open subset of A, we have

$$(X\backslash A) \subset (X\backslash G)$$

So, $\left(\overline{X/A}\right) \subset \left(\overline{X/G}\right) = (X\backslash G)$ which is closed.

Taking complement, $G \subset X\backslash\left(\overline{X/A}\right)$ and this is true for any open subset G of A we have $\text{Int}A \subset X\backslash\left(\overline{X/A}\right)$ ... (1.3.2)

Combining (1.3.1) and (1.3.2) we have (i) The other relation (ii) follows from (i) if we replace A by $(X\backslash A)$ and take complement.

Definition 1.3.4 Given a subset A in a metric space, Boundary of A denoted by Bdr(A) is defined as

$$\text{Bdr}(A) = \overline{A} \setminus \text{Int}(A).$$

Note that $\text{Bdr}(A) = \overline{A} \setminus \text{int}(A)$

$$= \overline{A} \cap (X \setminus \text{Int}A)$$

$$= \overline{A} \cap (X \setminus A) \text{ (by Theorem 1.3.5)}$$

So Bdr(A) is a closed set.

Example 1.3.3 In a metric space *(X,d)* let G be an open set. For any subset A of X show that $G \cap A = \phi$ if and only if $G \cap \overline{A} = \phi$.

Solution : Here $G \cap \overline{A} = \phi \Rightarrow G \cap A = \phi$ always. Let $G \cap A = \phi$. If possible, let $G \cap \overline{A} \neq \phi$ and take $u \in (G \cap \overline{A})$. Thus $u \in \overline{A}$ but $u \notin A$. So u is a limit point of A. G is open and contains u. Clearly, G is a neighbourhood of u, and therefore $G \cap A \neq \phi$ – a contradiction. Hence $G \cap A = \phi \rightarrow G \cap \overline{A} = \phi$.

Definition 1.3.5 (a) A subset H in a metric space (X,d) is said to be dense (everywhere dense) in X if $\overline{H} = X$.

(b) A subset B in *(X,d)* is said to be no-where dense in X if $\text{Int}(\overline{B}) = \phi$.

For example (a) the set Q of all rationals in R of reals with usual metric is dense in R *i.e.*, $\overline{Q} = R$.

(b) The set N of all natural numbers in R with usual metric is a no-where dense in R, and so is a finite subset of R.

Definition 1.3.6 A metric space *(X,d)* is said to be separable if there is a countable dense set C in X *i.e.*, $\overline{C} = X$.

For example, the real space R with usual metric is separable, because the set Q of **all** rationals in R is dense in R, and so is the case with Enclidean n-space R^n where the subset consisting of those points of R^n with rational co-ordinates is dense in R^n.

Definition 1.3.7 A metric space (X,d) is said to be discrete if every subset in X is both open and closed (clopen).

For example, the metric space (X,d) with trivial metric d is discrete; because every subset including singletons is clopen. But there are discrete metric spaces with non-trivial metric. Consider N of all natural numbers as a metric space with usual metric of reals. This metric is non-trivial, nevertheless N is a discrete metric space.

§ 1.4 Sub-spaces

Let (X,d) be a metric space and Y be a non-empty subset of X. They d_Y = restriction of d to Y×Y is also a metric on Y; because $d_Y(u,v)$ as $(u,v) \in (Y\times Y)$ satisfies all metric axioms (M.1)-(M.3). So (Y,d_Y) is a metric space, called a subspace of (X,d).

Theorem 1.4.1 Let (Y,d_Y) be a sub-space of (X,d). Then a subset G of Y is open in (Y,d_Y) and if and only if $G = Y \cap O$, where O is an open set in (X,d).

Proof : First we note that if $B_r^Y(y)$, $y \in Y$ and $r>0$ is an open ball in (Y,d_Y), then we have,

$B_r^Y(y) = Y \cap B_r^X(y)$, where $B_r^X(y)$ is an open ball in (X,d).

Now suppose G is an open set in (Y,d_Y),

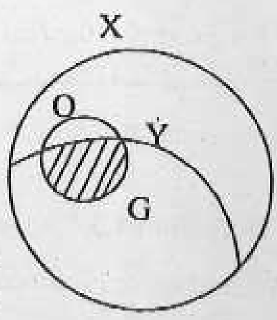
then G is Union of some open balls (Y,d_Y), say

$G = \cup B_r^Y(y)$ where $y \in Y$ and $r>0$

$= \cup(Y \cap B_r^X(y))$ where $y \in Y$ and $r>0$

$= Y \cap \{\cup B_r^X(y)\}$ where $y \in Y$ and $r>0$

$= Y \cap O$ where O is an open set in (X,d).



Conversely, Let $G = Y \cap O$ where O is open X and let $y \in G$, then $y \in O$ and y is an interior point of O relative to (X,d), and we find an open ball $B_r^X(y)$ $(r>0)$ such that $B_r^X(y) \subset O$; Now put $B_r^Y(y) = Y \cap B_r^X(y) \subset Y \cap O = G$. Where $B_r^Y(y)$ is an open ball centred at $y \in G$ relative to (Y,d_Y). Hence $y(\in G)$ is an interior point of G relative to (Y,d_Y). So G is open in Y.

Remark 1 Theorem 1.4.1 remains valid if the words 'open' are replaced by 'closed'.

Remark 2 There may an open set in (Y,d_Y) without being open in (X,d). For example, consider the sub-space as an interval $(0,1] = Y$ of the space R of reals with usual metric.

Then $\left(\frac{1}{2},1\right]$ is an open set in sub-space Y without being open in R.

Theorem 1.4.2 Any sub-space of a separable metric space is separable.

Proof : Let (Y,d_Y) be a sub-space of (X,d) and (X,d) be separable. Take $A = \{x_1,x_2,\ldots,x_n,\ldots\}$, a countable set in X such that A is dense in X *i.e.*, $\overline{A} = X$. So for each $y \in Y$, and each natural number m, the open ball $B_{\frac{1}{m}}(y)$ attracts elements of A, and say

$x_n \in \left(A \cap B_{\frac{1}{m}}(y)\right)$. That means $B_{\frac{1}{m}}(x_m) \cap Y \neq \phi$ (because y is there).

Put $\Lambda = \left\{(n,m) : Y \cap B_{\frac{1}{m}}(x_n) \neq \phi\right\}$. Clearly $\Lambda \neq \phi$. Now for each member $(n,m) \in \Lambda$

take a member, say $= y_{n,m} \in \left(Y \cap B_{\frac{1}{m}}(x_n) \right)$. Then $B = \{y_{n,m} : (n,m) \in \Lambda\}$ becomes a countable set in Y. We now check that B is dense in Y. Take $y \in Y$, for each +ve r, choose a natural number m to satisfy $\frac{1}{m} < r/2$. As stated above, there is an integer n such that $x_n \in B_{\frac{1}{m}}(y)$. So (n,m) is a member of Λ by choice, and we deduce that $d_y(y, y_{nm}) \le d(y, x_m) + d(x_n, y_{nm}) < \frac{1}{m} + \frac{1}{m}$

$< r$, showing that $y_{nm} \in B^Y(y,r)$.

That means $y \in Y$ -closure (B) and proof is complete.

Chapter-I

Exercise (A)

Short answer type questions

1. In Euclidean plane R^2 draw open unit ball $B_1\left(\underset{\sim}{O}\right)$ centred at $\underset{\sim}{O} = (0,0) \in R^2$ with respect to (*i*) usual metric $\tau\left(\underset{\sim}{x}, \underset{\sim}{y}\right) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$, (ii) metric $\rho\left(\underset{\sim}{x}, \underset{\sim}{y}\right) = \max(|x_1 - y_1|, |x_2 - y_2|$ and (iii) metric $\sigma\left(\underset{\sim}{x}, \underset{\sim}{y}\right) = |x_1 - y_1| + |x_2 - y_2|$ where $\underset{\sim}{x} = (x_1, x_2)$, $\underset{\sim}{y} = (y_1, y_2) \in R^2$.

2. Examine if $X = (a,b,c)$ and $d : X \times X \to$ Reals given by, $d(a,a) = d(b,b) = d(c,c) = (a,b) = d(b,a) = 0$, $d(b,c) = d(c,b) = 5$ and $d(c,a) = d(a,c) = 6$ is a metric on X, where a, b, c are different.

3. A finite set G in a metric space (X,d) is closed because (a) G has no limit point, (b) every point of G is limit point of G. Which one is correct? Give reason.

4. A metric space is discrete if every subset in it is both open and closed. Give an example with justification a discrete metric space (X,d) with a non-trivial metric d.

5. In each of following cases give an example in support of a set G in a metric space $(X.d)$ such that derived set G' satisfies,

   (i) $G' \subset G$ (ii) $G \subset G'$ (iii) neither (i) nor (ii).

6. The closed interval $[a,b]$ $(a<b)$ of the metric space R | of all reals with usual metric is a complete metric space because (a) $[a,b]$ is bounded (b) $[a,b]$ is not open (c) $[a,b]$ is a closed set of R. Which one is correct? Give reason.

7. If $G = \left\{\frac{1}{2^n} + \frac{1}{3^m}\right\}$ where m and n are natural numbers obtain (i) derived set G' of G, and (ii) derived set $(G')'$ of G'.

8. For two subsets A and B in a metric space if derived set A' = derived set B', does if follow that A = B? Give reasons for answer.

9. A subset F is a closed set in a metric space if (i) $F \subset$ derived set F' (ii) derived set $F' \subset F$. Which one is false? Give reasons.

10. In the metric space $C_R[0,1]$ with sup metric ρ obtain a second degree polynomial ρ, such that $\rho(p,0) = 1$.

Exercise (B)

1. If (X,d) is a metric space, and $x_1 x_2 \dots, x_n$ are members of X, then show that $d(x_1, x_n) \le d(x_1, x_2) + d(x_2, x_3) + \dots + d(x_{n-1}, x_n)$.

2. In a metric space (X,d) if $x \in X$, and $\aleph_x$ denotes the family of all neighbourhoods of x in X, Show that $\cap\{N : N \in \aleph_x\} = \{x\}$.

3. If (X,d) is a metric space, verify that $(X, \min(1,d))$ is a metric space.

4. If $(X_1,d_1), (X_2,d_2), \dots, (X_n,d_n)$ are n metric spaces, and

   $X = X_1 \times X_2 \times \dots \times X_n$, show that (X,d) is a metric space when

   (i) $d\left(\underset{\sim}{x}, \underset{\sim}{y}\right) = \sum_{i=1}^{n} d_i(x_i, y_i)$, (ii) $d\left(\underset{\sim}{x}, \underset{\sim}{y}\right) = \max_{1 \le i \le n} d_i(x_i, y_i)$ for any two members

   $\underset{\sim}{x} = (x_1, x_2, \dots, x_n)$ and $\underset{\sim}{y} = (y_1, y_2, \dots, y_n)$ of X.

5. If $X \ne \phi$, and $e : X \times X \to$ Reals in a function to satisfy (i) $e(x,y) = 0$ if and only if $x = y$ in X, and (ii) $e(x,z) \le e(x,y) + e(y,z)$ for all $x, y, z \in X$, Examine if (X,e) is a metric space.

6. If R is the set of all reals, examine if (R, σ) is a metric space when $\sigma(x,y) = |x-y|^2$ for all $x, y \in R$.

7. If $X \ne \phi$ and (X, d_k) is a metric space for k = 1,2,... such that for any two distinct members x,y in X, $d_k(x,y) > 0$ for some k. Let $\rho : X \times X \to$ Reals be defined as $\rho(u,v) = \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{d_k(u,v)}{1 + d_k(u,v)}$ for all u, v in X.

   Show that (X, ρ) is a metric space.

8. Let A be a subset of a metric space (X,d).

   Show that (i) Bdr A $= \overline{A} \cap \left(\overline{X/A}\right)$

   $= X/(A^0 \cap (X/A)^0)$, bar denoting the closure and 0 indicating Interior.

(ii) $\overline{A} = A^0 \cup \text{Bdr } A$, and

$X = A^0 \cup (X \setminus A)^0 \cup \text{Bdr } A$.

(iii) Obtain Bdr A where $A = \{(x,y) \varepsilon R^2 : y = o\}$.

9. (a) If A and B are two subsets in a metric space *(X,d)*, prove that $\overline{(A \cup B)} = \overline{A} \cap \overline{B}$, =bar denoting closure.

(b) Give an example of subsets A, B of reals with usual metric with $\overline{(A \cap B)} \neq \overline{A} \cap \overline{B}$.

(c) If two subsets A and B in a metric space satisfy $\text{Bdr } A \cap \text{Bdr } B = \phi$, then prove that $\overline{(A \cap B)} = \overline{A} \cap \overline{B})$ and $= (A \cup B)^0 = A^0 \cup B^0$.

10. Give an example of a metric space where closure of an open ball $B_r(x_0)$ is not equal to the closed ball $\overline{B}_r(x_0)$.

11. In Euclidean 2-space R^2 if $\rho((x_1,x_2),(y_1,y_2)) = |x_1 - y_1| + |x_2 - y_2|$ for all (x_1,x_2), $(y_1,y_2) \in R^2$, show that (R^2, ρ) is a metric space, and describe all open balls $B_r(0,0)$ $(r>0)$ centred at (0,0) of R^2.

12. Verify that collection $C_R[0,1]$ of all real-valued continuous functions over the closed interval [0,1] is a metric space with metric $\rho(f,g) = \int_0^1 |f - g|\, dt$ for all f, $g \in C_R[0.1]$. Find a second degree polynomial $p \in C_R[0,1]$ satisfying $\rho(0.p) = 1$.

13. Let Y be a sub-space of a metric space *(X,d)*; For $A \subset Y$, show that Y-closure (A) = $Y \cap X$ -closure (A).

14. In a metric space (X,d) if $A \subset X$, show

that Closure (A) = $\cap\{G$: G is a closed set in X containing A$\}$.

15. Let X denote collection of all sequences of reals having almost one non-zero term, and let the zero sequence be denoted by O. Further let the member of X having non-zero nth term x be marked as $x|^n$ $(n = 1,2,..)$. Define p : X×X→Reals by,

$P(x|^n, y|^n)$ = min $[1,|x-y|]$ if $x \le y$

= 1 if $x > y$

For $m \neq n$, Let $p(x|^n, |y|^m) = 1$.

Show that p satisfies all metric axioms except that of symmetry.

Unit 6 ❑ Complete Metric Spaces, Examples, Castors Theorem, Baire Theorem and Equivalent Metrics

(Convergent sequence, cauchy sequence in a metric space; Complete metric space, examples of complete metric spaces : R^n, $\ell_p (1 < p < \infty)$, $C_R[a.b]$; Incomplete metric space. Bounded sets. Diameter of a set, Cantor's Intersection Theorem. Baire Category Theorem; Equivalent metrics.)

§ 2.1 Let *(Y,d)* be a metric space, and $\{x_n\}$ be a sequence in X.

Definition 2.1.1. $\{x_n\}$ is said to be a convergent sequence in *(X,d)* if there is a member $u \in X$ such that $\lim_{n\to\infty} d(u,x_n) = 0$ or equivalently, given $a + ve \in$ there is an index N satisfying $d(u,x_n) < \varepsilon$ when $n \geq N$.

Remark : If $\{x_n\}$ is a convergent sequence in *(X,d)* with $u \in X$ and $\lim_{n\to\infty} (u,x_n) = 0$, u is called the limit of $\{x_n\}$; We write as $\lim_n x_n = u$

Theorem 2.1.1. $\lim_{n\to\infty} = u \in X$, then u is unique.

Proof : Let $\{x_n\}$ converge to u and to v aswell $(u,v \in X)$ where $u \neq v$. Then $d(u, ')>0$; Since *(X,d)* is Hausdorff we find two open balls $B_\epsilon(u)$ and $B_\epsilon(v)$ centered at u and v respectively with $0 < \varepsilon < \frac{1}{2} d(u,v)$ such that

$$B_\epsilon(u) \cap B_\epsilon(v) = \phi \qquad (2.1.1)$$

Since $\lim_n x_n = u$ and $\lim_n x_n = u = v$, corresponding to this +ve ϵ, we find an index N such that

$d(u,x_n) < \varepsilon$ *and* $d(v,x_n) < \varepsilon$ for $n > N$

or, $x_n \in B_\epsilon(u)$ and $x_n \in B_\epsilon(v)$ for $n \geq N$

This contradids (2.1.1) and Theorem is proved.

Definition 2.1.2. A sequence $\{x_n\}$ is said to be a cauchy sequence in (X,d) if $d(x_n,x_m) \to 0$ as $n,m \to \infty$, or equivalently if given a + ve $\in$, there is an index N satisfying $d(x_n,x_m) < \varepsilon$ whenever $n,m \geq N$.

Theorem 2.1.2. Any convergent sequence is a cauchy sequence, but converse is false in a metric space.

Proof : Let $\{x_n\}$ be a convergent sequence in (X,d) with $\lim_n x_n = u \in X$. Given $a + ve \in$, we find an index N satisfying

$$d(u,x_n) < \varepsilon/2 \text{ whenever } n > N, \quad (2.1.2)$$

Suppose $n,m \geq N$, then $d(x_n,x_m) \leq d(x_n,u) + d(u,x_m) < \varepsilon/2 + \varepsilon/2 = \varepsilon$ from (2.1.2) So $\{x_n\}$ is a cauchy sequence in (X,d).

The converse is not true. Take a metric space as $(0,1]$ and a sequence $\{x_n\}$ in $(0,1)$ as $x_n = \frac{1}{n}$ $(n = 1,2,\ldots)$. Then with usual metric for reals we have

$d(x_n,x_m) = |x_n - x_m| = \left|\frac{1}{n} - \frac{1}{m}\right| \leq \frac{1}{n} + \frac{1}{m} \to 0$ as $n,m \to \infty$. So $\{x_n\}$ is a cauchy sequence in $(0,1]$, and $\left\{x_n = \frac{1}{n}\right\}$ fails to converge to any member of $(0,1]$.

Theorem 2.1.3. If a cauchy sequence has a convergent sub-sequence in (X,d), then the cauchy sequence is convergent in (X,d).

Proof : Suppose $\{x_n\}$ is a cauchy sequence in (X,d), and $\{x_{nk}\}$ be a subsequence of $\{x_n\}$ such that $\{x_{nk}\}$ is convergent in (X,d). Let $u = \lim_k x_{n_k}; u \in X$. Given $a + ve \in$, there are indices N and L such that

$$d(x_m, x_n) < \in/2 \quad \text{whenever } n,m \geq N \quad \dots \quad (2.1.3)$$

because $\{x_n\}$ is cauchy, and

$$d(x_u, x_{nk}) < \in/2 \quad \text{whenever } k \geq L \quad \dots \quad (2.1.4)$$

because $\lim_k x_{n_k} = u$.

Take $M = \max(N,L)$: Then for $m \geq M$ (and hence $n_m \geq m \geq M$) from (2.1.3) and (2.1.4) we have

$d(u,x_m) \leq d(u,x_{n_m}) + d(x_{n_m},x_m)$

$< \varepsilon/2 + \varepsilon/2 = \varepsilon$, showing $\lim_n x_n = u$.

Definition 2.1.3. (X,d) is said to be a complete metric space if every cauchy sequence in (X,d) is a convergent sequence in (X,d).

Remark : The interval $(0,1] = \{x: 0 < x \leq 1\}$ is a metric space with usual metric of reals and it is not a complete metric space. However we have

Theorem 2.1.4. The space R of all reals with usual metric is a complete metric space.

Proof : The proof is exactly that of cauchy's General Principle of convergence in R.

Example 2.1.1 The Euclideon n-space R^n of all ordered n-tuples of reals $\underset{\sim}{x} = (x_1, x_2, \dots x_n)$. ($x_i$ is real) is a complete metric space with usual metric

$d\left(\underset{\sim}{x},\underset{\sim}{y}\right)=\left(\sum_{i=1}^{n}|x_i-y_i|^2\right)^{1/2}$ when $\underset{\sim}{x} = (x_1, x_2, ... x_n)$ and $\underset{\sim}{y} = (y_1, y_2, ... y_n)$ are any two members of R^n.

Proof : We know that with d as given (R^n, d) is a metric space. Let $\left\{\underset{\sim m}{x}\right\}$ be a cauchy sequence in R^n. So given $\varepsilon > 0$, we find and index N satisfying $d\left\{\underset{\sim m}{x}, \underset{\sim r}{x}\right\} < \varepsilon$ when $m, r > N$

Taking $\underset{\sim m}{x} = \left(x_1^{(m)}, x_2^{(m)}, ..., x_n^{(m)}\right)$ = (n fixed),

above gives $\left(\sum_{i=1}^{n}\left|x_i^{(m)}-x_i^{(r)}\right|^2\right)^{\frac{1}{2}} < \varepsilon$ when $m, r \geq N$. ... (2.1.5)

Now each $i=1,2,...n$ gives $\left|x_i^{(m)}-x_i^{(r)}\right|^2 \leq \sum_{i=1}^{n}\left|x_i^{(m)}-x_i^{(r)}\right| < \varepsilon^2$ when $m, r \geq N$.

or, $\left|x_i^{(m)}-x_i^{(r)}\right| < \varepsilon$ when $m, r \geq N$.

So, Cauchy's general Principle convergence says for each $i = 1,2,...,n$ the sequence $\left\{x_i^{(r)}\right\}$ (a sequence of reals with running index m) of reals is convergent and put

$\lim_{r \to \infty} x_i^{(r)} = x_i^{(0)}$; $x_i^{(0)} \in R$. ... (2.1.6)

Clearly n-tuple $\left(x_1^{(0)}, x_2^{(0)}, ..., x_n^{(0)}\right) = \underset{\sim 0}{x}$ (say) $\in R^n$ and from (2.1.5) on passage to limit as $r \to \infty$ and using (2.1.6) we deduce that $\left(\sum_{i=1}^{n} \left|x_i^{(m)} - x_i^{(0)}\right|^2\right)^{\frac{1}{2}} \le \varepsilon$ when $m \ge N$.

That means, $\left(\underset{\sim}{x}_m, \underset{\sim}{x}_0\right) \to 0$ as $m \to \infty$. Thus cauchy sequence $\left\{\underset{\sim}{x}_m\right\}$ as given is a convergent sequence in R^n. So (R^n, d) is a complete metric space.

Example 2.1.2 The sequence space $\ell_p(1<P<\infty)$ consisting of all real sequences $\underset{\sim}{x}=(x_1, x_2, \ldots, x_n, \ldots)$ with $\left(\sum_{i=1}^{\infty} |x_i|^p\right)^{\frac{1}{p}} < \infty$ is a complete metric space with respect to the metric $\rho\left(\underset{\sim}{x}, \underset{\sim}{y}\right) = \left(\sum_{i=1}^{\infty} |x_i - y_i|^p\right)^{\frac{1}{p}}$ for $\underset{\sim}{x} = (x_1, x_2, \ldots)$, $\underset{\sim}{y} = (y_1, y_2, \ldots) \in \ell_p$.

Solution : We have seen that the sequence ℓ_p is a metric space with ρ as a metric defined above. Now suppose $\left\{\underset{\sim}{x}_n\right\}$ be a cauchy sequence in ℓ_p where $\underset{\sim}{x}_n = \left(x_1^{(n)}, x_1^{(n)}, \ldots\right) \in \ell_p (n = 1, 2, \ldots)$.

So $\rho\left(\underset{\sim}{x}_n, \underset{\sim}{x}_m\right) \to 0$ as $n, m \to \infty$, and given $\epsilon > 0$, we find an index N such that $\rho\left(\underset{\sim}{x}_n, \underset{\sim}{x}_m\right) < \varepsilon$ when $n, m \ge N$.

or, $\left(\sum_{i=1}^{\infty} \left|x_i^{(n)} - x_i^{(m)}\right|^p\right)^{\frac{1}{p}} < \varepsilon$ when $n, m \ge N$ ... (2.1.7)

Then for each $i = 1,2,...$

$$\left|x_i^{(n)} - x_i^{(m)}\right|^p \leq \sum_{i=1}^{\infty}\left|x_i^{(n)} - x_i^{(m)}\right|^p < \varepsilon p \text{ when } n,m \geq N$$

or $\left|x_i^{(n)} - x_i^{(m)}\right| < \varepsilon$ when $n,m \geq N$ ... (2.1.8)

Now cauchy's general Principle of convergence says from (2.1.8) that for each $i=1,2,...$ the real sequence $\{x_i^{(m)}\}$ *(m* as running index) is convergent and let $\lim_{m\to\infty} x_i^{(m)} = x_i^{(0)}$, (say), $x_i^{(0)} \in R$.

Put $\underset{\sim 0}{x} = \left(x_1^{(0)}, x_2^{(0)}, ...\right)$; we check that $\underset{\sim 0}{x}$ is a member of ℓ_p. For each +ve integer k we have from (2.1.7)

$$\sum_{i=1}^{k}\left|x_i^{(n)} - x_i^{(m)}\right|^p \leq \sum_{i=1}^{\infty}\left|x_i^{(n)} - x_i^{(m)}\right|^p < \varepsilon^p \text{ when } m,n \geq N. \quad ... \quad (2.1.9)$$

Passing on limit as $m \to \infty$ in (2.1.9) we deduce that $\sum_{i=1}^{k}\left|x_i^{(n)} - x_i^{(0)}\right|^p \leq \varepsilon^p$ when $n \geq N$.

This is true for each +ve integer k and hence

$$\sum_{i=1}^{\infty}\left|x_i^{(n)} - x_i^{(0)}\right|^p \leq \varepsilon^p \text{ when } n \geq N. \quad ... \quad (2.1.10)$$

Now $\left(\sum_{i=1}^{\infty}\left|x_i^{(0)}\right|^p\right)^{\frac{1}{p}} = \left(\sum_{i=1}^{\infty}\left|x_i^{(0)} - x_i^{(n)} + x_i^{(n)}\right|^p\right)^{\frac{1}{p}} \leq \left(\sum_{i=1}^{\infty}\left(x_i^{(0)} - x_i^{(n)}\right)^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{\infty}\left|x_i^{(n)}\right|^p\right)^{\frac{1}{p}}$, by Minkowski inequality; and in view of (2.1.10) r.h.s. is a finite + ve quantity for large n. That means $\left(\sum_{i=1}^{\infty}\left|x_i^{(0)}\right|^p\right)^{\frac{1}{p}} < \infty$ and $\underset{\sim 0}{x} = \left(x_1^{(0)}, x_2^{(0)}, \ldots\right) \in \ell_p$.

Finally, from (2.1.10) we have $\left(\sum_{i=1}^{\infty}\left|x_i^{(n)} - x_i^{(0)}\right|^p\right)^{\frac{1}{p}} \le \varepsilon$ when $n \ge N$ and this shows $\lim_{n\to\infty} \underset{\sim n}{x} = \underset{\sim 0}{x} \in \ell_p$.

Thus the given cauchy sequence $\left\{\underset{\sim n}{x}\right\}$ in ℓ_p is a convergent sequence in ℓ_p and ℓ_p complete.

Example 2.1.3 The metric space $C_R[a,b]$ consisting of all real-valued continuous functions over the closed interval $[a,b]$ with sup metric is a complete metric space.

Solution . We know that $C_R[a,b]$ is a metric space with respect to the sup metric

$$\sigma(f,g) = \sup_{a \le t \le b} (l.u.b) |f(t) - g(t)| \text{ for } f,g \in C_R[a,b].$$

Suppose $\{f_n\}$ is a cauchy sequence of elements f_n in $(C_R[a,b],\sigma)$. Then we have $\sigma(f_n,f_m) \to 0$ as $n,m \to \infty$.

So given a +ve ε we find an index N satisfying $\sigma(f_n, f_m)<\varepsilon$ when $n,m \ge N$.

or $\sup_{a \le t \le b} |f_n(t) - f_m(t)| < \varepsilon$ when $n,m > N$.

For each t in $[a,b]$ we have $|f_n(t) - f_m(t)| \le \sup_{a \le t \le b} |f_n(t) - f_m(t)| < \varepsilon$ when $n,m \ge N$.

or $|f_n(t) \quad f_m(t)| < \varepsilon$ when $n,m \ge N$; and $a \le t \le b$ .. (2.1.11)

Inequality in (2.1.11) confirms that the sequence $\{f_n(t)\}$ of real-valued continuous functions converges uniformly to some function, say $= f_0(t)$ in $[a,b]$, and we know that this uniform limit function f_0 is continuous in $[a,b]$. So $f_0 \in C_R[a,b]$ and in (2.1.11) we pass on to limit as $m \to$ co. We have $|f_n(t) - f_0(t)| \le \varepsilon$ when $n \ge N$ and $a \le -t \le b$.

So $\sup_{a\le t\le b} |f_n(t) - f_0(t)| \le \varepsilon$ when $n \ge N$.

or, $\sigma(f_n, f_0) \le \varepsilon$ when $n \ge N$.

That means $\lim_{n\to\infty} f_n = f_0 \in C_R[a,b]$ and proof is complete.

Example 2.1.4 Open interval (0,1) of reals with usual metric is an incomplete metric space.

Solution : Here $\left\{\frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots\right\}$ is a sequence in (0,1). which is cauchy without being convergent in (0, 1).

Example 2.1.5 If $\wp[0,1]$ denotes the collection of all real polynomials in [0,1], show that $\wp[0,1]$ is a metric space with respect to sup metric and it is incomplete.

Solution : $\wp[0,1]$ is a sub-space of $C_R[0,1]$ and becomes a metric space with sup metric of $C_R[0,1]$. Consider a sequence $\{p_n\}$ of members (polynomials) of $\wp[0,1]$ where $P_n(t) = 1 + \frac{t}{1!} + \frac{t^2}{2!} + \ldots + \frac{t^n}{n!}$ $(n = 1,2,\ldots)$ in $0 \le t \le 1$.

If $n > m$ we have

$$d(P_n, P_m) = \sup_{0\le t\le 1}\left|\frac{t^{m+1}}{(m+1)!}+\frac{t^{m+2}}{(m+2)!}+\ldots+\frac{t^n}{n!}\right| \le \frac{1}{(m+1)!}+\frac{1}{(m+2)!}+\ldots+\frac{1}{n!};$$

Now *r.h.s.* goes to 0 as $m \to \infty$ by convergence of exponential series $1+\frac{1}{1!}+\frac{1}{2!}+\ldots$

Hence $\{P_n\}$ is a cauchy sequence in $\wp[0,1]$, but there is no real polynomial $\in \wp[0,1]$ to which $\{P_n\}$ converges in sup metric. Infact, $\lim_{n\to\infty} P_n(t) = e^t$ in respect of sup metric where e^t is not a polynomial. So $\wp[0,1]$ is incomplete.

§ 2.2.

Let A be a non-empty subset of a metric space (X,d).

Definition 2.2.1 (a) The set A is called bounded if $\sup_{x,x'\in A} \{d(x,x')\} < \infty$

(b) Diameter of A, denoted by Diam (A) is defined as Diam (A) $= \sup_{x,x'\in A} \{d(x,x')\} < \infty$

Clearly, A is bounded if and only if Diam (A) has a finite value (≥ 0).

Example 2.2.1 Any finite subset A of a metric space is bounded and Diam (A) equal to distance between a pair of elements in A that are at maximum distance apart.

Example 2.2.2

(a) An open ball of radius r in Euclidean 2-space R^2 which is an open circular disc in R^2 is bounded with its Diametral value = $2r$ (This is true if word 'open' is replaced by 'closed'.)

(b) The set $A = \{(x,y) \in R^2 : 2x - 3y - 5 = 0\}$ that represents the line of equation $2x - 3y - 5 = 0$ is an unbounded set in R^2 with Diam (A) = $+\infty$.

Theorem 2.2.1 A non-empty set A in a metric space (X,d) has Diam (A) = 0 if and only if A is a singleton.

Proof : If A is a singleton and = $\{u\}$ say, then Diam (A) = $d(u,u)$ = 0.

Conversely if Diam (A) = 0, and x,y are two elements in A, we have $d(x,y) \leq \sup_{u,v \in A} \{d(u,v)\}$= Diam (A) =0; Since $d(x,y) \geq 0$; always; we have $d(x,y) \geq 0$; so $x = y$;

That means A is a singleton.

Theorem 2.2.2 For any set G in (x,d), Diam (G) = Diam $(\overline{G})$,

Proof : Since $G \subset \overline{G}$, we have $\sup_{(x,y) \in G} d(x,y) \leq \sup_{(u,v) \in \overline{G}} d(u,v)$

or, Diam (G) $\leq$ Diam $(\overline{G})$ ... (1)

If Diam (G) = $+\infty$, then Diam $(\overline{G})$ = $+\infty$. So we take Diam (G) < ∞.

Let $\varepsilon > 0$ be arbitrary and $u, v \in \overline{G}$. Then we find $x, y \in G$ such that $d(u,x) < \frac{\varepsilon}{2}$ and $d(v,y) < \frac{\varepsilon}{2}$. Thus $d(u,v) \leq d(u,x) + d(x,y) + d(y,v) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} + d(x,y)$

or $d(u,v) < \varepsilon + d(x,y) \leq \varepsilon$ + Diam G. So $\sup_{(u,v) \in \overline{G}} d(u,v) \leq \varepsilon$ + Diam (G)

or Diam $(\overline{G}) \leq \varepsilon$ + Diam (G); As $\in > 0$ is arbitrary,

That means Diam $(\overline{G}) \leq$ Diam (G)

Combining (1) and (2) we have Diam (G) = Diam $(\overline{G})$. The proof is complete.

Remark Diam (A) = Diam (B) close not imply that A=B. For example (i) The set Q of all rationals has Diam (Q) = ∞ = Diam (R), but Q $\neq$ R, the set of all real; and *(ii)* The set {1,2,3} has its Diam–value = 2 = Diam[0,2], and {1,2,3} $\neq$ [0.2].

Theorem 2.2.3 If A and B are two sets in *(X,d)* with $A \cap B \neq \phi$, then Diam $(A \cup B)$ < Diam (A) + Diam (B).

Proof : If any of A and B is nubounded then $A \cup B$ is unbounded, and we are done. So let us suppose Diam (A) < ∞ and Diam (B) < ∞. If $u, v \in (A \cup B)$, then taking $\omega \in (A \cap B)$, we have $d(u,v) \leq d(u,\omega) + d(\omega,v) \leq$ Diam (A) + Diam (B) < ∞, and

$\sup\limits_{u,v \in (A \cup B)} d(u,v) \leq$ Diam (A) + Diam (B). That is to say,

Diam $(A \cup B) \leq$ Diam (A) + Diam (B).

Remark The condition $A \cap B \neq \phi$ can not be dropped for validity of Theorem above. For example, let us take A = the closed interval [0,1] and B = closed interval [2,5]. Then we have Diam $(A \cup B)$ = Diam $([0,1] \cup [2,5])$

= 5 $\not\leq$ 4 = 1+3 = Diam (0,1] + Diam [2,5] = Diam (A) + Diam (B). Here $A \cap B = \phi$

Example 2.2.3 Every cauchy sequence in *(X,d)* is bounded but converse is not true.

Solution : Let $\{x_n\}$ be a cauchy sequence in *(X,d)*; Taking $\varepsilon = 1$, we find an index

N such that $d(x_n, x_m) < 1$ for $n,m > N$; and put $\lambda = \max_{1 \le i,j \le N} (d(x_i, x_j))$; So that λ is a finite non-negative number such that $d(x_i, x_j) \le \lambda + 1$ for all i,j *i,e,* Diam $\{x_n\} < +\infty$, and the sequence $\{x_m\}$ is bounded.

However taking $\{x_n = (-1)^n\}$, *a* real sequence in R with usual metric we find $d(x_n, x_{n+1}) = 2$ for all n and hence $\{x_n\}$ is not a cauchy sequence. Nevertheless it is *a* bounded sequence of reals.

Remark : Every convergent sequence in (x,d) is bounded but converse is not true.

Theorem 2.2.4 (Cantor's intersection Theorem) A metric space (X,d) is complete if and only if every decreasing sequence of non-empty closed set $\{F_n\}$ with Diam $(F_n) \to 0$ as $n \to \infty$ has $\bigcap_{n=1}^{\infty} F_n$ as a singleton.

Proof : Let (X,d) be *a* complete metric space and let $\{F_n\}$ be *a* decreasing sequence of non-empty closed sets with Diam $(F_n) \to 0$ as $n \to \infty$. Take $u_n \in F_n$ $(n=1,2,....)$. Since $F_1 \supset F_2 \supset ... \supset F_n \supset F_{n+1} \supset ...$, for $m > n$, $u_m \in F_m \subset F_n$ and therefore $d(d_m, u_n) <$ Diam (F_n) (because $u_m, u_n \in F_n$). and hence as $n \to \infty$, (and $m \to \infty$),

$d(u_m, u_n) \to 0$ since Diam $(F_n) \to 0$ as $n \to \infty$.

Since role of m and n may be interchanged, we have $d(u_m, u_n) \to 0$ as $m,n \to \infty$; so $\{u_n\}$ is cauchy sequence in (X,d) which is complete, and Let $\lim_{n \to \infty} = u \in X$. Now $u_{n+p} \in F_{n+p} \subset F_n$ $(P = 1,2,...)$; As F_n is closed, $\lim_{p \to \infty} u_{n+p} = u \in F_n$. This is true for all n and $u \in \bigcap_{n=1}^{\infty} F_n$. If $v \in \bigcap_{n=1}^{\infty} F_n$, Then $u,v \in F_n$ for all n and $d(u,v) \le$ Diam $(F_n) \to 0$ as $n \to \infty$.

Thus $u = v$ and $\bigcap_{n=1}^{\infty} F_n$ is shown as *a* singleton.

Conversely, Suppose condition of Theorem is true. Take $\{x_n\}$ as *a* cauchy sequence in (X,d).

Now put $H_n = \{x_n, x_{n+1}, \ldots\}$; Then $\{H_n\}$ is *a* decreasing sequence of non-empty sets in X, and hence $\{\overline{H}_n\}$ is so. Given $\epsilon > 0$, as $\{x_n\}$ is cauchy, we find an index N satisfying $d(x_n, x_m) < \varepsilon$ when $n, m \geq N$. This gives Diam $(H_n) \leq \varepsilon$ when $n \geq N$.

or Diam $\{\overline{H}_n\}$ = Diam $(H_n) \leq \varepsilon$ when $n \geq N$.

That means $\lim_{n \to \infty}$ diam $\{\overline{H}_n\} = 0$; so $\{\overline{H}_n\}$ forms *a* decreasing sequence of non-empty closed sets with Diameters tending to zero, and hence we have $\bigcap_{n=1}^{\infty} \overline{H}_n$ consists of *a* single element $= u_0$ (say) $\in X$. As $x_n \in H_n \subset \overline{H}_n$ and $u_0 \in \overline{H}_n$ we have

$d(u_0, x_n) \leq$ Diam $(\overline{H}_n) \to 0$ as $n \to \infty$. Hence $\{x_n\}$ is convergent in (X,d) *i.e.*, (X,d) is complete and proof is complete.

Remark

(1) The condition that Diam $(F_n) \to 0$ as $n \to \infty$ can not be dropped in order to make Theorem 2.2.3 stand. For example, take $F_n = [n, \infty]$ as a closed set of reals with usual metric, where $F_1 \supset F_2 \supset \ldots$ and $F_n \neq \phi$, but Diam $(F_n) \to 0$. Here $\bigcap_{n=1}^{\infty} F_n = \phi$.

(2) Neither the condition that the sets F_n are closed can be dispensed with. For example, take $F_n = \left(0, \frac{1}{n}\right]$ for $n = 1,2,...$; then F_n are not closed, and Diam $(F_n) = \frac{1}{n} \to 0$ and $F_1 \supset F_2 \supset ...$; Here $\bigcap_{n=1}^{\infty} F_n = \phi$.

Definition 2.2.2 A subset B of *(X,d)* is said to be nowhere dense if Int (closure (B)) = ϕ; or equivalently given any open ball in X, inside which there is another open ball which free from points of B. For example (1) any finite set of reals with usual metric is a nowhere dense set, and (2) The set N of all natural numbers is a nowhere dense set in space reals with usual metric.

Definition 2.2.3 A subset in *a* metric space is called *a* set of first category if it is *a* countable union of nowhere dense set.

A subset in *a* metric space is called *a* set of second category if it is not *a* set of first category.

Example 2.2.4 The set of all rationals in real number space with usual metric is *a* set of first category. Because we know that the set of all rationals is *a* countable set, say, equal to $\{\gamma_1, \gamma_2, ...\gamma_n...\} = \{\gamma_1\} \cup \{\gamma_2\} \cup ...\{\gamma_n\} \cup ....$, which is countable union of singletons that are nowhere dense sets.

Example 2.2.5 The set R of all reals with usual metric is *a* second category. This statement is *a* special case of Theorem 2.2.4.

Example 2.2.6

(a) Every set of second category is everywhere dense in X.

(b) The complement of a set of first category in a complete metric space is a set of second category.

(c) Every subset of *a* set of first category is *a* set of first category in *a* metric space.

(d) Every set containing *a* set of second category is *a* set of second category in *a* metric space.

Theorem 2.2.4 Every complete metric space is of second catagory.

Proof : Since F_1 is nowhere dense, we find an open ball say $B_{r_1}(x_1)$ such that Since $B_{r_1}(x_1) \cap F_1 = \phi$. Since F_2 is nowhere dense there is an open ball $B_{r_2}(x_2)$ satisfying (i) $B_{r_2}(x_2) \subset B_{\frac{r_1}{2}}(x_1)$, (ii) $r_2 \leq \frac{r_1}{2}$ and (iii) $B_{r_2}(x_2) \cap F_2 = \phi$. .. (1)

Similarly as F_3 is nowhere dense there is an open ball $B_{r_3}(x_3)$ satisfying

(i) $B_{r_3}(x_3) \subset B_{\frac{r_2}{2}}(x_3)$, (ii) $r_3 \leq \frac{r_2}{2} \leq \frac{r_1}{2^2}$ and (iii) $B_{r_3}(x_3) \cap F_3 = \phi$. .. (2)

We continue this process to construct a sequence of open balls $\leq \frac{r_2}{2} \leq \frac{r_1}{2^2}$ satisfying

(i) $B_{r_n}(x_n) \subset B_{\frac{1}{2}r_{n-1}}(x_{n-1})$, (ii) $r_n \leq \frac{1}{2} r_{n-1} \leq \ldots \leq \frac{1}{2^{n-1}} r_1$

and (iii) $B_{r_n}(x_n) \cap F_n = \phi$

From (1), (2) and (3) it is clear that for any positive integer p, $d(x_{n+p}, x_n) < \frac{1}{2} r_n$ which $\rightarrow 0$ as $n \rightarrow \infty$, ... (4)

we show that $\{x_n\}$ is cauchy in X by completeness of which let $\lim_{n\rightarrow\infty} x_n = u \in X$.

From (4) we take $p \rightarrow \infty$, and we have $d(u, x_n) \leq \frac{1}{2} r_n < r_n$, and that shows $u \varepsilon$ $B_{r_n}(x_n)$ for $n = 1,2,\ldots$ As $B_{r_n}(x_n)$ is disjoint with F_n for $n = 1,2\ldots$ we see $u \varepsilon$ $(F_1 \cup F_2 \cup \ldots) = X$, which is not true. And proof is complete.

The following is an equivalent statement of Barie's category Theorem.

Theorem 2.2.5 If $\{G_n\}$ is *a* countable family of open and everywhere dense sets in *a* complete metric space *(X,d)*, then $\bigcap_{n=1}^{\infty} G_n$ is everywhere dense in X *i.e.*, $\overline{\bigcap_{n=1}^{\infty} G_n} = X$.

(Hint) Take $F_n = (F/G_n)$. Then F_n is closed and since G_n is everywhere dense, F_n becomes nowhere dense for $n = 1,2,...$ If $G = \bigcap_{n=1}^{\infty} G_n$, then $(X/G) = \bigcup_{n=1}^{\infty}(X/G_n) = \bigcup_{n=1}^{\infty} F_n$ which is a set of first category in X. Theorem 2.2.4. says X is of second category in X. and $X = G \cup (X/G)$ shows that G is of second category in X, and therefore G is everywhere dense.

The readers now try converse part.

Theorem 2.2.5 Let (X,d) be a complete metric space and $Y \subset X$. The subspace (Y,d_Y) is complete if and only if Y is a closed subset of (X,d).

Proof : First let Y be a closed subset of (X,d) which is complete metric space. With relativised metric d_Y (Y, d_Y) is a metric space and take $\{y_n\}$ as a cauchy sequence in (Y, d_Y). Then $\{y_n\}$ becomes a cauchy sequence in (X,d) and by completeness of (X,d), Let $\lim_n y_n = y \in X$. Since Y is closed in (X,d) it follows that $y \in Y$. Hence $\lim_n y_n = y \in Y$, and (Y, d_Y) is complete.

Conversely, Let (Y,d_Y) be complete and let $\{y_n\} \subset Y$ be a sequence with $\lim_n y_n = y$. Clearly $\{y_n\}$ is cauchy in (Y,d_Y) by completeness of which $\lim_n y_n = \in y$ or $y \in Y$. So Y is closed in (X,d)

§ 2.3

Definition 2.3.1. Two metrics d_1 and d_2 over same X are called equivalent if $\tau_{d_1} = \tau_{d_2}$ (*i.e.*, if metric Topologies τ_{d_1} and τ_{d_2} generated respectively by d_1 and d_2 are the same).

Explanation : We know that all open balls in metric space (X, d_1) constitute a base of the metric Topology τ_{d_1}. Hence $\tau_{d_1} = \tau_{d_2}$ if given any $x \in X$, any d_1 open ball centred at x contains a d_2 -open ball centred at x and vice-versa.

Theorem 2.3.1. If two metrics d_1 and d_2 over X satisfy

$a d_1(x,y) \leq d_2(x,y) < b\, d_1(x,y)$ for all $x, y \in X$ where a and b are fixed +ve reals, then d_1 and d_2 are equivalent metrics.

Proof : Let $x_0 \in X$. We show that $a\ d_1$ -open ball centred at $x_0 \subset a\ d_2$ -open ball centred at x_0, and vice-versa.

For a + *ve* real r we have following inclusion relations :

$d_1\text{-}B_r(x_0) \subset d_2 - B_{rb}\, B(x_0)$ and $d_2\text{-}B_r(x_0) \subset d_1\text{-}B_{r/a}(x_0)$

Therefore τ_{d_1} and τ_{d_2} shall be equal and proof is complete.

Example 2.3.1 If d_1 and d_2 are two metrics in Euclidean 2-space R^2 are given by

d_1 = usual metric for R^2

and $d_2\left(\underset{\sim}{x}, \underset{\sim}{y}\right) = \text{Max}\{|x_1 - y_1|, |x_2 - y_2|\}$, as $\underset{\sim}{x} = (x_1, x_2), \underset{\sim}{y} = (y_1, y_2) \in R^2$,

Then d_1 and d_2 are equivalent metrics in R^2.

Solution : We have $|x_1 - y_1|^2 \leq |x_1 - y_1|^2 + |x_2 - y_2|^2$

and $|x_2 - y_2|^2 \leq |x_1 - y_1|^2 + |x_2 - y_2|^2$

So $d_2\left(\underset{\sim}{x}, \underset{\sim}{y}\right) = \max\{|x_1 - y_1|, |x_2 - y_2|\} \leq \sqrt{\{|x_1 - y_1|^2 + |x_2 - y_2|^2\}} = d_1\left(\underset{\sim}{x}, \underset{\sim}{y}\right)$ (1)

Again $d_1^2\left(\underset{\sim}{x}, \underset{\sim}{y}\right) = |x_1 - y_1|^2 + |x_2 - y_2|^2$

$$\leq 2\left(\text{Max}\{|x_1 - y_1|, |x_2 - y_2|\}\right)^2$$

$$= 2d_2^2\left(\underset{\sim}{x}, \underset{\sim}{y}\right)$$

or, $d_1\left(\underset{\sim}{x}, \underset{\sim}{y}\right) \leq \sqrt{2}d_2\left(\underset{\sim}{x}, \underset{\sim}{y}\right)$

Combining (1) and (2) we have $d_2\left(\underset{\sim}{x}, \underset{\sim}{y}\right) \leq d_1\left(\underset{\sim}{x}, \underset{\sim}{y}\right) \leq \sqrt{2}d_2\left(\underset{\sim}{x}, \underset{\sim}{y}\right)$ for all $\underset{\sim}{x}, \underset{\sim}{y} \in R^2$. So Theorem 2.3.1 applies for desired conclusion.

Theorem 2.3.2 Given any metric space *(X,d)*. There is a metric *e* on X with *e(x,y)* ≤ 1 for all *(x,y)* $\in (X_x X)$ such that *d* and *e* are equivalent.

Proof : Put *e(x,y)* = min[1,*d(x,y)*] for all $x,y \in X$. It is a routine excercise to verify that (X,*e*) is a metric space. Clearly *e(x,y)* ≤ 1 for all $x,y \in X$.

Now given $x_0 \in X$, if $r > 0$, then we have following inclusion relations :

$d - B_r(x_0) \subset e - B_r(x_0)$

and $e - B_\delta(x_0) \subset d - B_r(x_0)$ Where $\delta = \text{Min}[1, r]$.

Thus $\zeta_d = \zeta_e$ and the metrics d and e are equivalent.

Example 2.3.2 If (X,d) is a metric space show that $\left(X, \frac{d}{1+d}\right)$ is a metric space and two metrics are equivalent.

Solution : It had shown before that $\left(X, \frac{d}{1+d}\right)$ is a metric space. We need showing $\zeta_d = \zeta_{d/1+d}$. For this take an element $x_0 \in X$, and r a + ve real. Put $e = \frac{d}{1+d}$. We have the following

$$d - B_r(x_0) = e - B_{r/1+r}(x_0)$$

Because $u \in d - B_r(x_0)$ if (if and only if)

$$d(x_0, u) < r \quad \text{if} \quad \frac{d(x_0,u)}{1+d(x_0,u)} < \frac{r}{1+r}$$

i.e., iff $\quad e(x_0,u) < \frac{r}{1+r}$

i.e., iff $\quad u \in e - B_{r/1+r}(x_0)$.

Thus $\zeta_d = \zeta_e = \zeta_{d/1+d}$ and metrics d and $\frac{d}{1+d}$ are equivalent.

Chapter II

Exercise (A)

Short answer type Questions

1. If $\{x_n\}$ is a cauchy sequence in a metric space show that $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$. Is the converse true?

2. In a metric space Diam (A) = Diam (B) for two subsets A and B. Is it true that A = B? Give reason for answer.

3. If X denotes the set of all positive integers and $\rho(m,n) = \left|\frac{1}{m} - \frac{1}{n}\right|$ for $m, n \in X$, show that (X, ρ) is a metric space and it is incomplete.

4. If O is an open set in a metric space, and $\overline{O}$ denotes its closure, show that $\overline{O}|O$ is a nowhere dense set.

5. If each of $F_1, F_2, \ldots$ is a set of first category in (X,d), show that $F = \bigcup_{n=1}^{\infty} F_n$ is a set of first category in (X,d).

6. Show that an isometric image of a complete metric space is complete.

7. If $\{x_n\}$ and $\{y_n\}$ are two cauchy sequences in a metric space (X,d), show that real sequence $\{d(x_n, y_n)\}$ is convergent.

8. Let (X,d) be a metric space and $x_0 \in X$ and $0 < r < R$. Show that the set $\{x \in X : r < d(x, x_0) < R\}$ is an open set in (X,d).

9. If d_1 and d_2 are two metrics on $X \neq \phi$, verify that $2d_1 + 3d_2$ is also a metric on X.

Excercise (B)

1. Show that the sequence $\left\{x_n = 1+\frac{1}{1!}+\frac{1}{2!}+\ldots+\frac{1}{n!}\right\}$ is a cauchy sequence in metric space of reals with usual metric and examine its convergence.

2. Show that $\left\{\left(1+\frac{1}{n}\right)^n\right\}$ is a convergent sequence in real number space with usual metric, and hence obtain $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{2n}$

3. Show that a closed set F in a metric space (X,d) is a nowhere dense set if and only if (X/F) is an everywhere dense set in (X,d).

4. If $\sigma(x,y) = \left|\tan^{-1}x - \tan^{-1}y\right|$ for all reals $x,y \in R$, show that (R,σ) is a metric space and it is not complete.

5. Obtain the closure of the set $\left\{(x,y): y=\sin\frac{1}{x} \text{ and } 0<x\le 1\right\}$ in R^2 with usual metric.

6. If R^n is Enclidean n-space with two metrics d_0 and d_1 where

$$d_0\left(\underset{\sim}{x},\underset{\sim}{y}\right) = \underset{1\le i\le n}{\text{Max}}\left\{\left|x_i - y_i\right|\right\}, \text{ and}$$

$d_1\left(\underset{\sim}{x},\underset{\sim}{y}\right) = \sum_{i=1}^{n}\left|x_i - y_i\right|$, $\underset{\sim}{x} = (x_1, x_2, \ldots x_n)$ and $\underset{\sim}{y}(y_1, y_2, \ldots y_n) \in R^n$, Show that $d_0\left(\underset{\sim}{x},\underset{\sim}{y}\right) \le d_1\left(\underset{\sim}{x},\underset{\sim}{y}\right) \le n\ d_0\left(\underset{\sim}{x},\underset{\sim}{y}\right)$. Hence verify that $\zeta_{d_0} = \zeta_{d_1}$

7. If two metric ρ and ρ^* on $X(\neq \phi)$ are equivalent, show that *a* cauchy sequence in (X,ρ) is *a* cauchy sequence in (X,ρ^*) and vice-versa.

8. Let X denote the collection of all real sequences $\underset{\sim}{x} = (x_1, x_2, \cdots)$, $x_i \in R$ with only a finite number of terms of $\underset{\sim}{x}$ being non-zero, and let $\rho\left(\underset{\sim}{x}, \underset{\sim}{y}\right) = \left(\sum_{n=1}^{\infty} |x_i - y_i|^2\right)^{\frac{1}{2}}$ where $\underset{\sim}{x} = (x_1, x_2, \cdots)$, $\underset{\sim}{y} = (y_1, y_2, \cdots) \in X$. shbw that (X,ρ) is *a* metric space and examine if it is complete.

Unit 7 ❒ Continuous Functions over Metric Spaces, Uniform Continuity, Contraction Theorem, Werstrom approximation Theorem

(Continuous functions over metric spaces, uniform continuity, Banach Contraction principle. Its application as Picards Theorem in O.D.E.. Distance of a point from a set, its properties, distance between two sets, Normality, Completion of a metric space, Weirstrass approximation Theorem.)

§ 3.1 Let (X,d) and (Y,ρ) be two metric spaces.

Definition 3.1.1 A function $f:(X,d)\to(Y,\rho)$ is said to be continuous at a point of $x_0\in X$ if given a +ve ϵ, there is a +ve δ such that

$\rho(f(x), f(x_0))<\varepsilon$ whenever $d(x,x_0)<\delta$.

Explanation : In Definition 3.1.1 δ depends on x_0 and ε. And $\rho(f(x), f(x_0))<\varepsilon$ whenever $d(x,x_0)<\delta$ may be restated as $f(x)\in B_\epsilon(f(x_0))$ in (Y,ρ) as $x\in B_\delta(x_0)$, where $B_\delta(x_0)$ denotes open ball centred at x_0 with radius $=\delta$ in (X,d). or, equivalently, $f(B_\delta(x_0))\subset B_\epsilon(f(x_0))$. or equivalently $B_\delta(x_0)\subset f^{-1}(B_\epsilon(f(x_0)))$. Further if f is a real-valued function of real variable *i.e.*, if $f:R\to R$ where R is the metric space of reals with usual metric, then usual (ε,δ) Definition of Continuity of f at $x_0\in R$ *i.e.*, $|f(x)-f(x_0)|<\varepsilon$ whenever $|x-x_0|<\delta$ is in agreement with Definition 3.1.1 above. Also if (X,d) is a discrete metric space then any function $f:(X,d)\to(Y,\rho)$ becomes continuous at any point of X.

Definition 3.1.2 A function $f:(X,d)\to(Y,\rho)$ is said to be continuous function (over the whole space X) if f is continuous at each point of X.

Theorem 3.1.1 A function $f:(X,d)\to(Y,\rho)$ is continuous if and only if inverse image of an open set (a closed set) in (Y,ρ) under f is an open set (a closed set) in (X,d).

Proof : Let $f:(X,d)\rightarrow(Y,\rho)$ be a continuous function, and take G to be an open set in (Y,ρ) and take $u\in f^{-1}(G)\subset X$; So $f(u)\in G$ and since G is open in Y, we find an open ball, say $B_{\varepsilon}(f(u))$ in (Y,ρ) with $B_{\delta}(u)$ in (X,d) such that

$$f(B_{\delta}(u))\subset B_{\varepsilon}(f(u))\subset G$$

or, $$B_{\delta}(u)\subset f^{-1}(G)$$

That means u is an interior point of $f^{-1}(G)$; and therefore $f^{-1}(G)$ is open in X because u is an arbitrary member of it.

Conversely, Let f^{-1}(an open set in Y) be an open set in X. If $x_0\in X$, and $f(x_0)\in Y$; Given a +ve ε, consider the open ball $B_{\varepsilon}(f(x_0))$ which is also an open set in (Y,ρ), and by supposed condition $f^{-1}(B_{\varepsilon}(f(x_0)))$ is an open set in (X,d) with $x_0\in f^{-1}(B_{\varepsilon}(f(x_0)))$. Thus there is an open ball say, $B_{\delta}(x_0)$ in X satisfying,

$$B_{\delta}(x_0)\subset f^{-1}(B_{\varepsilon}(f(x_0)))$$

or, $$f(B_{\delta}(x_0))\subset B_{\varepsilon}(f(x_0)).$$

That means f is continuous at x_0, and choice of x_0 in X being arbitrary f is a continuous function over X.

Finally, note that if F is a closed set in (Y,ρ) then Y/F is an open set in Y, and we have for continuous function f,

$f^{-1}(Y\setminus F)$ is an open set in X or, $X\setminus f^{-1}(F)$ is an open set X

or, $f^{-1}(F)$ is an closed set in X; and

Converse holds as a routine exercise. So Theorem is proved completely.

Theorem 3.1.2 A function $f:(X,d)\to(Y,\rho)$ is continuous at a point $x_0\in X$ if and only if $\{f(x_n)\}$ is convergent with $\lim_n f(x_n)=f(x_0)$ in (Y,ρ) whenever $\{x_n\}$ is a convergent sequence in (X,d) with $\lim_n x_n=x_0$.

The proof is easy and left to readers.

Theorem 3.1.3 If $f:(X,d)\to(Y,\rho)$ is a continuous function, when following statements are equivalent :

(a) f is continuous

(b) $f^{-1}(G)$ is an open set in (X,d) for any open set G in (Y,ρ)

(c) $f^{-1}(F)$ is a closed set in (X,d) for any closed set F in (Y,ρ).

(d) $\overline{f^{-1}(B)}\subset f^{-1}(\overline{B})$, for any subset B of (Y,ρ)

(e) $f(A)\subset\overline{f(A)}$, for any subset A of (X,d), bar denoting the closure.

Proof : Parts (a), (b) and (c) are equivalent. Suppose (c) holds; Take any subset B of (Y,ρ); Then $\overline{B}$ is a closed set in (Y,ρ) and by (c) we have $f^{-1}(\overline{B})$ is a closed set in (X,d); Since $B\subset\overline{B}$, we have $f^{-1}(B)\subset f^{-1}(\overline{B})$

This gives $\overline{f^{-}(B)}\subset\overline{f^{-1}(\overline{B})}=f^{-1}(\overline{B})$ because $f^{-1}(\overline{B})$ is closed. This is part (*d*).

Now assume (*d*), and suppose A is a subset of (X,d). Let us put $f(A) = B$. So $A \subset f^{-1}(B)$ and hence $\overline{A} \subset \overline{f^{-1}(B)} \subset f^{-1}(\overline{B})$ (by (d)). That means $f(\overline{A}) \subset \overline{B} = \overline{f(A)}$ which is (e). Finally assume (e) and take G as a closed set in (Y,ρ). So $f^{-1}(G) \subset X$. From (d) $f\left(\overline{f^{-1}(G)}\right) \subset \overline{ff^{-1}(G)} = \overline{G} = G$ because G is closed. So, $\overline{f^{-1}(G)} \subset f^{-1}(G)$; That means $f^{-1}(G)$ is a closed set in X *i.e.*, inverse image of a closed set in (Y,ρ) under f is a closed set in (X,d). So f is continuous and the cycle of implication is complete.

Example 3.1.1 Let f and $g:(X,d)\to(Y,\rho)$ be continuous functions, then the set $\{x \in X: f(x) \neq g(x)\}$ is an open set in X.

Solution : Let $G = \{x \in X: f(x) \neq g(x)\}$ where f and g are continuous functions : $X \to Y$. If $u \in G$, we find $f(u) \neq g(u)$ in (Y,ρ) which is a Hausdorff space. So there are open sets P and Q in (Y,ρ) containing $f(u)$ and $g(u)$ respectively such that

$$P \cap Q = \phi \qquad (1)$$

Clearly, $u \in f^{-1}(P) \cap g^{-1}(Q)$ where $f^{-1}(P)$ and $g^{-1}(Q)$ are open sets in (X,d) by continuity of f and g. Now $f^{-1}(P) \cap g^{-1}(Q)$ is an open set containing u in (X,d) and hence we find an open ball, say, $B_\varepsilon(u) \subset f^{-1}(P) \cap g^{-1}(Q)$; Further $v \in B_\varepsilon(u)$ implies $f(v) \in P$ and $g(v) \in Q$ and from (1) we see $f(v) \neq g(v)$; Hence $B_\varepsilon(u) \subset G$ *i.e.*, u is an interior point of G. Thus G is shown as an open set in X.

Remark 1 The set $\{x \in X: f(x) = g(x)\}$ is a closed set in (X,d).

Definition 3.1.3 A function $f:(X,d)\to(Y,\rho)$ is said to be a homeomorphism if f is 1–1 and onto (bijective) and that f and f^{-1} are continuous.

If $f:(X,d)\to(Y,\rho)$ is a homeomorphism, then metric spaces (X,d) and (Y,ρ) are said to be homeomorphic.

Example 3.1.2 Let $X = [0,1]$ and $Y=[a,b](a<b)$ be closed intervals of reals with usual metric d of reals, then (X,d) and (Y,d) are homeomorphic.

Solution : Let $f:X\to Y$ be a function where

$$f(x) = a + x\,(b-a); \qquad 0\le x\le 1.$$

Then by routine check up we see that f is bijective and that f and f^{-1} are both continuous; f^{-1} being given by $f^{-1}(x) = (x-a)/(b-a)$; $a\le x\le b$. Thus f is a homeomorphism; So (X,d) and (Y,d) homeomorphic.

Definition 3.1.4 A function $\phi:(X,d)\to(Y,\rho)$ is said to be an Isometry if $\rho(\phi(u),\phi(v)) = d(u,v)$ for all $u,v\in X$.

If $\phi:(X,d)\to(Y,\rho)$ is an Isometry, then metric spaces (X,d) and (Y,ρ) are called isometric.

Remark : An isometry is a homeomorphism.

Example 3.1.3. A homeomorphic image of a complete metric space may not be a complete metric space.

Take $X = \{1,2,3,\cdots,n,\cdots\}$ and $Y=\left\{1,\frac{1}{2},\frac{1}{3},\ldots,\frac{1}{n},\ldots\right\}$ as metric spaces with usual metric of reals. Here $f: X\to Y$ given by $f(n)=\frac{1}{n}$; $n=1,2,\cdots$ is homeomorphism; and X being a closed set of reals is complete; but Y is not so.

Remark 2 If two continuous functions :$(X,d) \to (Y,\rho)$ agree over a dense set in (X,d), then they become identical.

§ 3.2

Definition 3.2.1 A function $f:(X,d) \to (Y,\rho)$ is said to be uniformly continuous if given a +ve ϵ, we find a +ve δ such that $\rho(f(x), f(x')) < \varepsilon$ whenever $d(x, x') < \delta$, $x, x \in = (X,d)$.

Explanation : In case of uniform continuity of a function +ve δ depends only on ε. Uniform continuity of f implies its continuity. Because if $x_0 \in X$, taking $x' = x_0 \in X$, Definition of continuity of f at x_0 follows from Definition 3.2.1, namely $\rho(f(x), f(x_0)) < \varepsilon$ whenever $d(x,x_0) < \delta$. However, converse is false. Take $f:(0, 1] \to (0,\infty)$ as $f(x) = \frac{1}{x}$ in $0 < x \leq 1$. This function f is a continuous function without being uniformly continuous therein; Because, Suppose the contrary and take $\varepsilon = 1$; if there is a +ve δ as wanted in Uniform continuity, choose natural number n so large that $\frac{1}{n(n+2)} < \delta/2$. Taking $x = \frac{1}{n}$ and $x' = \frac{1}{n+2}$ we find x,x' in $(0,1]$ such that $|x - x'| = \frac{2}{n(n+2)} < \delta$, and $|f(x) - f(x')| = |n - (n+2)| > 1 = \varepsilon$ contradiction. Hence f fails to be uniformly continuous in $(0,1]$.

Theorem 3.2.1 Every uniformly continuous function transforms a Cauchy sequence into a Cauchy sequence.

Proof : Let (X,d) and (Y,ρ) be metric spaces and $f:X \to Y$ be a uniformly continuous function, and Let $\{x_n\}$ be cauchy sequence in (X,d). Given $\varepsilon > 0$, there is a positive δ such that $\rho(f(x), f(x')) < \varepsilon$ whenever $d(x,x') < \delta$ by uniform continuity of f. By cauchyness of $\{x_n\}$ we find an index K satisfying

$d(x_n, x_m) < \delta$ whenever $n, m \geq K$.

And therefore $\rho(f(x_n)), f(x_m)) < \varepsilon$ for $n, m \geq K$.

That means $\{f(x_n)\}$ is Cauchy in (Y, ρ):

Remark : (a) Every uniformly continuous function sends a convergent sequence into a convergent sequence.

Because every uniformly continuous function f is a continuous function and sequential continuity property of f gives the result.

(b) A continuous function may however fail to transform a Cauchy sequence into a similar such sequence.

For example, take $X = (0, 1]$ and $f:(0, 1] \to$ Reals with associated metric as usual metric of reals, such that $f(x) = \frac{1}{x}$ as $0 < x \leq 1$. Then f is a continuous function without being uniformly continuous, and $f\left(\frac{1}{n}\right)$ for $n = 1, 2, \cdots$; Here $\left\{\frac{1}{n}\right\}$ is a cauchy sequence in X whereas $\left\{f\left(\frac{1}{n}\right) = n\right\}$ is. not a cauchy sequence of reals.

§ 3.3 Let A be a non-empty subset of (X, d) and $x \in X$.

Definition 3.3.1 Distance of x from A, denoted by $\text{dist}(x, A) = \inf\{d(x, a) : a \in A\}$.

Here, $0 \leq \text{dist}(x, A) < \infty$.

Remark : Since infimum is taken over a set of non-negative reals, inf. value exists; x may or may not be a member of A.

Theorem 3.3.1 For any subset $A \subset X$ and $x \in X$ $\text{dist}(x, A) = \text{dist}(x, \overline{A})$, bar denoting the closure.

Proof : Since $A \subset \overline{A}$, we have

$\inf\{d(x,a):a\in\overline{A}\}\le\inf\{d(x,a):a\in A\}$ or, $\text{dist}(x,\overline{A})\le\text{dist}(x,A)$

Let ε be an arbitrary +ve number; if $a\in\overline{A}$, we find $u\in A$ satisfying $d(u,a)<\varepsilon$.

Now $\text{dist}(x,A) = \inf\{d(x,w): w\in A\}\le d(x,u)$

$\le d(x,a) + d(a,u) < \varepsilon + d(x,a)$

i.e., $\text{dist}(x,A) < \varepsilon + d(x,a)$, where a is any member of $\overline{A}$, and hence we have $\text{dist}(x,A)\le\varepsilon + \text{dist}(x,\overline{A})$, where $\varepsilon>0$ is arbitrary positive. This gives

$$\text{dist}(x,A)\le\text{dist}(x,\overline{A}) \qquad (2)$$

Combining (1) and (2) we have $\text{dist}(x,A) = \text{dist}(x,\overline{A})$.

Theorem 3.3.2 Given $A(\neq\phi)\subset X$, $\text{dist}(x,A)$ is a continuous function of $x\in X$.

Proof : Let $x,y\in X$. Given $\epsilon>0$, we find $a\in A$ such that,

$$d(y,a)<\text{dist}(y,A)+\varepsilon.$$

Now, $\text{dist}(x,A)\le d(x,a)\le d(x,y) + d(y,a) < d(x,y) + \varepsilon + \text{dist}(y,A)$

or, $\text{dist}(x,A)< \text{dist}(y,A) + d(x,y)+\varepsilon$.

Here $\varepsilon>0$ is arbitrary, and this implies,

$$\text{dist}(x,A)\le\text{dist}(y,A)+ d(x,y)$$

or, $\text{dist}(x,A) - \text{dist}(y,A)\le d(x,y)$ (1)

Interchanging x and y we similarly obtain

$$\text{dist}(y,A) - \text{dist}(x,A)\le d(y,x) = d(x,y) \qquad (2)$$

Combining (1) and (2) we get

$$|\text{dist}(y,A) - \text{dist}(y,A)|\le d(x,y)$$

This inequality shows if $\{x_n\}$ is a sequence in (X,d) with $\lim_n x_n= x$ then taking

x_n in place of y, we have $\lim_n$ dist(x_n,A) = dist (x,A). The proof is now complete.

Theorem 3.3.3 Given A⊂X, $u\in\overline{A}$ if and only if dist (u,A) = 0.

Proof : Let dist (u,A) = 0; if $u\in A$, then $u\in\overline{A}$ and we are done. Suppose $u\notin A$. Given a +ve ε, we find a member $a\in A$ such that $d(u,a)<\varepsilon$ or, $a\in B_\varepsilon(u)$

or, $B_\varepsilon(u)\cap A\neq\phi$.

That means u is a limit point of A and $u\in\overline{A}$. Converse part follows by a similar argument.

Theorem 3.3.4 If A and B are two disjoint closed sets in a metric space (X,d), there is a continuous function $f:(X,d)\to[0,1]$ satisfying

$f(a)=0$ for all $a\in A$ and $f(b)=0$ for all $b\in B$.

Proof : Let $f:(X,d)\to[0,1]$ be defined by

$$f(x)=\frac{\text{dist}(x,A)}{\text{dist}(x,A)+\text{dist}(x,B)} \text{ for } x\in X.$$

Since A and B are closed disjoint sets in X, we have denominator

dist (x,A) + dist (x,B) > 0 for all $x\in X$. Further f is continuous with $0\le f(x)\le 1$ for $x\in X$. Now if $x\in A$, we have dist(x,A) = 0, while dist(x,B)>0. So $f(x)=0$ and if $x\in B$, dist (x,B) = 0 while dist (x,A)>0; So $f(x)=1$. Thus $f(x)=0$ for all $x\in A$ and $f(x)=1$ for all $x\in B$.

Remark : This property of a metric space as in Theorem says that a metric space is a Normal space.

Definition 3.3.2 Distance between two sets A and B, denoted by dist(A,B)= inf $\{d(a,b):a\in A,\ b\in B\}$ in a metric space (X,d).

Clearly, dist (A,B) = dist (B,A)≥0.

Remark : (a) If $A \cap B \neq \phi$, then dist(A,B) = 0

(b) If $A \cap B = \phi$, then dist(A,B) may or may not be zero. For example, Take $A = (0,1)$ and $B = (1,2)$ as two disjoint open intervals; here dist(A,B) = 0, the metric being taken as that of reals.

Again taking $A = (0,1)$ and $B = (2,3)$ as two disjoint open intervals with usual metric of reals, we have dist(A,B) > 0.

Theorem 3.3.5 If A and B are two subsets of (X,d), then $\text{dist}(\overline{A},\overline{B}) = \text{dist}(A,B)$, bar denoting the closure.

Proof : We have $A \subset \overline{A}$ and $B \subset \overline{B}$, So that

$\inf\{d(u,v): u \in \overline{A},\ v \in \overline{B}\} \leq \inf\{d(a,b): a \in A, b \in B\}$.

or, $$\text{dist}(\overline{A},\overline{B}) \leq \text{dist}(A,B) \qquad (1)$$

Let $\varepsilon > 0$ be given. There are members $u \in \overline{A}$ and $v \in \overline{B}$

satisfying $$d(u,v) < \text{dist}(\overline{A},\overline{B}) + \varepsilon/2 \qquad (2)$$

Again we find $a \in A$ and $b \in B$ such that

$d(u,a) < \varepsilon/4$ and $d(v,b) < \varepsilon/4$. So that

$d(a,b) \leq d(a,u) + d(u,v) + d(v,b) < d(u,v) + \varepsilon/4 + \varepsilon/4$

$< \text{dist}(\overline{A},\overline{B}) + \varepsilon/2 + \varepsilon/2 = \text{dist}(\overline{A},\overline{B}) + \varepsilon$. (by (2)).

Now $\text{dist}(A,B) < d(a,b)$

$< \text{dist}(\overline{A},\overline{B}) + \varepsilon$

Since $\varepsilon > 0$ is arbitrary, we have

$\text{dist}(A,B) \leq \text{dist}(A,B)$ (3)

Combining (1) and (3) we have $\text{dist}(\overline{A},\overline{B}) = \text{dist}(A,B)$.

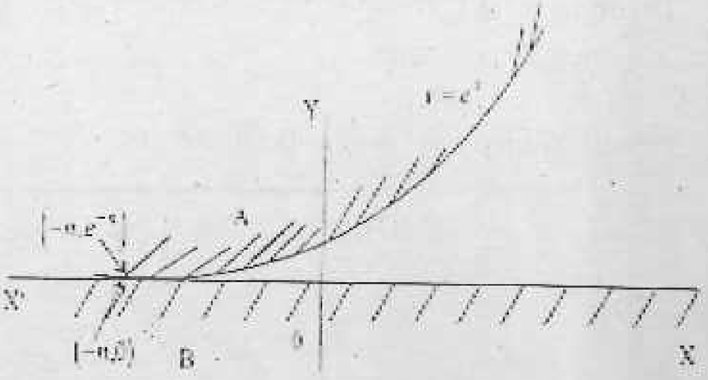
Corollary : If $\overline{A} \cap \overline{B} \neq \phi$, then dist(A,B) = 0

Because $\overline{A} \cap \overline{B} \neq \phi$, implies $\text{dist}(\overline{A},\overline{B}) = 0$ and Theorem 3.3.5 applies to give dist (A,B) = 0.

However converse is not true.

Example 3.3.1 Take A and B as subsets of R^2 as $A = \{(x,y): y > e^x\}$,

$B = \{(x,y): y \leq 0\}$. Here each A and B is a closed set in R^2 and $A \cap B = \overline{A} \cap \overline{B} = \phi$;



But

$\text{dist}(A,B) \leq \sqrt{(-n-(-n))^2 + (0-e^{-n})^2} = \sqrt{e^{-2n}} = e^{-n} \to 0$ as $n \to \infty$ and hence dist (A,B)=0

§ 3.4

Definition 3.4.1 Let (X,d) be a metric space. A function T:X→itself is said to be a contraction if there is a real α such that $d(T(x),T(y)) \leq \alpha d(x,y)$ and $0<\alpha<1$, for all $x, y \in X$.

Example 3.4.1 A function f:R→R, where R is the metric space of reals with usual metric defined by $f(x) = \frac{1}{2}x$ for all $x \in R$, is a contraction.

Theoerem 3.4.1 Every contraction in a metric space is a uniformly continuous function.

Proof : Let $T:(X,d)\to$ itself be a contraction, and $\varepsilon > 0$ be given. Let us take a +ve δ to satisfy $\delta < \frac{\varepsilon}{2\alpha}$. Now if $d(x, y) < \delta$,

we have $d(T(x),T(y)) \le \alpha d(x,y) < \alpha\delta < \varepsilon/2 < \varepsilon$. Hence T is uniformly continuous over (X,d).

Corollary : Every contraction in a metric space is a continuous function.

Theorem 3.4.2 (Banach contraction Principle) :

Let T be a contraction in a complete metric space (X,d). Then there is a unique element $u \in X$, such that $T(u) = u$.

Proof : Let a contraction $T:(X,d)\to$ itself satisfy $d(T(x),T(y)) \le \alpha d(x,y)$, for $0 < \alpha < 1$ and for all $x, y \in X$. Take any point, say, $x_0 \in X$, and Let the sequence $\{x_n\}$ with $x_n = T(x_{n-1})$ for $n = 1,2,\cdots$ be defined by induction. We show $\{x_n\}$ to be a cauchy sequence in (X,d). We have $d(x_2,x_1) = d(T(x_1),T(x_0)) \le \alpha d(x_1,x_0) = \alpha d(T(x_0),x_0)$.

and $\quad d(x_3,x_2) = d(T(x_2),T(x_1)) \le \alpha^1 d(x_2,x_1) \le \alpha^2 (T(x_0),x_0)$.

and, in general $d(x_{k+1}, x_k) \le \alpha d(x_k,x_{k-1}) \le \alpha^2 d(x_{k-1},x_{k-2})$

$\le \cdots \le \alpha^k d(T(x_0), x_0)$. (1)

Let m,n be +ve integers with $m>n$. We have

$d(x_m, x_n) \le d(x_m, x_{m-1}) + d(x_{m-1}, x_{m-2}) + \cdots + d(x_{m+1}, x_n)$

$\leq(\alpha^{m-1}+\alpha^{m-2}+\cdots+\alpha^{n})d(T(x_0),x_0)$ from (1)

$=\alpha^{n}(\alpha^{m-n-1}+\alpha^{m-n-2}+\cdots+1)d(T(x_0),x_0)$

$$<\frac{\alpha^{n}}{1-\alpha}d(T(x_0),x_0)\rightarrow u \text{ as } \alpha^{n}\rightarrow 0 \text{ as } n\rightarrow\infty,$$

because α is a +ve fraction.

In case $n>m$, by similar argument the same conclusion is reached, and therefore $d(x_m,x_n)\to 0$ as $m, n\to\infty$. Thus $\{x_n\}$ is a cauchy sequence in (X,d) which is complete. So $\{x_n\}$ is convergent in (X,d); Let $\lim_n x_n = u\in X$. By continuity of T we have,

$$\lim_n T(x_n) = T(u) \quad \text{or,} \quad \lim_n x_{n+1} = T(u) \quad \text{or,} \quad u = T(u).$$

For uniqueness of u, let $v\in X$ be such that $T(v) = v$.

Then $d(u,v) = d(T(u),T(v))\leq\alpha(u,v)$; As $0<\alpha<1$, we have $d(u,v)=0$ *i.e.*, $u = v$ and proof is complete.

Remark 1 Because $T(u)=u$ with u as a unique element of X, shows that u is a unique fixed point under T, and also for each +ve integral index K.T^K is a contraction in (X,d) with same unique fixed point as that of T.

Remark 2 Proof of Theorem 3.4.2 describes existence, determination and uniqueness of fixed point of a contraction in (X,d).

Theorem 3.4.3 Let T be a continuous function of a complete metric space (X,d) into itself and for some +ve integer K, T^K be contraction. Then T has unique fixed point $u\in X$.

Proof : By Theorem 3.4.3 T^K has a unique fixed point u in X with $\lim_n (TK)^n (x_0)=u$, $x_0\in X$ being any member.

So. $u=\lim_n (TK)^n (T(x_0))$

$= \lim_n T_{(TK)^n}(x_0)$

$= T\left(\lim_n (T^K)^n(x_0)\right)$ by continuity of T.

$= T(u)$.

And uniqueness of u follows from the fact that each fixed point of T is also a fixed point of T^K.

Theorem 3.4.4 (Picard's Theorem on the existence of solution of ordinary Differential Equation).

Let D be an open set in R^2 with $(x_0,y_0) \in D$. Let f be a real-valued continuous function in D, and satisfy a Lipbschitz condition :
$|f(x,y_1) - f(x,y_2)| < M|x_1 - y_2|$ when $(x_1, y_1), (x_2,y_2) \in D$.
Then there exists a $t > 0$, and a function ϕ continuous and differentiable in the closed
interval $[x_0 - t, x_0 + t]$ such that (i) $\phi(x_0) = y_0$, and (ii) $y = \phi(x)$ satisfies the Differential Equation.

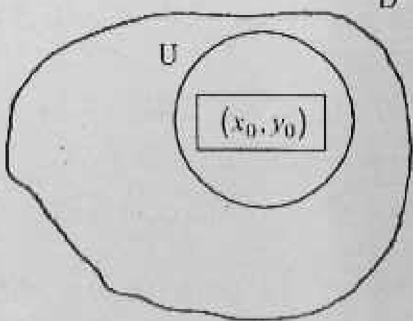
$$\frac{dy}{dx} = f(x,y) \qquad \text{for } x \in [x_0 - t, x_0 + t]$$

Proof: It suffices to show ϕ continuous in $[x_0 - t, x_0 + t]$ satisfies

$$\phi(x) = y_0 + \int_{x_0}^{x} f(u, \phi(u))du \quad (x_0 - t \le x \le x_0 + t) \text{ with}$$

$f(x,\phi(x)) - \in D \ (x_0 t \le x \le x_0 + t)$.

Let U denote a closed circular disc centred at (x_0,y_0), with radius +ve, and $U \subset D$.



Let $m = \sup_{(x,y)\in U} |f(x,y)|$; since f is continuous and U is a closed and bounded, m is finite.

Now choose t and δ such that

(i) $0 < t \le \frac{1}{M}$ and

(ii) The closed rectangle $\{(x,y):|x - x_0|\leq t, |y - y_0|<\delta\}$ is contained in U; and

(iii) $mt<\delta$.

We know that the collection $C_R[x_0- t, x_0 + t]$ of all real-valued continuous functions over the closed-interval $[x_0- t, x_0+t]$ is a complete metric space with respect to sup metric.

Denote $E = \{\phi\in C_R[x_0 - t, x_0+t]:\phi:[x_0- t, x_0+ t]\to(y_0-\delta,y_0+\delta)$ is continuous$\}$. Then it is easy to check that E is a closed set in $C_R[x_0- t, x_0+ t]$, and hence E is a complete metric space with sup metric.

Define a function T over E by saying $T(\phi) = \psi$ for $\phi\in E$ where,

$$\psi(x) = y_0 + \int_{x_0}^{x} f(u,\phi(u))\,du$$

Then ψ is continuous in $[x_0 - t, x_0 +t]$ such that

$|\psi(x) - y_0| \leq \sup\limits_{x_0-t\leq u\leq x_0+t} |f(u,\phi(u))|\int_{x_0}^{x} dt \leq m.t < \delta$ by choice of δ.

Hence T Transforms members of E into themselves, *i.e.* $T : E\to E$; Further, we verify that T is a contraction.

Let $\phi_1,\phi_2\in E$ with $\psi_1 = T(\phi_1)$ and $\psi_2 = T(\phi_2)$. So,

$$|\psi_1(x) - \psi_2(x)| = \left|\int_{x_0}^{x} \{f(u,\phi_1(u)) - f(u,\phi_2(u))\}du\right| \leq M. \sup_{\in[x_0-t,x_0+t]} \left|\phi_1(u) - \phi_2(u)\right| \times \int_{x_0}^{x} dt$$

$< t\, Md(\phi_1,\phi_2)$, where d denotes sup metric.

Taking sup in L.H.S. we set

$d(\psi_1,\psi_2)< t\, Md(\phi_1,\phi_2)$

or, $\quad d(T(\phi_1),T(\phi_2))< t\, Md(\phi_1,\phi_2), \qquad 0< t\, M<1$

Hence verification is done to show that T is a contraction on E.

By Contraction Principle there is member $\phi \in E$ with $T(\phi) = \phi$ *i.e.*,

$$\phi(x) = y_0 + \int_{x_0}^{x} f(u, \phi(u))du \text{ in } x_0 - t \leq u \leq x_0 + t.$$

Example 3.4.2 Let $X = \{x \in R : x \geq 1\}$ and function $T : X \to X$ be defined as, $T(x) = \frac{x}{2} + \frac{1}{x}$ for $x \in X$. Show that T is a contraction and obtain the fixed point under T.

Solution : Here X is a complete metric space with usual metric of reals.

Let $x, y \in X$; Then $|T(x) - T(y)| = \left|\frac{x}{2} + \frac{1}{x} - \frac{y}{2} - \frac{1}{y}\right| = \left|\frac{1}{2}(x-y) - \left(\frac{1}{y} - \frac{1}{x}\right)\right|$

$= \left|\frac{1}{2}(x-y) - \frac{1}{xy}(x-y)\right| = \left|(x-y)\left(\frac{1}{2} - \frac{1}{xy}\right)\right| = \left|\frac{1}{2} - \frac{1}{xy}\right| |x-y|$; As $x \geq 1$ and $y \geq 1$, we have $\frac{1}{xy} \leq 1$; So $\frac{1}{xy}$ lies either between 0 and $\frac{1}{2}$ or between $\frac{1}{2}$ and 1 and in each of these cases $\left|\frac{1}{2} - \frac{1}{xy}\right| \leq \frac{1}{2}$ Hence $|T(x) - T(y)| \leq \frac{1}{2}|x - y|$. So T is a contraction and Banach Contraction Principle says that T has a unique fixed point u given by $u = \frac{u}{2} + \frac{1}{u}$ or $u^2 = 2$ or $u = \sqrt{2}$.

Example 3.4.3 Let $[a,b]$ be a closed interval and $Q : [a,b] \to [a,b]$ be a function with derivative $Q'(x)$ satisfying $|Q'(x)| < \alpha < 1$ in $a \leq x \leq b$. (derivative at end points to be taken unilateral derivative), show that $Q(x) - x = 0$ has exactly one root in $[a,b]$.

Solution : Here $[a,b]$ is a complete metric space with usual metric of reals. If $x_1, x_2 \in [a,b]$ with $x_1 < x_2$ we have by Mean-Value Theorem of Differential Calculus, $Q(x_2) - Q(x_1) = (x_2 - x_1)\, Q'(u)$ for some u between x_1 and x_2. Hence

$|Q(x_1) - Q(x_2)| = |Q'(u)||(x_1 - x_2| \leq \alpha|x_1 - x_2|$ where $0<\alpha<1$. So Q is a contraction and Banach Contraction Principle says that there is exactly one member $u\in[a,b]$ satisfying $Q(u) = u$ or $Q(u) - u = 0$ or $Q(x) - x = 0$ has exactly one root in $[a,b]$.

Definitions 3.4.2 A complete metric space (X^*, d^*) is said to be a completion of a given metric space (X,d) if (X,d) is isometric to a dense sub-space of (X^*, d^*)

Theorem 3.4.5 Every metric space has a completion.

Proof : Let (X,d) be a given metric space. If it is complete it is its own completion. So assume (X,d) to be incomplete. We define a binary relation of 'being equivalent' among family ζ of all cauchy sequences in (X,d). Two cauchy sequences $\{x_n\}$ and $\{x'_n\}$ in X are said to be equivalent if

$$\lim_n d(x_n, x'_n) = 0$$

This binary relation on ζ is an equivalence relation which partitions ζ into disjoint equivalent classes of cauchy sequences in X. If X^* denotes the set of all representative members, say, $x^*, y^*, \cdots$ of equivalent classes of cauchy sequences in (X,d), we are now ready to define a metric d^* on X^* where $d^*(x^*, y^*) = \lim_n d(x_n, y_n)$ where $\{x_n\}\in x^*$, $\{y_n\}\in y^*$. To this end we must be satisfied that (i) r.h.s. limit exists and (ii) it is independent of choice of member sequence $\{x_n\}, \{y_n\}$.

(i) Here $d(x_n, y_n) \leq d(x_n, x_m) + d(x_m, y_m) + d(y_m, y_n)$

or $$d(x_n, y_n) - d(x_m, y_m) < d(x_n, x_m) + d(y_m, y_n) \qquad \cdots \quad (1)$$

interchanging n and m we have

$$d(x_m, y_m) - d(x_n, y_n) \leq d(x_m, x_n) + d(y_n, y_m) \qquad \cdots \quad (2)$$

From (1) and (2) we get

$$|d(x_m, y_m) - d(x_n, y_n)| \leq d(x_n, x_m) + d(y_n, y_m)$$

$\to 0$ as $n, m \to \infty$.

That means real sequence $\{d(x_n, y_n)\}$ is a cauchy sequence of reals, and by cauchy general principle of convergence we have

$\lim_n d(x_n, y_n)$ exists.

(ii) Part is a routine exercise and left out. Now that d^* is a metric on X^* follows upon verification of metric aixoms (M.1) – (M.3). We now show that (X^*, d^*) is a complete metric space. Take $\{x^*_1, x^*_2, \ldots, x^*_n, \cdots\}$ be a cauchy sequence in (X^*, d^*). Out of the equivalent class represented by x^*_n take a cauchy sequence $\{x_1^{(n)}, x_2^{(n)}, \cdots\}$ of (X, d); We can find a +*ve* index K_n so that

$$d\left(x_m^{(n)}, x_{k_n}^{(n)}\right) \le \frac{1}{n} \quad \text{for } m \ge K_n \qquad \ldots \qquad (4)$$

Now consider the sequence $\left\{x^{(1)}K_1, x^{(2)}K_2, \ldots x^{(n)}K_n, \ldots\right\}$. This is a cauchy sequence. For $x \in X$, constant sequence $\{x, x, x, \cdots\}$ is obviously a cauchy sequence, and belongs to a certain equivalent class represented by a member of X^*.

Also $d^*(x^*, y^*) = d(x, y)$ when $\{x, x, \cdots\} \in x^*$ and $\{y, y, \cdots\} \in y^*$ (4)

For each natural number n, constant sequence $\left\{xK_n^{(n)}, xK_n^{(n)}, \cdots\right\} \in xK_n^{(n)*}$;

$$\text{Thus } d^*\left(x^*_n, x^*_{k_n}\right) = \lim_{K \to \infty} d\left(x_k^{(n)}, x_{k_n}^{(n)}\right) \le \frac{1}{n} \qquad (5)$$

Suppose $\epsilon > 0$ is given. Since $\{x^*_n\}$ is cauchy choose n_0

$$\text{So that } d^*(x^*_n, x^*_m) < \varepsilon/3 \text{ and also } \frac{1}{n}, \frac{1}{m} \text{ each } < \varepsilon/3 \qquad (6)$$

From (4), (5) and (6) we find for $n, m \le n_0$

$$d\left(xK_n^{(n)}, xK_n^{(n)}\right) = d^*\left(xK_n^{(n)*}, xK_m^{(n)*}\right)$$

$$\le d\left(xK_n^{(n)*}, x_n^*\right) + d^*\left(x_n^*, x_m^*\right) + d^*\left(x_m^*, xK_m^{(m)}\right) \le d^*\left(x_n^*, x_m^*\right) + \frac{1}{n} + \frac{1}{m} < \varepsilon$$

So, $\left\{xK_1^{(1)}, xK_2^{(2)}, \cdots xK_n^{(n)} \cdots\right\}$ is cauchy and therefore belongs to an equivalence class represented by a member $x^* \in X^*$. We finally show that, $x^* = \lim_n x^*_n$.

Now $d\left(x_{K_p^{(p)}}, x_{K_n^{(n)}}\right) < \varepsilon/2$ for $p,n \geq N_1$ because $\left\{x_{K_n^{(n)}}\right\}$ is cauchy in (X,d). (7)

From (5) and (7) taking $n \geq N_1$ we have

$$d^*\left(x^*, x_n^*\right) \leq d^*\left(x^*, x_{K_n^{(n)}}^*\right) + d^*\left(x_{K_n^{(n)}}^*, x_n^*\right)$$

$$\leq d^*\left(x^*, x_{K_n^{(n)}}^*\right) + \frac{1}{n}$$

$$\leq \lim_{p\to\infty} d\left(x_{K_p^{(p)}}, x_{K_n^{(n)}}\right) + \frac{1}{n}$$

$$\leq \varepsilon/2 + \frac{1}{n} \text{ which is } < \varepsilon \text{ for large values of } n.$$

That means, $\lim_n x_n^* = x^* \in X^*$.

It remains to check that X is isometric to a sub-space of X^* which is dense in X^*. Let x_0^* denote the collection of **all** constant sequences $(x,x,...)$ as $x \in X$. Then $X_0^* \subset X^*$; And there is 1–1 correspondence between members of X_0^* and those of X, with property $d(x^*,y^*) = d(x,y)$ as $(x,x,\cdots) \in x^*$ and $(y,y,\cdots) \in y^*$ when $x,y \in X$.

Thus X_0^* and X become isometric. Lastly, take $x^* \in X^*$, cauchy sequence $\{x_1, x_2, \cdots\} \in x^*$. Let $\varepsilon > 0$ be given then there is an index n such that

$d(x_m, x_n) < \varepsilon$ if $m > n$ (8)

Denote by $x_\varepsilon^* \in X_n$ to represent constant sequence $\{x_n, x_n, \cdots\}$. From (8) we have

$$d^*\left(x^*, x_\varepsilon^*\right) = \lim_m (x_m, x_n) < \varepsilon$$

Hence X_0^* is everywhere dense in X^*.

The weirstrass approximation Theorem

Before we take up the Theorem let us consider a polynomial $p(x) = a_0 + a_1x + \cdots + a_nx^n$ with coefficients a_i as reals over a closed interval $[a,b]$(a < b). We know that every such polynomial is a continuous function over $[a,b]$; and it is also well-known that limit of any uniformly convergent sequence of such polynomials becomes a continuous function in $[a,b]$ Weirstrass theorem says that converse is also true.

Theorem 3.4,6 Given a continuous function f in $[a,b]$ and $\varepsilon>0$, there is a real polynomial p satisfying $|f(x)-p(x)|<\varepsilon$ for all x in $[a,b]$

Proof : It suffices to prove this Theorem in the closed unit interval [0,1] because; of reasons that $x = x'(b-a)+a$ gives a continuous transformation of [0,1] onto [a,b]; so that g taken as $g(x') = f(x'(b-a)+a)$ is continuous in [0,1], Theorem if proved for [0,1] shall give a polynomial p' in [0,1] satisfying $|g(x')-p'(x')|<\varepsilon$ for all $x'\in[0,1]$, and in terms of x, one obtains $\left|f(x)-p'\left(\frac{x-a}{b-a}\right)\right|<\varepsilon$ for all x in $[a,b]$; thus putting $p(x)=p'\left(\frac{x-a}{b-a}\right)$ our purpose is done.

Binomial coefficients ${}^nC_k$ (given a +ve integer n and an integer k with $0\le k\le n$)$=\frac{n!}{k!(n-k)!}$ is shortened as $\binom{n}{k}$

Put $B_n(x)=\sum_{k=0}^{n}\binom{n}{k}x^k(1-x)^{n-k}f\left(\frac{k}{n}\right)$; these are called

Bernstein polynomials associated with f. We need following identities :–

$$\sum_{k=0}^{n}\binom{n}{k}x^k(1-x)=(x+(1-x))^n=1 \tag{1}$$

Differentiating w.r.t x we get

$$\sum_{k=0}^{n}\binom{n}{k}\left(kx^{k-1}(1-x)^{n-k}-(n-k)x^k(1-x)x^{n-k-1}\right)=\sum_{k=0}^{n}\binom{n}{k}x^{k-1}(1-x)^{n-k-1}(k-nx)=0$$

Now multiply both sides by $x(1-x)$ and get,

$$\sum_{k=0}^{n}\binom{n}{k}x^k(1-x)^{n-k}(k-nx)=0 \tag{2}$$

Differentiating w.r.t x and taking $x^k(1-x)^{n-k}$ as a factor to apply product rule one obtains

$$\sum_{k=0}^{n}\binom{n}{k}(-nx^k(1-x)^{n-k}+x^{k-1}(1-x)^{n-k-1}(k-nx)^2)=0 \tag{3}$$

Applying (1) to (3) we have

$$\sum_{k=0}^{n}\binom{n}{k}x^{k-1}(1-x)^{n-k-1}(k-nx)^2=n$$

multiplying throughout by $x(1-x)$ we have

$$\sum_{k=0}^{n}\binom{n}{k}x^{k}(1-x)^{n-k}(k-nx)^2=nx(1-x)$$

and division by n^2 yields

$$\sum_{k=0}^{n}\binom{n}{k}x^{k}(1-x)^{n-k}\left(x-\frac{k}{n}\right)^2=\frac{x(1-x)}{n} \tag{4}$$

Using (1) we find $f(x)-B_n(x)=\sum_{k=0}^{n}\binom{n}{k}x^{k}(1-x)^{n-k}\left(f(x)-f\left(\frac{k}{n}\right)\right)$.

So that $|f(x)-B_n(x)| \le \sum_{k=0}^{n}\binom{n}{k}x^{k}(1-x)^{n-k}\left|f(x)-f\left(\frac{k}{n}\right)\right|$ (5)

Since f is uniformly continuous in [0,1] there is a + ve δ such that $\left|x-\frac{k}{n}\right|<\delta$ implies $\left|f(x)-f\left(\frac{k}{n}\right)\right|<\frac{\varepsilon}{2}$. Let us split the sum in (5) in two parts say by $\sum$ and $\sum'$, where $\sum$ is the sum of those terms for which $\left|x-\frac{k}{n}\right|<\delta$ stands o.k. and where $\sum'$ is the sum of those left-over. Clearly $\sum<\frac{\varepsilon}{2}$. The proof is completed by showing $\sum'$ made smaller than $\frac{\varepsilon}{2}$ independent of x for sufficiently large n. As f is bounded we find a + ve K so that $|f(x)|\le K$ In all $x\in[0,1]$. Thus we have

$$\textstyle\sum' \le 2K\sum\binom{n}{k}x^{k}(1-x)^{n-k},$$

Where the sum on *r.h.s*, say, $\sum''$ is taken for all k such that $\left|x-\frac{k}{n}\right| \geq \delta$. Now it suffices to show $\sum''$ smaller than $\varepsilon/4K$ independent of x by taking n appropriately large. Identity (4) gives

$$\delta^2 \sum'' \leq \frac{x(1-x)}{n} \qquad \text{so } \sum'' \leq \frac{x(1-x)}{\delta^2 n}$$

Now $\max_{0<x\leq 1}(x(1-x)) = 1/4$; and hence $\sum'' \leq \frac{1}{4\delta^2 n}$.

So taking $n > \frac{K}{\delta^2 \varepsilon}$, we have $\sum'' < \frac{\varepsilon}{4K}$, and hence $\sum' < \frac{\varepsilon}{2}$ and consequently $|f(x) - B_n(x)| < \varepsilon$ for all $x \in [0,1]$. The proof is now complete.

Example 3.4.4 A necessary and sufficient condition that a function f is continuous in $[a,b]$ is that converponding to $\varepsilon > 0$, there is a polynomial p on $[a,b]$ such that $|f(x) - p(x)| < \varepsilon$ for all $x \in [a,b]$.

Solution : Condition is necessary : Let f be a continuous function in $[a,b]$ and Weirstrass Theorem takes care of this part.

Condition is sufficient: Let f be a function $[a,b]$ and given $\varepsilon > 0$, there is a polynomial p in $[a,b]$ such that

$$|f(x) - p(x)| < \frac{\varepsilon}{3} \text{ for all } x \in [a,b] \tag{1}$$

Take $x = c$ any point in $[a,b]$, since polynomial p is continuous at $x = c$, we find a +ve δ such that

$$|p(x) - p(c)| < \frac{\varepsilon}{3} \text{ whenever } c - \delta < x < c + \delta \tag{2}$$

Taking $x = c$ in (1) we have

$$|f(c) - p(c)| < \frac{2}{3} \tag{3}$$

Now $|f(x) - f(c)| \le |f(x) - p(x)| + |p(x) - p(c)| + |p(c) - f(c)|$

$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$ whenever $|x - c| < \delta$ from (1), and (2) and (3).

That means f is continuous at $x = c$. Hence conclusion follows.

Excercise-(A)

Short answer type questions

1. If A and B are bounded sets in a metric space show that $A \cap B$ is a bounded set.

2. Which of the following sets in R with usual metric is bounded ?

   (a) The set E of all irrationals

   (b) open interval (a,b)

   (c) $\left\{1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots\right\}$.

   (d) The closed interval $[2,3] \cup \left\{3 + \frac{1}{2^n}\right\}, n = 1, 2, \ldots$

3. Show that $[0,\infty)$ is an unbounded closed set in R with usual metric.

4. Show that for a continuous function f :R→R with usual metric of reals, and for $0 < r < R$, the set $\{x \in R : r < |f(x)| < R\}$ is an open set.

5. Show that a constant function from one metric space into another is a continuous function.

6. Let $[X,d]$ and $[Y,p]$ be two metric space and $[X,d]$ is discrete. Show that any function f :X→Y is a continuous function.

Excercise-(B)

1. Show that $f:[0,1]\to[a,b]$ given by $f(x)=a+(b-a)x$; $0\le x\le 1$ is a homeomorphism.

2. Give an example to illustrate that a homeomorphic image of a complete metric space may not be complete.

3. Obtain the closure of the set $\left\{(x,y): y=\sin\frac{1}{x} ; 0\le x\le 1\right\}$ in R^2 with usual metric.

4. Let $f:(X,d)\to$Reals be a function such that for any rational α, sets $\{x\in X: f(x)<\alpha\}$ and $\{x\in X: f(x)>\alpha\}$ are open. Show that f is a continuous function.

5. If $\{x_n\}$ and $\{y_n\}$ are two cauchy sequences in (X,d), show that real sequence $\{d(x_n,y_n)\}$ is convergent.

6. If T is a contraction in a complete metric space X and $x\in X$, show that

$$T\left(\lim_n T^n(x)\right)=\lim_n T^{n+1}(x).$$

7. Show that $C_R[0,1]$ = Collection of all real-valued continuous functions on the closed interval $[0,1]$ is a metric space with respect a metric

$$\delta(f,g)=\int_0^1 |f(t)-g(t)|dt, \text{ where } f,g\in C_R[0,1].$$

Examine if $(C_R[0,1],\delta)$ is a complete metric space.

(Hint : Take $f_n : [0,1]\to$Reals as

$$f_n(t)=1 \quad \text{if} \quad 0\le t<\frac{1}{2}$$

$$=1-2^n\left(t-\frac{1}{2}\right) \text{ if } \frac{1}{2}\le t<\frac{1}{2}+\frac{1}{2^n},$$

$= 0 \qquad\qquad \text{if} \quad \frac{1}{2}+\frac{1}{2^n} \le t \le 1.$

and show that $\{ f_n\}$ is cauchy without being convergent in metric space $(C_R[0,1],\delta)$.

8. Let (X,d) be a complete metric space, and for each +ve integer n function $T_n:X\to X$ is a contraction with same contraction parameter. Such that $T(x) = \lim_n T_n(x)$ for $x\in X$. If $u,u_1,u_2,...$ are fixed points of $T, T_1,T_2,...$ respectively, show that $u = \lim_n u_n$.

9. Examine if $f(x) = x^2$ is a uniformly continuous function over the space R of all reals with usual metric.

10. If $\{x_n\}$ and $\{y_n\}$ are two convergent sequences in a metric space (X,d), show that $\lim_n \{d(x_n,y_n)\} = d((\lim_n x_n, \lim_n y_n)$.

11. Give an example of a function f of R into R (R = space of reals with usual metric) such that the set $\{x: f(x)\ge 0\}$ is not equal to the closure of the set $\{x: f(x)>0\}$.

12. Let $\{G_\alpha\}_{\alpha\in\Lambda}$ be a family of closed sets in (X,d) with F.I.P such that for some $\alpha_0\in\Delta$, $G_{\alpha 0}$ compact. Show that $\bigcap_{\alpha\in\Delta} G_\alpha \ne \phi$.

13. If A and B are two subsets in (X,d) and B is compact, show that dist $(A,B) = 0$ if and only if $\overline{A}\cap B\ne \phi$.

14. If a continuous function $f: R\to R$ (The space R of all reals having usual metric) satisfies the property $f(x + y) = f(x) + f(y)$ for all $x,y\in R$. Show that $f(x) = xf(1)$ for all $x\in R$.

15. Let $f:(X,d)\to(Y,p)$ be a function. Let A and B be two closed subsets of X with $X = A\cup B$. If restrictions f/A and f/B are continuous function over A and B respectively, show that f is a continuous function.

Unit 8 ❐ Compactness and Connectedness in Metric Spaces and Applications

(Open cover in *a* metric space, compactness, compact sets are bounded closed sets, Finite Intersection property. ε–nets, total boundedness, sequential compactness, continuous image of compact space, separated sets, connectedness, properties of connected sets, connected sets of reals, continuous image of connected metric space. Local connectedness.)

§ 4.1

Definition 4.1.1 **(a)** A family $\varphi\{A_i\}_{i\in\Delta}$ of open sets A, in *a* metric space (X,d) is said to be an open cover for X if every element of X belongs to at least one member A_i, of φ; That is, if $X=\bigcup_{i\in\Delta} A_i$.

(b) A sub-family of an open cover for (X,d) which is by itself an open cover for X is called sub-cover for X.

(c) (X,d) is called *a* compact metric space if every open cover for X has *a* **finite** subcover for X.

Example : By *a* finite sub-cover we mean the sub-cover consisting of *a* finite number of members only. The family $\varphi=\{(-n,n)\}$ where n is a natural number, is *a* family of open intervals (and hence open sets) of reals that covers R = space of **all** real numbers; because $R=\bigcup_{n=1}^{\infty}(-n,n)$. So φ is an open cover for R, Similarly, the family $\wp=\{(x-1,x+1)\}$ where x is any real number is an open cover for R. But neither of these open covers has got *a* **finite** sub-cover for R. That is why, real space R with usual metric is not *a* compact metric space.

Definition 4.1.2 If G is *a* subset of *a* metric space (X,d) then G is said to be compact if as *a* sub-space with relativised metric becomes compact as per

Definition 4.1.1

§ 4.1.1

For example, we know that R with usual metric fails to be a compact space; if B is a **finite** subset of R, then B becomes compact.

Theorem 4.1.1 A compact subset of a metric space is bounded and closed.

Proof : Suppose G is *a* compact set in *a* metric space (X,d). If $x_0 \in G$, consider the family of open balls $\{B_n(x_0)\}$ all centred at x_0 having radius $= n$ $(n = 1,2,3,...)$. Then it is easy to see that the family $\{B_n(x_0)\}_{n=1,2,3,...}$ is an open cover for G. *i.e.*, $G \subset \bigcup_{n=1}^{\infty} B_n(x_0)$; By compactness of G, we obtain *a* finite sub-family, say,

$= \{B_{n_1}(x_0), B_{n_2}(x_0), ...\ B_{n_k}(x_0)\}$ to cover G.

As these open balls are concentric, there is one among them with maximum radius $= N$ say, so that $G \subset B_N(x_0)$. Hence G is bounded.

For G to be shown as closed take $x_0 \in (X \setminus G)$, consider the closed concentric balls $B_{\frac{1}{n}}(x_0)$ all centred at x_0 having decreasing radius $= \frac{1}{n}$, $n = 1,2,...$; Then the family of open sets $\left\{\left(X \setminus \overline{B}_{\frac{1}{n}}(x_0)\right)\right\}$ as $n = 1,2,...$ becomes an open cover for G; and by compactness of G we obtain *a* finite sub-cover, say $\left\{X \setminus \overline{B}_{\frac{1}{n_1}}(x_0)\right\}, ... \left\{X \setminus \overline{B}_{\frac{1}{n_k}}(x_0)\right\}$;

If $\frac{1}{N}$ is the radius of the smallest closed ball associated with these k members, we have $\left(G \cap B_{\frac{1}{N}}(x_0)\right) \subset \left(G \subset \overline{B}_{\frac{1}{N}}(x_0)\right) = \phi$. That means x_0 is neither in G **nor** *a* limit point of G; As : x_0 is any member of $(X \setminus G)$ we have shown G to be closed.

Remark : Converse of Theorem 4.1.1 is not true. Because the closed unit ball in metric space like $C_R[a,b]$ (with sup metric) or the sequence space ℓ_2 is not compact.

Theorem 4.1.2 A closed subset of *a* compact metric space is compact.

Proof : Take F to be *a* closed subset of (X,d) which is compact. Assume that $\Re$ is an open cover for F. Then Since $(X\backslash F)$ is open, we see that $\Re\cup(X\backslash F)$ is an open cover for the whole space X. Since (X,d) is compact, This family has *a* finite sub-cover, say $\Re_0$ for X possibly with $(X\backslash F)$ as *a* member. Thus $\Re_0$ minus $(X\backslash F)$ becomes still *a* finite sub-family of $\Re$ to serve as *a* cover for F. So F is compact.

Theorem 4.1.3 A compact metric space is separable.

Proof : Let (X,d) be *a* compact metric space. For each +ve integer n the family of open balls $B_{\frac{1}{n}}(x)$ centred at x *as* $x\in X$ is an open cover for X. So there is *a* finite sub-cover, say, $\left\{B_{\frac{1}{n}}(x_{1n}), B_{\frac{1}{n}}(x_{2n}), \ldots, B_{\frac{1}{n}}(x_{mn})\right\}$ for X. Put $G=\{x_{jn}\in X : j=1,2,\ldots, m(n) ; n=1, 2, \ldots\}$.

So G is *a* countable set in X. Let $B_r(x)$ be an open ball centred at $x\in X$ with radius = r. Choose +ve integer n so large that $\frac{1}{n}<r$. As x enters into one of members of open cover, we find some k with $1\le k\le m(n)$ such that $x\in B_{\frac{1}{n}}(x_{kn})$.

So, $x_{kn}\in B_{\frac{1}{n}}(x)\subset B_r(x)$.

That is to say that G is dense in X.

Definition 4.1.2 A family $\Re$ of subsets in *a* metric space is said to have finite intersection property (F.I.P) if and only if every **finite** sub-family of $\Re$ has *a* non-empty intersection.

For example, every decreasing sequence of non-empty bounded closed subsets like $\left\{I_n=\frac{-1}{n}\le x\le\frac{1}{n}\right\}$ of closed intervals of reals with usual metric has F.I.P.

Theorem 4.1.4 (X,d) is a compact metric space if and only if each family of closed sets in X with F.I.P has non-empty intersection.

Proof : Let (X,d) be a compact metric space and $\Re$ be *a* family of closed sets in X with F.I.P. If possible, Let $\bigcap_{F\in\Re} F = \phi$: so by De-Morgan's rule $\bigcup_{F\in\Re}(X \setminus F) = X$. Therefore family $\{(X\setminus F) : F\in\Re\}$ is an open cover for X. By compactness of (X,d) we obtain *a* finite sub-family, say = $\{(X\setminus F_1),(X\setminus F_2),\cdots,(X\setminus F_n)\}$ of this family to cover X. That is to say $\bigcup_{i=1}^{n}(X \setminus F_i) = X$, and hence $\bigcap_{i=1}^{n} F_i = \phi$ – *a* contradiction that $\{\Re\}$ has F.I.P.

So we have $\bigcap_{F\in\Re}^{n} F \neq \phi$.

Conversely, let the condition hold in (X,d) but (X,d) be not compact. Then there is an open cover say = $\mathcal{G}$ of open sets G for X but $\mathcal{G}$ has no finite sub-cover for X. Thus every finite sub-family of g fails to cover X. *i.e.*, every finite sub-family of $\mathcal{G}'$ consisting of closed sets $(X\setminus G)$ as $G\in\mathcal{G}$ has non-empty intersection. Clearly the family $\mathcal{G}'$ of closed sets has F.I.P and by assumption $\bigcap_{G\in g}(X \setminus G) \neq \phi$.

or $\bigcup_{G\in g} G \neq X$, a contradiction that $\mathcal{G}$ is an open cover for X. Hence (X,d) is compact. The proof is complete.

Definition 4.1.3 A metric space (X,d) is called sequentially complete if every sequence in X has *a* convergent sub-sequence in X.

Theorem 4.1.5 A compact metric space is sequentially compact.

Proof : Let (X,d) be a compact metric space and $\{x_n\}$ is any sequence in X. Put $G_n=\{x_n,x_{n+1},\ldots\}$ Then the family $\{\overline{G}_n\}_{n=1,2,\ldots}$ of closed sets is *a* decreasing chain of closed sets in (X,d) which is compact. Now by decreasing property every finite sub-family of $\{\overline{G}_n\}$ has non-empty intersection, and by compactness of the space this family $\{\overline{G}_n\}$ has F.I.P. and hence $\bigcap_{n=1}^{\infty}\overline{G}_n \neq \phi$; Take $u\in\bigcap_{n=1}^{\infty}\overline{G}_n$. Given $\varepsilon>0$ as $u\in\overline{G}_n$ for every n, we have *a* member x_{m_n} with $m_n\geq n$ in G_n satisfying $d(u,x_{m_n})<\varepsilon$; That means $\lim_n x_{m_n}=u\in X$.

Hence $\{x_n\}$ has *a* convergent subsequence in X.

i.e., (X,d) is sequentially compact.

Remark : The converse of Theorem 4.1.5 is true. To prove the same we have to incorporate some new Definitions.

Definition 4.1.4

(a) Give a + ve ε, a finite subset A = $(a_1, a_2, \cdots, a_n)$of (X,d) is said to be a finite ε-net for X if $X = \bigcup_{i=1}^{n} B_\varepsilon(a_i)$.

(b) (X,d) is called totally bounded if there ia a finite ε-net for X for every +ve ϵ

Explanation : If (X,d) has a finite ϵ-net (ε>0), it means there is a finite cover of open balls for X. A subset G of (X,d) is totally bounded if G as a metric sub-space of (X,d) becomes totally bounded.

Theorem 4.1.6 If G is a totally bounded subset of (X,d), then G is bounded.

Proof: Let ε>0 be given, then there is a finite ε-net, say, $A=(a_1, a_2, \ldots, a_n)$ for G. Clearly Diam(A)<∞ because A is a finite set. Further $G \subset \bigcup_{i=1}^{n} B_\varepsilon(a_i)$. Take $u, v \in G$, we find two corresponding members, say, a_i and a_j from A such that $u \in B_\varepsilon(a_i)$ and $v \in B_\varepsilon(a_j)$ (a_i may be equal to a_j). So we have $d(u,a_i)<\varepsilon$ and $d(v,a_j)<\varepsilon$; Now

$d(u,v) \leq d(u,a_i) + d(a_i,a_j) + d(a_j,v) < 2\epsilon +$ Daim.(A) ; Taking sup on L.H.S over all $u, v \in G$, we obtain Diam(G)≤2ε + Diam(A) < + ∞. The proof is now complete.

However converse of Theorem 4.1.6 is not true, consider an example in support.

Example 4.1.4 Take (X,d) as the sequence space ℓ_2 consisting of all real sequences

$x = (x_1, x_2, \cdots, x_n, \cdots)$ with $\left(\sum_{n=1}^{\infty} |x_n|^2\right)^{\frac{1}{2}} < \infty$ with metric $d\left(\underset{-}{x}, \underset{-}{y}\right) = \left(\sum_{i=1}^{\infty} |x_i - y_i|^2\right)^{\frac{1}{2}}$ as

$\underset{\sim}{x}, \underset{\sim}{y} \in \ell_2$; and take a subset G = $\left\{\underset{\sim}{x_1}, \underset{\sim}{x_2}, \ldots \underset{\sim}{x_n}, \ldots\right\}$ of ℓ_2 where $\underset{\sim}{x_1} = (1,0,0,\cdots)$,

$\underset{\sim}{x_2} = (0,1,0,\ldots), \ldots$; so that $d\,(\underset{\sim}{x_i}, \underset{\sim}{x_j})(i \neq j) = \sqrt{1+1} = \sqrt{2}$ (1)

and hence G is a bounded set, because $\text{Diam}(G) \leq \sqrt{2}$.

We now check that G is not totally bounded.

Take ε as $0 < \varepsilon < \frac{1}{2}\sqrt{2}$. If possible, let $\left(\underset{\sim}{u_1}\ \underset{\sim}{u_2}, \ldots \underset{\sim}{u_k}\right)$ be a finite ε-net for G in ℓ_2.

That is, $G \subset \sum_{\ell=1}^{k} B_\varepsilon\left(\underset{\sim}{u_\ell}\right)$.

or, $\left\{\underset{\sim}{x_1}, \underset{\sim}{x_2}, \underset{\sim}{x_n}, \ldots\right\} \subset \bigcup_{\ell=1}^{k} B_\varepsilon\left(\underset{\sim}{u_\ell}\right)$.

Since L.H.S. is not a finite set, there are two distinct members, say, $\underset{\sim}{x_i}, \underset{\sim}{y_j} \in G$ to enter the same open ball say, $B_\varepsilon\left(\underset{\sim}{u_\ell}\right)$ of R.H.S. This gives $d\left(\underset{\sim}{u_\ell}, \underset{\sim}{x_j}\right) < \varepsilon$ and $d\left(\underset{\sim}{u_\ell}, \underset{\sim}{x_j}\right) < \varepsilon$; So, $d\left(\underset{\sim}{u_\ell}, \underset{\sim}{x_j}\right) \leq d\left(\underset{\sim}{u_i}, \underset{\sim}{x_\ell}\right) + d\left(\underset{\sim}{u_j}, \underset{\sim}{x_\ell}\right) < 2\,\varepsilon < \sqrt{2}$ – a contradiction of (1).

Hence our assertion stands.

Theorem 4.1.7 Every sequentially compact metric space is totally bounded and complete.

Proof: Let (X,d) be sequentially compact, and Let $\varepsilon > 0$ be given. Take any member, say, $a_1 \in X$ and the open ball $B_\varepsilon(a_1)$. If $X = B_\varepsilon(a_1)$, then $\{a_1\}$ serves a finite ε-net for X; If not, take a member, say, $a_2 \in X \setminus B_\varepsilon(a_1)$. Clearly $d(a_1, a_2) > \varepsilon$. If $\{a_1, a_2\}$ serves as an ε-net for X, we are done ; or else continue this process to obtain a member $a_3 \in X \setminus \{B_\varepsilon(a_1) \cup B_\varepsilon(a_2)\}$; So that $d(a_i, a_3) \geq \varepsilon$, $i = 1, 2,$. Unless this process terminates at some nth stage when we arrive at a desired ε – net for X, we construct inductively a sequence $\{a_n\}$ in X satisfying $d(a_i, a_n) \geq \varepsilon$ for $i = 1, 2, \ldots, (n-1)$ for every n. Now such a sequence $\{a_n\}$ in (X,d) does not have a convergent subsequence in X– *a* contradiction that (X,d) is sequentially compact. So (X,d) is totally bounded. Finally, if $\{x_n\}$ is cauchy sequence in X, then by sequential compactness $\{x_n\}$ has a convergent subsequence. and therefore $\{x_n\}$ is itself convergent. So (X,d) is complete.

Theorem 4.1.8 Every totally bounded and complete metric space is compact.

For the proof the reader is referred to Brown & page 'Functional Analysis'.

Proof : Let (X,d) be totally bounded and complete metric space without being compact. So there is an open cover say, $\mathcal{A}$ of X that has no finite sub-cover. We construct a sequence $\{x_n\}$ in X such that for each n

(i) $d(x_n, x_{n-1}) < \frac{1}{2^{n+1}}$, and (ii) the open ball $B_{\frac{1}{2^{n-1}}}(x_n)$ can not be covered by a finite sub-family of $\mathcal{A}$.

Now (X,d) is totally bounded ; So taking $\varepsilon = 1$, there is a finite 1–net in X ; call it $(u_1, u_2, \ldots, u_k)$. So that $X = \bigcup_{j=1}^{k} B_1(x_j)$; As $\mathcal{A}$ fails to produce a finite sub-cover for X; So one of the open balls on r.h.s can not be covered by a finite sub-family of $\mathcal{A}$; say $B_1(x_j)$ is one such with j minimum. We re-name x_j as x_1. Then condition (ii) is satisfied for this x_1 and (i) is vacuously satisfied. Suppose $x_1, x_2, \ldots, x_n$ have been so chosen. Since the whole space (X,d) is totally bounded so is every open ball of (X,d) and in particular so is open ball $B_{\frac{1}{2^{n-1}}}(x_n)$ for which say a finite $\frac{1}{2^n}$ – net is $(v_1, v_2, \cdots, v_m)$ *i.e.*, $B_{\frac{1}{2^{n-1}}}(x_n) \subset \bigcup_{j=1}^{m} B_{\frac{1}{2^n}}(v_j)$, and by an argument as above, we find j with minimum value so that $B_{\frac{1}{2^n}}(v_j)$ fails to be covered by a finite sub-family of $\mathcal{A}$.

Put x_{n+1} = such av_j. Here stipulations (i) and (ii) are *ok* in favour of $B_{\frac{1}{2^{n-1}}}(x_n)$. Hence sequence $\{x_n\}$ is determined inductively.

Now if $m > n$, we have

$$d(x_m, x_n) \le d(x_m, x_{m-1}) + d(x_{m-1}, x_{m-2}) + \cdots + d(x_{n+1}, x_n)$$

$$\le \frac{1}{2^{m-2}} + \frac{1}{2^{m-3}} + \cdots + \frac{1}{2^{n-1}} \le \frac{1}{2^{n-2}} \to 0 \text{ as } n \to \infty$$

(hence $m \to \infty$).

Thus $\{x_n\}$ is a cauchy sequence in (X,d) which is complete. Let $\lim_n x_n = w \in X$. Now w enters into some member A of $\mathcal{A}$ and A is open, we get an open ball $B_r(w) \subset A$. Now choose N such that $d(x_n, w) < \frac{1}{2}r$ and $\frac{1}{2^{n-r}} < \frac{1}{2}r$ for $n \ge N$. So if $y \in B_{\frac{1}{2^{N-1}}}(x_N)$, We have $d(y,w) \le d(y,x_N) + d(x_N, w)$.

$$< \frac{1}{2^{N-1}} + \frac{1}{2}r < \frac{1}{2}r + \frac{1}{2}r = r.$$

In other words, we verify that $B_{\frac{1}{2^{N-1}}}(x_N) \subset B_r(w) \subset A$ — a contradiction of (ii) above. The proof is complete.

Now Theorem 4.1.7 and 4.1.8 give following Theorem immediately.

Theorem 4.1.9 Every sequencially compact metric space is compact.

Theorem 4.4.10. Continuous image of a compact metric space is compact.

Proof : Let $f:(X,d) \to (Y,p)$ be continuous and (X,d) is compact.

Let $\mathcal{B} = (B_\alpha)_{\alpha \in \Delta}$ be an open cover for $f(X)$ in (Y, p).

Put $B_\alpha = f(X) \cap H_\alpha$ where H_α is an open set in (Y, p) for each $\alpha \in \Delta$.B, continuity of f we have $f^{-1}(H_\alpha)$ is an open set in X and we have $f(X) \subset \bigcup_{\alpha \in \Delta} B_\alpha = \bigcup_{\alpha \in \Delta} f(X) \cap H_\alpha)$

and $X \subset f^{-1}\left(\bigcup_{\alpha \in \Delta} (f(X) \cap H_\alpha) \right) = \bigcup_{\alpha \in \Delta} (X \cap f^{-1}(H_\alpha))$

So the family $\{f^{-1}(H_\alpha)\}$ is an open cover for X ; Since (X,d) is compact, we find a finite sub-cover, say $\{ f^{-1}(H_1), f^{-1}(H_2), \ldots f(H_n)\}$ for X. *i.e.*,

$$X \subset \bigcup_{k=1}^{n} f^{-1}(H_k)$$

or $$f(X) \subset \bigcup_{k=1}^{n} H_k = \bigcup_{k=1}^{n} (f(X) \cap H_k) = \bigcup_{k=1}^{n} B_k.$$

Hence given open cover has a finite sub-cover for $f(X)$.

So $f(X)$ is compact.

Corollary : Every real valued continuous function f over a closed interval $[a,b]$ (with usual netric of reals) is bounded and attains its bounds in $[a,b]$.

Because closed interval $[a,b]$ being a bounded closed set of reals is compact and if $f:[a,b] \to$ Reals is continuous, then $f[a,b]$ is a compact set of reals, and it is a bounded closed set. Thus f becomes a bounded function and by closure property of $f[a,b]$ we see that f attains its bounds (upper and lower bonds) in $[a,b]$.

4.5

Definition 4.5.1 (a) A metric space (X,d) is called connected if X is **never** a Union of two nonempty disjoint open or clsoed sets in X.

(b) If there is such a decomposition of X, then (X,d) is called disconnected.

(c) A subset G of X is said to be connected or disconnected if G as a metric sub-space of (X,d) becomes connected or disconnected.

Explanation : Connectedness in a metric space (X,d) means that ϕ and X are its only subsets that are both open and closed. If (X, d) is not connected, then there is subset other than ϕ and X *i.e.*, a non-empty and proper subset in X which is clo-open (Closed and open); and corresponding to each such clo-open set we have a decomposition of X as a disconnection.

Example 4.5.1 The set Q of all rationals in R with usual metric of reals is disconnected set.

Solution : Let us c as an irrational number. Then we have

$Q = \{(-\infty,c)\cap Q\} \cup \{(c,+\infty)\cap Q\}$; Since r.h.s. sets are each non-empty open subsets of Q and are disjoint, So above partition of Q is a disconnection for Q.

Since c can be taken in many ways, Q has many disconnections.

Theorem 4.5.1 A subset of reals with usual metric is connected if and only if it is an interval.

Proof : Suppose G is a connected set of reals with usual metric without being an interval. So there are reals a, b and c with $a<b<c$, and $a,c\subset G$ and $b\notin G$. Now we can write $G = \{(-\infty,b)\cap G\}\cup\{(b,\infty)\cap G\}$; then $r,h.s$ is a disconnection of G-a contradiction that G is taken as connected.

Conversely, Let J be an interval of reals suppose J has a disconnection, as

$J = A\cup B$ where A and B are non-empty **disjoint** closed subsets. Take $x\in A$ and $z\in B$; Since $A\cap B = \phi$, $x\neq z$; Without loss of generality let $x<z$. Since J is an interval containg x and $z(x<z)$, we have the closed interval $[x,z]\subset J$, thus each point of $[x,z]$ is either in A or in B.

Put $y = \sup\{[x,z]\cap A\}$. Hence $x\leq y\leq z$, and $y\in J$; and since A is closed,

we have $y\in A$ ... (1)

Thus $y<z$; given $\in >0$, by sup-Definition; for large +ve integer n we have $y<y+\frac{1}{n}<z$ and $\left(y+\frac{1}{n}\right)\subset B$; Finally, since B is closed we have $\lim_n\left(y+\frac{1}{n}\right)\in B$

or $y\in B$ ... (2)

Now (1) and (2) give $A\cap B\neq\phi$–a contradiction. So proof is complete.

Corollary : The space R with usual metric is connected.

Definition 4.5.2 Two non-empty disjoint subsets A and B of (X,d) are said to be separated if and only if $\overline{A}\cap B = \phi$ and $A\cap\overline{B} = \phi$.

For example disjoint open intervals (0,1) and (2,3) of reals with usual metric are separated; because $\overline{(0,1)}\cap(2,3) = [0,1]\cap(2,3) = \phi$ and $(0,1)\cap\overline{(2,3)} = (0,1)\cap[2,3] = \phi$.

Also there are disjoint sets without being separable. Let Q be the set of all rationals; Then Q and its complement (R\Q) are disjoint; As $\overline{Q}$ = set R of all reals, we have $\overline{Q} \cap (R\setminus Q) \neq \phi$ and hence Q and (R\Q) are not separated.

Theorem 4.5.2 Let A and B be separated sets in (X,d) and C is *a* connected set in (X,d) with $C\subset(A\cup B)$. Then either $C\subset A$ or $C\subset B$.

Proof : Now if $Y = A\cup B$, then have

$\overline{A}\cup Y = \overline{A}\cap(A\cup B)=(\overline{A}\cap A)\cup(\overline{A}\cap B) = A\cup\phi = A$; That means A is *a* closed set in subspace Y. Similarly, B is also *a* closed set in $Y = (A\cup B)$. As $A= (A\cup B)\setminus B$ and $B = (A\cup B)\setminus A$; So A and B are open sets in $(A\cup B)$. Since C is *a* connected subset of $(A\cup B)$, and now $C\cap A$ and $C\cap B$ are both open and closed in C, either $C\cap A = C$ in that case $C\subset A$ or $C\cap B = C$ in that case $C\subset B$.

Theorem 4.5.3 If $\{G_\alpha\}_{\alpha\in\Delta}$ is *a* family of connected sets in (X,d), such that $\bigcap_{\alpha\in\Delta} G_\alpha \neq \phi$. Then $\bigcup_{\alpha\in\Delta} G_\alpha$ is connected.

Proof : Let $\bigcup_{\alpha\in\Delta} G_\alpha$ be not connected, and let $\bigcup_{\alpha\in\Delta} G_\alpha = A\cup B$ be *a* disconnection where A and B are non-empty disjoint clo-open set. So A and B are separated. Now each G_α is connected, and $G\subset \bigcup_{\alpha\in\Delta} G_\alpha \subset (A\cup B)$; So Theorem 4.5.2 applies. Either $G_\alpha\subset A$ or $G_\alpha\subset B$.

Since $\bigcap_{\alpha\in\Delta} G_\alpha \neq \phi$; Take $u\in \bigcap_{\alpha\in\Delta} G_\alpha$. Then either $u\in A$ or $u\in B$. say, $u\in A$; clearly $u\in G_\alpha$ for every $\alpha\in\Delta$, we have $G_\alpha\subset A$ for $\alpha\in\Delta$; That means $\bigcup_{\alpha\in\Delta} G_\alpha \subset A$–a contradiction. The proof is now complete.

Theorem 4.5.4 If A is a connected set in (X,d), then its closure $\overline{A}$ is connected.

Proof : Let A be *a* connected set in (X,d), and let $\overline{A}$ be not connected. Suppose

$\overline{A} = B \cup C$ be *a* disconnection where B and C are non-empty disjoint clo-open sets, and hence separated. Now $A \subset \overline{A} = B \cup C$. So Theorem 4.5.2 applies. Suppose, for instance $A \subset B$; Then $\overline{A} \subset \overline{B} = B$; consequently $\overline{A} \cap C = \phi$; Since B and C are separated, that means $C = \phi$ – a contradiction. So proof is complete.

Theorem 4.5.5 A continuous image if *a* connected metric space is connected.

Proof : Let $f:(X,d) \to (X,p)$ be *a* continuous function and (X,d) be a connected metric space. If possible, let $f(X)$ be not connected in metric space (Y, p), and let

$f(X) = G \cup H$ be *a* disconnection where G and H are non-empty disjoint open sets in $f(X) \subset Y$.

Put $G = f(X) \cap A$ and $H = f(X) \cap B$ where A and B are open sets in (Y, p). By continuity of f we have $f^{-1}(A)$ and $f^{-1}(B)$ are each open sets in (X,d) and we have

$f(X) = (f(X) \cap A) \bigcup (f(X) \cap B)$

and $X \subset (X \cap f^{-1}(A)) \bigcup (X \cap f^{-1}(B))$

i.e., $X = f^{-1}(A) \cup f^{-1}(B)$, giving *a* disconnection of X– a contradiction. Hence proof is complete.

Corollary : Any real-valued continuous function over an interval possesses an intermediate-value property.

Because such a function sends an interval into *a* connected set *i.e.*, similar such set

i.e., an interval ; So if $f:[a,b] \to$ Reals is continuous, then $f[a,b]$ is an interval containing $[f(a), f(b)]$

Example 4.5.2 If $\{G_n\}$ is *a* sequence of connected sets in *a* metric space with

$G_n \cap G_{n+1} \neq \phi$ for all n, show that $\bigcup_{n=1}^{\infty} G_n$ is connected.

Solution Suppose $G = \bigcup_{n=1}^{\infty} G_n$ is not connected, and let $G = A \cup B$ be *a* disconnection, where A and B are non-empty **disjoint** open sets (closed sets).

Now for each n we have $G_n \subset G \subset A \cup B$; Since G_n is connected, and A and B are separated, we have either $G_n \subset A$ or $G_n \subset B$; say $G_n \subset A$; now for same reason either $G_{n+1} \subset A$ or $G_{n+1} \subset B$. If $G_{n+1} \subset B$, we see, $G_n \cap G_{n+1} \subset A \cap B = \phi$, *a* contradiction. Hence $G_{n+1} \subset A$. That is say, $G_n \subset A \Rightarrow G_{n+1} \subset A$ and this is true for all n. That means $\bigcup_{n+1}^{\infty} G_n \subset A$ or, $G \subset A$, leaving $B = \phi$, again *a* contradiction. So we have shown that G is connected.

Example 4.5.3 Show that the subset $\{(x,y): x>0, y = \sin \frac{1}{x}\} \cup H$ where $H = \{(0,y): -1 \leq y \leq +1\}$ of R^2 is connected.

Solution : Here consider *a* function $f:(0,\infty) \to R^2$ given by $f(x) = \left(x, \sin \frac{1}{x}\right)$ as $0 < x < \infty$

Then f is *a* continuous function over $(0,\infty)$ and $(0,\infty)$ is *a* connected set of Reals with usual metric and since continuous image of connected set is connected, it follows that $f(0,\infty)$ is connected set in R^2 with usual metric of R^2. Now $f(0,\infty) = \{(x,y): x>0, y = \sin \frac{1}{x}\}$ and closure of $f(0,\infty)$ becomes *a* connected set of R^2 ; That is to say the given set $\{(x,y) : x>0, y = \sin \frac{1}{x}\} \cup \{(0,y) : -1 \leq y \leq 1\}$ being the closure of $f(0,\infty)$ is *a* connected set in R^2.

Definition 4.5.2 A metric space (X,d) is called locally connected if x is any point of X, and G any neighbourhood of x, then G contains *a* connected neighbourhood of x.

Explanation : Equivalent to statement in Definition 4.5.2 is that each point of (X,d) has *a* neighbourhood base whose members are connected. For example, the real number space R with usual metric is Locally connected, because each point of

R has a neighbourhood base consisting of open intervals containing the point, and they are connected sets. Of course R is also a connected metric space. But one should have a caution. Notions of connectedness and Local connectedness are independent in the sense that Local connectedness neither implies nor is implied by connectedness.

Example 4.5.3 (a) Let $X = (0,1) \cup (2,3)$ be a metric with usual metric of reals. Then X is not connected because taking any number α between 1 and 2 ($1 < \alpha < 2$) one finds a disconnection of X as $X = \{(-\infty,\alpha) \cap X\} \cup \{(\alpha,\infty) \cap X\}$. However X is Locally connected, because every member of X has a neighbourhood base consisting of open intervals containing the member and contained in X, and they are connected.

Example 4.5.3 (b) Take example 4.5.3 where we see that the metric space with usual metric of R^2 is connected without being Locally connected; because a point like $\left(0, \frac{1}{2}\right)$ fails to attract a connected neighbourhood.

Example 4.5.3 (c) Continuous image of a Locally connected space may not be Locally connected.

Solution ; Take $X = \{0,1,2,\cdots\}$ and $Y = \left\{0,1,\frac{1}{2},\frac{1}{3},\ldots\right\}$ where X is a discrete metric space and Y is a metric space each with usual metric of reals relativised.

Then metric Topology on X is the discrete topology where every set is clo-open, and that on Y makes each singleton $\left\{\frac{1}{n}\right\}, n = 1,2,\cdots$ open without making $\{0\}$ an open set.

Now the function $f : X \to Y$ where $f(0) = 0$ and $f(n) = \frac{1}{n}$ $(n = 1,2,\cdots)$ is a bijective continuous function ; Here X is locally connected, but $Y = f(X)$ is not Locally connected, because member $0 \in Y$ does not possess a connected neighbourhood in Y.

Example 4.5.4 If $G_1, G_2, \cdots, G_n$ are compact sets in a metric space (X,d), show that $\bigcup_{i=1}^{n} G_i$ is a compact set of (X,d). Can you extend the result over an infinite number of such set in (X,d) ?

Solution : If G_1 and G_2 are compact sets, we show that $G_1 \cup G_2$ is also compact in a metric space (X,d). Let $A = \{A_\alpha\}_{\alpha\in\Delta}$ be an open cover for $G_1 \cup G_2$. Thus $(G_1 \cup G_2) \subset \bigcup_{\alpha\in\Delta} A_\alpha$. So $G_1 \subset (G_1\cup G_2) \subset \bigcup_{\alpha\in\Delta} A_\alpha$ Hence A become an open cover for G_1 which is compact, so we obtain a finite sub-cover, say $(A_1, A_2, \cdots, A_n)$ of A for G_1. Similarly, we find a finite cover, say, $(A_1', A_2', \ldots, A_m')$ of A for G_2.

Now $(G_1 \cup G_2) \subset (A_1 \cup A_2 \cup \ldots \cup A_n) \cup (A_1', A_2', \ldots, A_m')$; Thus, there is a finite sub-cover $\{A_1, A_2, L, A_n, A_1', A_2', \ldots, A_m'\}$ of A for $(G_1 \cup G_2)$, Hence $G_1 \cup G_2$ is a compact set in (X,d) and by induction we have $\bigcup_{i=1}^{n} G_i$ as a compact set in (X,d) when $G_1, G_2, \cdots G_n$ are so.

It is not possible to extend the result over a countable infinite number of compact sets. For example, we know that each of the closed intervals $[-1,+1], [-2,+2], \cdots [-n,+n], \cdots$ is a compact set of reals with usual metric. However $[-1,+1] \cup [-2,+2] \cup \cdots \cup [-n,+n] \cup \cdots$ which equals to the whole space R of reals and R is not a compact metric space with usual metric.

Example 4.5.5 Let A be a compact set in a metric space (X,d) and $x_0 \in (X \setminus A)$, show that there is a member $a \in A$ such that $d(x_0, a) = \text{dist}(x_0, A)$.

Solution : Here for a fixed $x_0 \in (X\setminus A)$, $d(x_0, x)$ becomes a real-valued continuous function of x in A which is compact; and we know that such a function assumes its $\inf_{x\in A} d(x_0, x)$ as a minimum value at some point of A, say $= a \in A$. Thus $d(x_0, a) = \inf_{x\in A} d(x_0, x) = \text{dist}\,(x_0, A)$.

We have seen that uniform continuity of a function over a metric space implies its continuity, but converse is false. However we have following theorem.

Theorem 4.5.6 Let $f : (X,d) \to (Y,p)$ be a continuous function where (X,d) is a compact metric space and (Y, p) is any metric space, then f is uniformly continuous.

Proof : Let $\varepsilon > o$ be given. If $x \in X$, by continuity of f at $x \in X$ we find a +ve $\delta(x)$ such that

$$d(f(x), f(y)) < \frac{1}{2}\in \text{ when } y \in B_{\delta(x)}(x) \tag{1}$$

Consider the family $\left\{B_{\frac{1}{2}\delta(x)}(x)\right\}_{x\in X}$ of open balls in X.

Clearly it is an open cover for X by compactness of which one finds a finite open sub-cover, say = $\left\{B_{\frac{1}{2}\delta(x_1)}(x_1), B_{\frac{1}{2}\delta(x_2)}(x_2), \ldots, B_{\frac{1}{2}\delta(x_n)}(x_n)\right\}$. for X.

Choose a +ve δ such that $\delta = \min\limits_{1\le i\le n}\left\{\frac{1}{2}\delta(x_i)\right\}$. Now if $u,v\in X$ with $d(u,v)<\delta$. So, $u \in B_{\frac{1}{2}\delta(x_k)}(x_k)$, say, for some k with $1\le k\le n$. Thus $d(x_k,u)<\frac{1}{2}\delta(x_k)$ and

$d(x_k,v)\le d(x_k,u)+d(u,v)<\frac{1}{2}\delta(x_k)+\delta \le \frac{1}{2}\delta(x_k)+\frac{1}{2}\delta(x_k)=\delta(x_k)$ by choice of δ, Therefore from (1) we obtain $d(f(u), f(v))$

$$< d(f(u), f(x_k)) + d(f(x_k), f(v)) < \frac{1}{2}\in + \frac{1}{2}\in = \varepsilon$$

That means we have shown that f is uniformly continuous over (X,d).

Corollary : If f is a real-valued continuous function over a closed interval $[a,b]$, then f is uniformly continuous.

Because a closed interval $[a,b]$ is a closed and bounded set of reals with usual metric, and it is compact. So Theorem 4.5.6 applies here. We know that if a sequence $\{f_n\}$ of real-valued continuous functions converges uniformly over an interval, then the uniform limit function is rendered continuous; but converse is false. However we have following theorem that we present to close the text.

Theorem 4.5.7 $\{f_n:(X,d)\cdot\text{Reals}\}$ be a sequence of continuous functions where (X,d) is a compact metric space and $\{f_n\}$ is monotone increasing *i.e.*, $f_n(x)< f_{n+1}(x)$

for all $x\in X$ and $n=1,2,\cdots$, Let $\lim_{n\to\infty}(f_n)=f(x)$ for $x\in X$ where $f:(X,d)\to$Reals is continuous. Then $\lim_n f_n=f$ is uniform (equivalently, $\{f_n\}$ converges to f uniformly over (X,d)).

Proof : Put $g_n(x)=f(x)-f_n(x)$ for $x\in X$ and $n=1,2,\ldots$

Then we have $0<g_{n+1}(x)<g_n(x)$ for $x\in X$; And each g_n is continuous because by assumption f is so over X.

Let $\varepsilon>0$ be given: and Let $X_n=\{x\in X : g_n(x)<\varepsilon\}$ for $n=1,2,\cdots$; By continuity of g_n we have each X_n is open in (X,d) and further $X=\bigcup_{n=1}^{\infty}X_n$. Clearly $\{X_n\}$ is an open cover for (X,d) which is compact. So there is a finite sub-cover say $\{X_{n1},X_{n2},\ldots,X_{nk}\}$ for (X,d). Also $0\le g_{n+1}(x)\le g_n(x)$ for all $x\in X$, gives $X_n\subset X_{n+1}$, and hence taking $N=\max(n_1,n_2,\cdots,n_k)$, we have $X_{nj}\subset X_N$ and $X=X_{n1}\cup X_{n2}\cdots\cup X_{nk}\subset X_N$.

i.e., $X=X_N$.

So $0\le g_n(x)<\varepsilon$ for $n\ge N$ and for all $x\in X$. Hence for $n\ge N$, we have $\sup_{x\in X}|f(x)-f_n(x)|<\varepsilon$.

i.e., $|f(x)-f_n(x)|<\varepsilon$ $n\ge N$ and for all $x\in X$. That means $\{f_n\}$ converges to f uniformly over X and proof is complete.

Excercise-(A)

Short answer type questions.

1. In a metric space (X,d) if $\lim_n x_n=x$, show that $\{x_1,x_2,\cdots,x_n,\ldots\}\cup\{x\}$ is compact.
2. Give an example of a subset in a metric space that is bounded without being compact.
3. Show that a singleton in a metric space is connected.
4. Let A be a compact set in (X,d) and $x_0\in(X\backslash A)$. Show that dist $(x_0,A)=d(x_0,a_0)$ for some $a_0\in A$.

(**Hint** : consider a function f:A→Reals (with usual metric) where $f(x) = d(x_0,x)$ as $x \in A$. Then f is a continuous function over a compact set A and hence $\inf_{x\in A} f(x)$ is attained at some point $a_0 \in A$. So $f(a_0) = \inf_{x\in A} f(x)$ or $d(x_0,a_0) = \inf_{x\in A} f(x_0,x)$ $=$ dist(x_0,A)).

5. Let A and B be two disjoint compact sets in (X,d). Show that there are two disjoint open sets G_1 and G_2 such that $A \subset G_1$ and $B \subset G_2$.

6. Give an example with reasons of a continuous function f over non-compact metric space such that f is not bounded.

7. Give an example of an 1–1 continuous function from a metric space (X,d) onto (Y,p) such that f^{-1} is not continuous.

Excercises (B)

1. Let $f:(X,d)\to(Y,p)$ be a continuous function where (X,d) is compact, show that $f(\overline{A}) = \overline{f(A)}$ for every subset A of X.

2. Suppose $f:(X,d)\to(Y,p)$ be a continuous surjection ; and if $\{A_n\}$ is decreasing sequence of compact sets in (X,d), show that $f\left(\bigcap_{n=1}^{\infty} A_n\right) = \bigcap_{n=1}^{\infty} f(A_n)$.

3. Show that Hilbert cube consisting of all those sequence $\underset{\sim}{x} = (x_1, x_2, x_n, ...)$ satisfying $|x_n| < \frac{1}{n} (n = 1,2,...)$ is a compact set in ℓ_2.

4. In Education n-space R^n with usual metric d, show that notions of boundedness and total boundedness concide.

5. Show that a continuous injection of (0,1) to Reals is a monotone function.

6. Let A and B be two disjoint sets in a metric space (X,d) such that A is compact and B is closed. Show that dist $(A,B)>0$. Also examine the result if A and B are only closed sets.

7. Show that no two of the intervals (a,b), $[a,b]$ and $[a,b)$ $(b>a)$ are homeomorphic.

8. Let f be a function on a compact metric space (X,d) into itself satisfying $d(f(x), f(y))<d(x,y)$ for all x,y with $x \neq y$ in X. Show that f has a unique fixed point in X. (Hint : consider the function $\varphi : X \to$ Reals where $\varphi(x)=d(x,f(x))$ for $x \in X$).

9. Let $f:[0,1] \to R$ be a bounded function with metric associated as usual metric of reals such that the set $\{(x,f(x)) : x \in [0,1]\}$ is closed in R^2. Show that f is continuous. Also show by an example that the condition of f being bounded can not be dropped in order to make the conclusion stand.

10. Let f be a real-valued continuous function on a compact metric space (X,d); show that there are point $x_1, x_2 \in X$ such that $f(x_1) \leq f(x) \leq f(x_2)$ for all $x \in X$.

11. Let $f:(X,d) \to (Y,p)$ be an 1–1 and onto continuous function, where (X,d) is compact. Show that $f^{-1}: Y \to X$ is continuous.

12. Verify that $f:R \to R$ where concerned metrics are usual metric of reals and $f(x) = x^2$ for $x \in R$ is continuous but not uniformly continuous.

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1. G. F. Simons- Modern Analysis
2. I. P Natanson- Theory of functions of Real variable-I, II
3. C. Goffman- Real functions
4. B. K. Lahiri- Functional Analysis
5. Brown and Page- Functional Analysis
6. J. L.- Kelley- Topology
7. Lecture notes on Functional analysis in T.I.F.R *by* Professor Bonsai.

GROUP–B

COMPLEX ANALYSIS

Unit : 1 ❑ Complex Numbers

1.1 Introduction :

There is no real number x which satisfies the polynomial equation $x^2 + 1 = 0$. To permit solutions of this and similar equations, the set of complex numbers is introduced. From a strictly logical point of view it is desirable to define a complex number as an ordered pair (a, b) of real numbers a and b subject to certain operational definitions. These definitions are as follows.

(i) $(a, b) + (c, d) = (a + c, b + d)$,

(ii) $(a, b) . (c, d) = (ac - bd, ad + bc)$,

(iii) $(a, b) = (c, d)$ iff $a = c$ & $b = d$,

(iv) $m(a, b) = (ma, mb)$.

Also we denote by 'i' the ordered pair (0, 1) and we identify the real number 'a' with the ordered pair $(a,0)$.

Now from the above definition we see that

$$\begin{aligned}(x, y) &= (x, 0) + (0, y)\\ &= x(1, 0) + y(0, 1)\\ &= x + iy.\end{aligned}$$

Further we see that

$$i^2 = (0, 1) . (0, 1) = (-1, 0) = -1.$$

1.2 Fundamental Operations with Complex Numbers :

(i) Addition : $(a + ib) + (c + id) = (a + c) + i(b + d)$.

(ii) Subtraction : $(a + ib) - (c + id) = (a - c) + i(b - d)$.

(iii) Multiplication : $(a + ib)(c + id) = (ac - bd) + i(ad + bc)$.

(iv) Division : $\dfrac{a+ib}{c+id} = \dfrac{(a+ib)(c-id)}{(c+id)(c-id)} = \dfrac{ac+bd}{c^2+d^2} + i\dfrac{bc-ad}{c^2+d^2}$.

From the above we can prove that if z_1, z_2, z_3 belong to the set S of complex numbers, then

(i) $z_1 + z_2$ and $z_1 z_2$ belong to S; Closure law

(ii) $z_1 + z_2 = z_2 + z_1$; Commutative law of addition

(iii) $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$; Associative law of addition

(iv) $z_1 z_2 = z_2 z_1$; Commutative law of multiplication

(v) $z_1(z_2 z_3) = (z_1 z_2)z_3$; Associative law of multiplication

(vi) $z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$; Distributive law.

(vii) $z_1 + 0 = 0 + z_1 = z_1$, $1.z_1 = z_1.1 = z_1$, 0 is called the identity with respect to addition, 1 is called the identity with respect to multiplication.

(viii) For any complex number z_1 there is a unique number z in S such that $z + z_1 = 0$; z is called the inverse of z_1 with respect to addition and is denoted by $-z_1$.

(ix) For any $z_1 \neq 0$ there is a unique number z in S such that $z_1 z = z z_1 = 1$; z is called the inverse of z_1 with respect to multiplication and is denoted by z_1^{-1} or $\frac{1}{z_1}$.

In general any set, such as S, whose members satisfy the above is called a field.

1.3 Complex Plane or Argand Plane :

Let us consider two mutually perpendicular axes X′OX and Y′OY called x-axis and y-axis on a plane. Since a complex number $z = x + iy$ can be considered as an ordered pair of real numbers, we can represent such members by points in the xy-plane, called the complex plane or Argand Plane. To each complex number $z = x + iy$ there corresponds one and only one complex number $z = x + iy$. Because of this we often referred to the complex number z, as the point z. Sometimes we referred to x and y as the real and imaginary parts of $z = x + iy$ respectively and so the x-axis and y-axis are sometimes called the real and imaginary axis respectively. The complex plane is generally denoted by C and is often called the z-plane.

The nonnegative number $|z|$, called modulus or absolute value of $z = (x, y)$ is the distance of the complex number z from the origin, and hence $|z| = \sqrt{x^2 + y^2}$ (see Fig. 1.1).

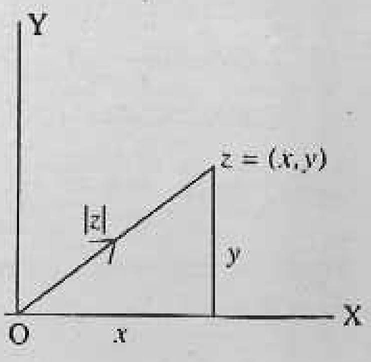


Fig. 1.1

The distance between two points $z_1 = x_1 + iy_1$, and $z_2 = x_2 + iy_2$ in the complex plane is given by

$$|z_1 - z_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

We shall frequently use the following simple inequalities.

$$x \le |x| \le \sqrt{x^2 + y^2} \Rightarrow \text{Re}(z) \le |\text{Re}(z)| \le |z|.$$

$$y \le |y| \le \sqrt{x^2 + y^2} \Rightarrow \text{Im}(z) \le |\text{Im}(z)| \le |z|.$$

Conjugate

If $z = x + iy$ is any complex number, then its complex conjugate denoted by $\bar{z}$ is $\bar{z} = x - iy$ or $\bar{z} = (x, -y)$. Obviously $\bar{z}$ is the mirror image of the complex point z into real axis. This indicates that $z = \bar{z} \Leftrightarrow z$ is purely a real number. Also $\bar{\bar{z}} = z$.

The following are the easy consequences of the above definition.

(i) $\overline{z_1 \pm z_2} = \bar{z}_1 \pm \bar{z}_2$.

(ii) $\overline{z_1 z_2} = \bar{z}_1 \, \bar{z}_2$.

(iii) $z + \bar{z} = 2 \, \text{Re}(z)$.

(iv) $z - \bar{z} = 2i \, \text{Im}(z)$.

(v) $z\bar{z}$ is real and positive unless $z = 0$.

(vi) $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\bar{z}_1}{\bar{z}_2}$.

Another entirely equivalent way of representing a complex number $z = x + iy$ is to use the vector $\overrightarrow{OP}$ joining the origin O of the complex plane to the point $P = (x, y)$, instead of using the point P itself. We discuss the sum and difference of two complex numbers as follows :

(i) Sum : Since a complex number $z = (x, y)$ can be represented by a vector in xy-plane, the sum of two complex numbers may be given by a vector which is diagonal of the parallelogram whose sides are represented by these vectors. (See Fig. 1.2).

(ii) Difference : Similarly, the difference of two complex numbers z_1, z_2 is the vector $z_1 - z_2$ joining z_2 to z_1 as illustrated in Fig. 1.3.

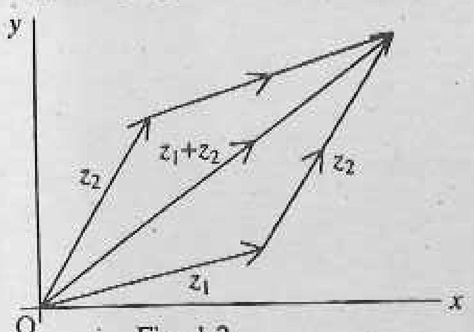


Fig. 1.2

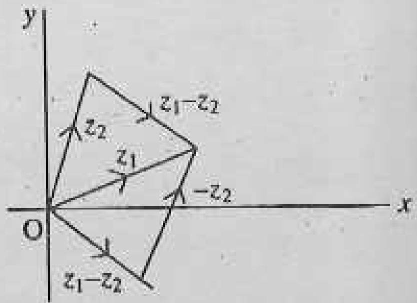


Fig. 1.3

1.4 Polar form of Complex Numbers :

If P is the point in the complex plane corresponding to the complex number $z = x + iy$, then we see from Fig. 1.4 that

$$x = r\cos\theta \;\&\; y = r\sin\theta$$

where $r = \sqrt{x^2 + y^2}$ is called the modulus or magnitude or absolute value of $z = x + iy$, denoted by $|z|$ and θ, called the amplitude or argument of $z = x + iy$, denoted by arg z, is the angle which the line OP makes with the positive x-axis. That is

$$\arg z = \theta = \tan^{-1}\frac{y}{x}.$$

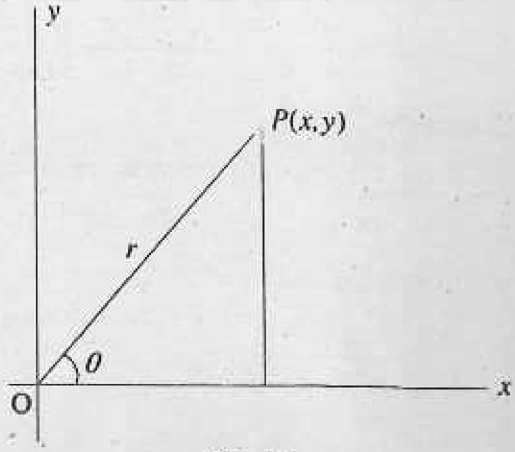


Fig. 1.4

If we restrict the angle θ in $0 \le \theta < 2\pi$, then it is called the principal argument of z. Now it follows from Fig. 1.4 that

$$z = x + iy = r\cos\theta + ir\sin\theta$$

which is called the polar from of the complex number and r and θ are called the polar coordinates.

Roots of a Complex Number

A number ω is called an nth root of a complex number z if $\omega^n = z$, and we write $\omega = z^{1/n}$. If n is a positive integer, then

$$z^{1/n} = \{r(\cos\theta + i\sin\theta)\}^{1/n}$$

$$= r^{1/n}\left[\cos\left(\frac{\theta+2k\pi}{n}\right)+i\sin\left(\frac{\theta+2k\pi}{n}\right)\right],\; k = 0, 1, 2, \ldots, n-1,$$

from which it follows that there are n different values for $z^{1/n}$, i.e. n different nth roots of z, provided $z \neq 0$.

Euler's Formula

By assuming that the infinite series expansion

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

of elementary calculus holds when $x = i\theta$, we can arive at

$$e^{i\theta} = \cos\theta + i\sin\theta, \quad e = 2.71828 \cdots \tag{1.1}$$

which is called Euler's formula. It is more convenient, however, simply to take (1.1) as a definition of $e^{i\theta}$. In general, we define

$$e^{z} = e^{x+iy} = e^{x}.e^{iy} = e^{x}(\cos y + i\sin y).$$

The nth Roots of Unity

The solutions of the equation $z^n = 1$ where n is a positive integer are called the nth roots of unity and are given by

$$z = \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n} = e^{\frac{2k\pi i}{n}}, \; k = 0, 1, 2, \cdots, n-1.$$

If we let $\omega = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n} = e^{\frac{2\pi i}{n}}$, the n roots are $1, \omega, \omega^2, \cdots, \omega^{n-1}$. Geometrically they represent the n vertices of a regular polygon of n sides inscribed in a circle of radius one with centre at the origin. This circle has the equation $|z| = 1$ and is often called the unit circle.

Point at Infinity

By means of the transformation $w = \frac{1}{z}$ the point $z = 0$ (i.e. the origin) is mapped into $\omega = \infty$, called the point at infinity in the ω-plane. Similarly we denote by $z = \infty$ the point at infinity in the z-plane. To consider the behaviour of $f(z)$ at $z = \infty$, it suffices to let $z = \frac{1}{\omega}$ and examine the behaviour of $f\left(\frac{1}{\omega}\right)$ at $\omega = 0$.

1.5 Extended Complex Plane :

By the extended complex number system, we shall mean the complex plane C along with ∞, the point at infinity, which satisfy the following properties :

(i) If $z \in C$, then we have $z + \infty = z - \infty = \infty$, $\frac{z}{\infty} = 0$.

(ii) If $z \in C$, and $z \neq 0$, then $z.\infty = \infty$ and $\frac{z}{0} = \infty$.

(iii) $\infty + \infty = \infty.\infty = \infty$.

(iv) $\frac{\infty}{z} = \infty \; (z \neq \infty)$.

The set $C \cup \{\infty\}$ is called extended complex plane.

The nature of the Argand plane at the point at infinity is made much clear by the use of Riemann's spherical representation of complex numbers, which depends on Stereographic Projection.

Stereographic Projection

Let C be the complex plane and consider a unit sphere $\bar{S}$ (radius one) tangent to C at $z = 0$. The diameter NS is perpendicular to C and we call the points N and S the north and south poles of $\bar{S}$ respectively. Corresponding to any point A on C we can construct the line NA intersecting $\bar{S}$ at a point A′. Thus to each point of the complex plane C there corresponds one and only one point of the sphere $\bar{S}$ and we can represent any complex number by a point on the sphere.

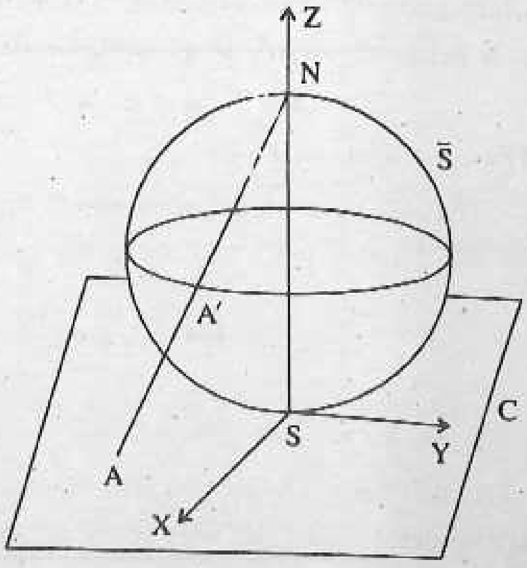


Fig. 1.5

For completeness we say that the point N itself corresponds to the point at infinity of the complex plane. The set of all points of the complex plane including the point at infinity is called the entire complex plane or the entire z-plane or the extended complex plane.

The above method for mapping the plane on to the sphere is called stereographic projection. The sphere is sometimes called the Riemann sphere.

Example 1.1. Find all the values of z for which $z^5 = -32$ and locate these values in the complex plane.

Solution : $z^5 = -32 = 32\,[\cos(2k+1)\pi + i\sin(2k+1)\pi]$, $k = 0, \pm 1, \pm 2, \cdots$

$$\text{i.e. } z = 2\left[\cos\frac{(2k+1)\pi}{5} + i\sin\frac{(2k+1)\pi}{5}\right],\ k = 0, 1, 2, 3, 4.$$

If $k = 0$, $z = z_1 = 2\left(\cos\frac{\pi}{5} + i\sin\frac{\pi}{5}\right)$.

If $k = 1$, $z = z_2 = 2\left(\cos\frac{3\pi}{5} + i\sin\frac{3\pi}{5}\right)$.

If $k = 2$, $z = z_3 = 2\left(\cos\frac{5\pi}{5} + i\sin\frac{5\pi}{5}\right) = -2$.

If $k = 3$, $z = z_4 = 2\left(\cos\frac{7\pi}{5} + i\sin\frac{7\pi}{5}\right)$.

If $k = 4$,

$$z = z_5 = 2\left(\cos\frac{9\pi}{5} + i\sin\frac{9\pi}{5}\right).$$

These are the only roots of the given equation. These five roots are called the fifth roots of -32 and are collectively denoted by $(-32)^{1/5}$.

The values of z are indicated in Fig. 1.6. Note that they are equally spaced along the circumference of a circle with centre at the origin and radius 2. Another way of saying this is that the roots are represented by the vertices of a regular polygon.

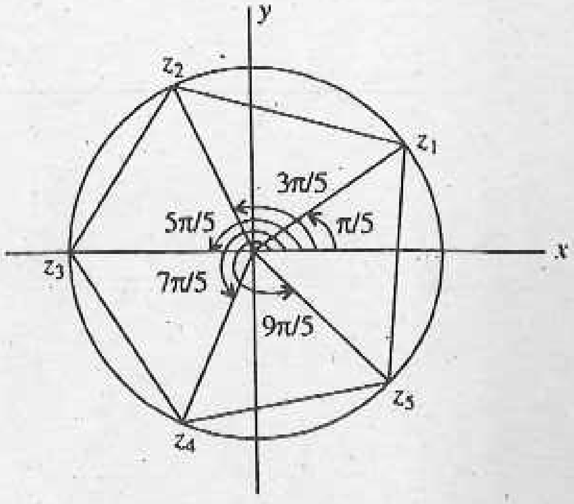


Fig. 1.6

Example 1.2. Find each of the indicated roots and locate them graphically.

$$\left(-2\sqrt{3} - 2i\right)^{\frac{1}{4}}$$

Solution : $-2\sqrt{3} - 2i = 4\left[\cos\left(2k\pi + \frac{7\pi}{6}\right) + i\sin\left(2k\pi + \frac{7\pi}{6}\right)\right]$

$$\therefore \left(-2\sqrt{3} - 2i\right)^{\frac{1}{4}} = 4^{\frac{1}{4}}\left[\cos\left(\frac{2k\pi + \frac{7\pi}{6}}{4}\right) + i\sin\left(\frac{2k\pi + \frac{7\pi}{6}}{4}\right)\right],$$

$$k = 0, 1, 2, 3.$$

If $k = 0$, $z_1 = \sqrt{2}\left(\cos\frac{7\pi}{24} + i\sin\frac{7\pi}{24}\right)$.

If $k = 1$, $z_2 = \sqrt{2}\left(\cos\frac{19\pi}{24} + i\sin\frac{19\pi}{24}\right)$.

If $k = 2$, $z_3 = \sqrt{2}\left(\cos\frac{31\pi}{24} + i\sin\frac{31\pi}{24}\right)$.

If $k = 3$, $z_4 = \sqrt{2}\left(\cos\frac{43\pi}{24} + i\sin\frac{43\pi}{24}\right)$.

These are represented graphically in Fig. 1.7

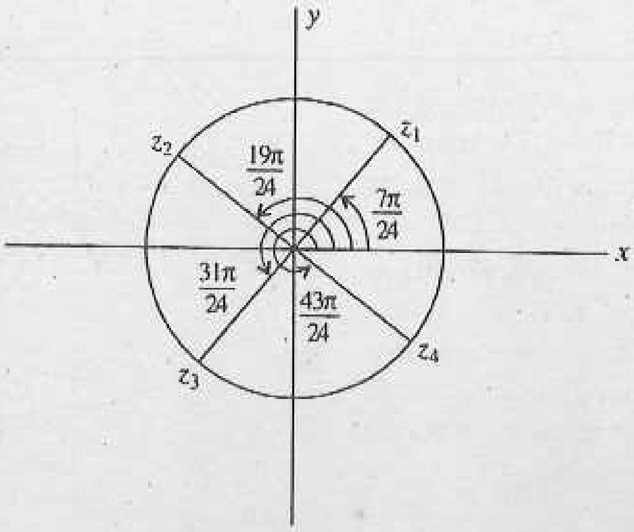


Fig. 1.7

Example 1.3. Establish the relation

$$\frac{n}{2^{n-1}} = \prod_{k=1}^{n-1} \sin\left(\frac{k\pi}{n}\right), \ n \geq 2.$$

Solution : Let 1, ω_1, ω_2, ..., ω_{n-1} be the n roots of unity, where

$$\omega_k = e^{\frac{2\pi ki}{n}}, \ k = 0, 1, 2, \cdots, n-1.$$

Then, $z^n - 1 = (z-1)(z-\omega_1)(z-\omega_2)\cdots(z-\omega_{n-1})$.

Dividing both sides by $z - 1$ and letting $z \to 1$, we get

$$n = (1-\omega_1)(1-\omega_2)\cdots(1-\omega_{n-1})$$

and hence $\quad n = \left(1-\overline{\omega}_1\right)\left(1-\overline{\omega}_2\right)\cdots\left(1-\overline{\omega}_{n-1}\right).$

Thus, $\quad n^2 = \prod_{k=1}^{n-1}(1-\omega_k)\left(1-\overline{\omega}_k\right)$

$$= \prod_{k=1}^{n-1}\left(1-e^{\frac{2\pi ki}{n}}\right)\left(1-e^{-\frac{2\pi ki}{n}}\right)$$

$$= \prod_{k=1}^{n-1} 2\left(1-\cos\frac{2k\pi}{n}\right)$$

$$= \prod_{k=1}^{n-1} 4\sin^2\left(\frac{k\pi}{n}\right)$$

$$\text{i.e. } n^2 = 4^{n-1} \prod_{k=1}^{n-1} \sin^2\left(\frac{k\pi}{n}\right)$$

$$\text{i.e. } \frac{n}{2^{n-1}} = \prod_{k=1}^{n-1} \sin\left(\frac{k\pi}{n}\right),\ n \geq 2,$$

taking non-negative square root of both sides.

Example 1.4. Find all the roots of $(1 + z)^5 = (1 - z)^5$.

Solution : Let $\omega = \dfrac{1+z}{1-z}$. Then the given equation becomes

$$\omega^5 = 1 = \cos 2k\pi + i \sin 2k\pi$$

$$\text{i.e. } \omega = \cos\frac{2k\pi}{5} + i \sin\frac{2k\pi}{5},\ k = 0, 1, 2, 3, 4.$$

$$= e^{\frac{2k\pi i}{5}}$$

Again from $\omega = \dfrac{1+z}{1-z}$ we obtain $z = \dfrac{\omega-1}{\omega+1}$.

Hence $z = \dfrac{e^{\frac{2k\pi i}{5}} - 1}{e^{\frac{2k\pi i}{5}} + 1}$, $k = 0, 1, 2, 3, 4$.

Example 1.5. If $f(z) = a_0 z^n + a_1 z^{n-1} + ... + a_{n-1} z + a_n$ is a polynomial in a complex variable z with real coefficients $a_0, a_1, ..., a_n$ then show that $\overline{f(z)} = f(\bar{z})$.

Solution : We have

$$\overline{f(z)} = \overline{a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n}$$

$$= \overline{a_0 z^n} + \overline{a_1 z^{n-1}} + \cdots + \overline{a_{n-1} z} + \bar{a}_n$$

$$= a_1 \bar{z}^n + a_1 \overline{z^{n-1}} + \cdots + a_{n-1} \bar{z} + a_n$$

$$[\because a_0, a_1, ..., a_n \text{ are real}]$$

$$= a_0 (\bar{z})^n + a_1 (\bar{z})^{n-1} + \cdots + a_{n-1} \bar{z} + a_n$$

$$= f(\bar{z}).$$

Example 1.6. For any two non-zero complex numbers z_1 and z_2 prove that

$$2(|z_1|+|z_2|) \geq (|z_1+z_2|)\left|\frac{z_1}{|z_1|}+\frac{z_2}{|z_2|}\right|$$

Proof. We have, $|z_1 + z_2| \left|\frac{z_1}{|z_1|}+\frac{z_2}{|z_2|}\right|$

$$= |z_1 + z_2| \left|\frac{\{z_1|z_2|+|z_1|z_2\}}{|z_1||z_2|}\right|$$

$$= |z_1 + z_2| \frac{|\{z_1|z_2|+|z_1|z_2\}|}{|z_1 z_2|}$$

$$\leq \frac{|z_1+z_2|}{|z_1 z_2|}\{|z_1|z_2||+||z_1|z_2|\}$$

$$= \frac{|z_1+z_2|}{|z_1 z_2|}\{|z_1 z_2|+|z_1 z_2|\}$$

$$= \frac{|z_1+z_2|}{|z_1 z_2|}\cdot 2|z_1 z_2| = 2|z_1+z_2|.$$

This proves the inequality.

1.6 Equation of a Straight Line in a Complex Plane :

We now consider the equation of a straight line in a complex plane taking the real and imaginary axes as the axes of coordinates.

The equation of any straight line can be written as

$$ax + by + c = 0 \qquad \text{... (1.2)}$$

where a, b, c are real numbers and a, b are not simultaneously zero. Let $z = (x,y)$, $\bar{z} = (x, -y)$. Then we can write $z = x + iy$ and $\bar{z} = x - iy$.

Putting $x = \frac{z+\bar{z}}{2}$ and $y = \frac{z-\bar{z}}{2i}$ in (1.2) we get

$$a\left(\frac{z+\bar{z}}{2}\right)+b\left(\frac{z-\bar{z}}{2i}\right)+c=0$$

i.e. $a(z+\bar{z})-ib(z-\bar{z})+2c=0$

i.e. $(a-ib)z+(a+ib)\bar{z}+2c=0$

i.e. $\alpha z+\overline{\alpha z}+c_1=0$,

where $\alpha = a - ib$ and $c_1 = 2c$.

Conversely, let us consider an equation of the form

$$\beta z + \overline{\beta z} + k = 0 \qquad \text{... (1.3)}$$

where β is an arbitrary non-zero complex number and k is real. Now if $\beta = (a, b)$ and $z = (x,y)$, then

$$\beta z + \overline{\beta z} + k = 0$$

$$\Rightarrow (a + ib)(x + iy) + (a - ib)(x - iy) + k = 0$$

$$\Rightarrow 2ax - 2by + k = 0$$

$$\Rightarrow a'x + b'y + c' = 0$$

where $a' = 2a$, $b' = -2b$ and $c' = k$. Hence equation (1.3) represents a straight line in the complex plane.

Equation of a straight line through two given points represented by complex numbers z_1 and z_2.

Let $P(z)$ be any point on the line AB so that from Fig. 1.8 we have

$$\arg \frac{z-z_1}{z_1-z_2} = 0 \text{ or } \pi$$

and hence $\frac{z-z_1}{z_1-z_2}$ is purely real.

So if $z^* = \frac{z-z_1}{z_1-z_2}$, then $z^* = \overline{z^*}$,

i.e. $\overline{\left(\frac{z-z_1}{z_1-z_2}\right)} = \frac{z-z_1}{z_1-z_2}$.

i.e. $(z-z_1)(\bar{z}_1-\bar{z}_2) = (z_1-z_2)(\bar{z}-\bar{z}_1)$

i.e. $z(\bar{z}_1-\bar{z}_2)-\bar{z}(z_1-z_2) = -z_1\bar{z}_2+\bar{z}_1z_2$

i.e. $z(\bar{z}_1-\bar{z}_2)-\bar{z}(z_1-z_2)+(z_1\bar{z}_2-\bar{z}_2z_1)=0$.

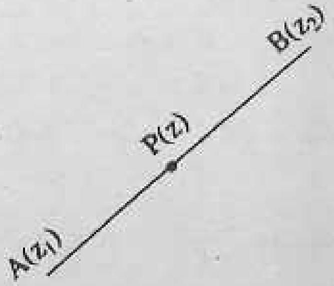


Fig. 1.8

Above represents the required equation of the line.

1.7 Equation of a Circle :

Let the complex number a represent the centre C of the circle in the Argand's plane and r be its radius. If $P(z)$ be any point on its circumference then $\overrightarrow{CP} = z - a$.

Therefore, $\left|\overrightarrow{CP}\right| = |z-a| = r$

i.e. $|z-a|^2 = r^2$

i.e. $(z-a)(\overline{z-a}) = r^2$

i.e. $(z-a)(\bar{z}-\bar{a}) = r^2$

i.e. $z\bar{z} - a\bar{z} - \bar{a}z + a\bar{a} = r^2$

i.e. $z\bar{z} - a\bar{z} - \bar{a}z + \left(|a|^2 - r^2\right) = 0,$

which represents the equation of the circle where a is a complex constant and $|a|^2 - r^2$ is real.

Equation of a circle through three given points.

Let $A(z_1)$, $B(z_2)$, $C(z_3)$ be the three points through which the circle passes and $P(z)$ be any other point on the circumference of the circle.

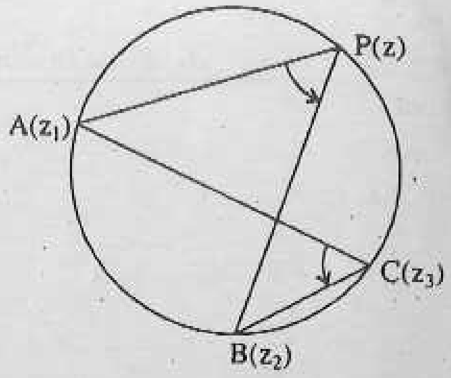


Fig. 1.9.

Then $\angle APB = \angle ACB$

as shown in Fig. 1.9. If P be taken in between A and B then

$$\angle APB - \angle ACB = \pi.$$

Thus, $\angle APB - \angle ACB = 0$ or π, in any case.

i.e. $\text{Arg}\,\dfrac{z-z_1}{z-z_2} - \text{Arg}\,\dfrac{z_3-z_1}{z_3-z_2} = 0$ or π

i.e. $\text{Arg}\,\dfrac{(z-z_1)(z_3-z_2)}{(z-z_2)(z_3-z_1)} = 0$ or π

Hence $\dfrac{(z-z_1)(z_3-z_2)}{(z-z_2)(z_3-z_1)}$ is purly real and so

$$\frac{(\overline{z-z_1})(\overline{z_3-z_2})}{(\overline{z-z_2})(\overline{z_3-z_1})} = \frac{(z-z_1)(z_3-z_2)}{(z-z_2)(z_3-z_1)} \quad \text{... (1.4)}$$

This is the equation of the desired circle.

Note 1.1. If we replace z by z_4 in (1.4), then we obtain

$$\frac{(\overline{z_4-z_1})(\overline{z_3-z_2})}{(\overline{z_4-z_2})(\overline{z_3-z_1})} = \frac{(z_4-z_1)(z_3-z_2)}{(z_4-z_2)(z_3-z_1)},$$

which is the condition for four points to be concyclic.

Example 1.7. Prove that $\left|\frac{z-1}{z+1}\right|$ = constant and amp $\left|\frac{z-1}{z+1}\right|$ = constant are orthogonal circles.

Proof. Given $\left|\frac{z-1}{z+1}\right| = \lambda$ (constant)

or, $\left|\frac{x-1+iy}{x+1+iy}\right| = \lambda$

or, $\frac{(x-1)^2+y^2}{(x+1)^2+y^2} = \lambda^2$

or, $x^2 + y^2 + 2\frac{\lambda^2+1}{\lambda^2-1}\,x + 1 = 0$

This is of the form

$$x^2 + y^2 + 2gx + 1 = 0 \qquad \text{... (1.5)}$$

which represents a circle.

Also amp $\left(\frac{z-1}{z+1}\right)$ = constant

or, amp $(x - 1 + iy)$ − amp $(x + 1 + iy)$ = constant

or, $\tan^{-1}\frac{y}{x-1} - \tan^{-1}\frac{y}{x+1}$ = constant

Therefore, $\tan^{-1}\frac{2y}{x^2+y^2-1}$ = constant

or, $\frac{2y}{x^2+y^2-1} = \mu$ (say)

or, $x^2 + y^2 + 2fy - 1 = 0 \qquad \text{... (1.6)}$

where $f = -\frac{2}{\mu}$, which also represents a circle.

Clearly the circles represented by (1.5) and (1.6) are orthogonal, because with the usual notation the condition of orthogonality $2g_1g_2 + 2f_1f_2 = c_1 + c_2$ is satisfied.

Example 1.8. If c is real, $b = p + iq$, $z = x + iy$, then show that the equation $b(z + \bar{z}) + \bar{b}(\bar{z} - z) + c = 0$ represents two straight lines in the xy-plane and find the angle between them.

Solution : We know that every linear equation in x and y with real coefficients represent a straight line in the xy-plane.

Now, $b(z+\bar{z})+\bar{b}(\bar{z}-z)+c=0$

$\Rightarrow (p+iq)2x+(p-iq)(-2iy)+c=0$

$\Rightarrow (2px-2qy+c)+i(2qx-2py)=0$

Equating the real and imaginary part we get

$$2px-2qy+c=0 \quad \dots (1.7)$$

$$2qx-2py=0 \quad \dots (1.8)$$

There represents two straight lines in xy-plane.

If θ be the angle between the straight lines (1.7) and (1.8), then

$$\tan\theta = \pm\frac{\frac{p}{q}-\frac{q}{p}}{1+\frac{p}{q}.\frac{q}{p}} = \pm\frac{p^2-q^2}{2pq}.$$

Hence the required angle is

$$\theta = \tan^{-1}\left(\pm\frac{p^2-q^2}{2pq}\right).$$

Example 1.9. If the sum and product of two complex numbers are both real, then the two numbers must be either real or conjugate.

Solution : Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$

Then $z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$

and $z_1z_2 = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)$.

By the given condition we have

$$y_1 + y_2 = 0 \text{ and } x_1y_2 + x_2y_1 = 0$$

This gives $y_2 = -y_1$ and $x_2 = x_1$ provided $y_1 \neq 0$.

Hence the result follows.

Example 1.10. Find the equation of the circle joining the points z_1 and z_2 as diameter.

Solution : Let $P(z)$ be any point on the circumference of the circle. (see Fig. 1.10). Then the angle between the lines AP and BP is $\frac{\pi}{2}$. Hence

$$\arg\frac{z-z_1}{z-z_2} = \frac{\pi}{2}.$$

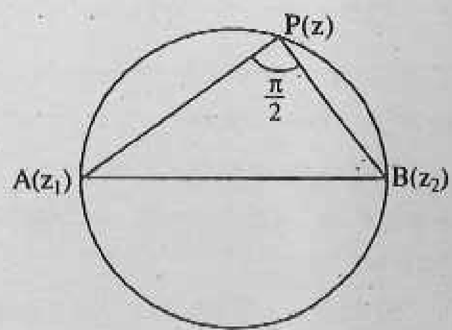


Fig. 1.10

This means $\dfrac{z-z_1}{z-z_2}$ is purely imaginary.

$$\text{i.e. } \frac{1}{2}\left[\frac{z-z_1}{z-z_2}+\overline{\frac{z-z_1}{z-z_2}}\right]=0$$

$$\text{i.e. } (z-z_1)(\bar{z}-\bar{z}_2)+(z-z_2)(\bar{z}-\bar{z}_1)=0$$

$$\text{i.e. } 2z\bar{z}-z(\bar{z}_1+\bar{z}_2)-\bar{z}(z_1+z_2)+z_1\bar{z}_2+z_2\bar{z}_1=0,$$

which is the required equation.

Example 1.11 : Find the equation for an ellipse with major axis of length 10 and foci at (–3,0) and (3,0).

Solution : From the given condition the sum of the distances from any point z on the ellipse to the foci must be equal to 10.

Hence the required equation is $|z + 3| + |z - 3| = 10$.

Example 1.12. If z_1, z_2 and z_3 are three complex numbers satisfying

$$z_1^2 + z_2^2 + z_3^2 - z_2z_3 - z_3z_1 - z_1z_2 = 0.$$

then show that $|z_2 - z_3| = |z_3 - z_1| = |z_1 - z_2|$.

Solution : We have

$$z_1^2 + z_2^2 + z_3^2 - z_2z_3 - z_3z_1 - z_1z_2 = 0$$

$$\text{i.e. } (z_1 - z_2)^2 = (z_1 - z_3)(z_3 - z_2)$$

$$\text{i.e. } |(z_1 - z_1)^2| = |(z_1 - z_3)(z_3 - z_2)|$$

$$\text{i.e. } |z_1 - z_2|^2 = |z_3 - z_1|\,|z_2 - z_3|$$

Similarly, we obtain

$$|z_2 - z_3|^2 = |z_1 - z_2|\,|z_3 - z_1|$$

$$\text{and } |z_3 - z_1|^2 = |z_1 - z_2|\,|z_2 - z_3|.$$

Hence

$$2|z_1 - z_2|^2 + 2|z_2 - z_3|^2 + 2|z_3 - z_1|^2$$

$$= 2|z_3 - z_1|\,|z_2 - z_3| + 2|z_1 - z_2|\,|z_3 - z_1| + 2|z_1 - z_2|\,|z_2 - z_3|$$

$$\text{i.e. } (|z_1 - z_2| - |z_2 - z_3|)^2 + (|z_2 - z_3| - |z_3 - z_1|)^2 + (|z_3 - z_1| - |z_1 - z_2|)^2 = 0.$$

This gives $|z_1 - z_2| = |z_2 - z_3| = |z_3 - z_1|$, as

$|z_1 - z_2|$, $|z_2 - z_3|$ and $|z_3 - z_1|$ are reals.

Exercise - 1

1. Show that $Im(iz) = \text{Re}(z)$ and $\text{Re}(iz) = -Im(z) = |z|^2 Im(z^{-1})$.
2. Solve the equation : $|z| - z = 2 + i$.
3. If z_1, z_2, z_3 be nonzero complex numbers such that $z_1 \neq z_2$, $|z_1| = a$, $|z_2| = b$, $|z_3| = c$. If $a^3 + b^3 + c^3 = 3abc$, show that

$$2 \arg \frac{z_3 - z_1}{z_2 - z_1} = \arg \frac{z_3}{z_2}.$$

4. Let z_1, z_2, z_3 be the vertices of a triangle on the complex plane. If
   i. the triangle is equilateral, then show that
   $$z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1;$$
   ii. the triangle is isosceles and right angled at z_3, then show that
   $$z_1^2 + z_2^2 + 2z_3^2 = 2(z_1 + z_2)z_3.$$
5. Find the area of a triangle in the complex plane whose vertices are z_1, z_2 and z_3.
6. Find all the roots of $(-8 - 8\sqrt{3}\, i)^{1/4}$ and exhibit them geometrically.
7. Find all the 5th roots of unity.
8. Solve : $6z^4 - 25z^3 + 32z^2 + 3z - 10 = 0$.
9. If $p + iq$ is a root of $a_0z^n + a_1z^{n-1} + \cdots + a_{n-1}z + a_n = 0$ where $a_0 \neq 0$, $a_1, \ldots, a_{n-1}, a_n, p$ and q are real, then prove that $p - iq$ is also a root.
10. Find each of the indicated roots and locate them graphically $(-1 + i)^{1/3}$.
11. Find an equation of a circle of radius 2 with centre at $(-3, 4)$.
12. Find two complex numbers whose sum is 4 and product is 8.
13. Prove that for any integar $m > 1$,

$$\cot\frac{\pi}{2m} \cot\frac{2\pi}{2m} \cot\frac{3\pi}{2m} \cdots \cot\frac{(m-1)\pi}{2m} = 1.$$

14. Find that point on the line joining points z_1 and z_2 which divides it in the ratio $p : q$.
15. Find all the roots of the equation
$$z^4 - (1-z)^4 = 0.$$
16. Show that the equaiton $z\bar{z} + b\bar{z} + \bar{b}z + c$ represents a circle when c is real number and $|b|^2 > c$; show further that the centre represents complex number $-b$ and radius is $\sqrt{|b|^2 - c}$.
17. Let the roots z_1, z_2, z_3 of the equation $x^3 + 3ax^2 + 4bx + c = 0$, in which a, b, c are complex numbers, correspond to the points A, B, C in the Argand plane. Show that it will be equilateral if $a^2 = b$.

Unit : 2 ❑ Functions, Limits and Continuity

2.1 Introduction :

Students are very well conversant with the definitions of limits and continuity for functions of a real variable. In this unit we shall deal with corresponding definitions for functions of a complex variable and these definitions are analogous to the definition we are already familiar with.

2.2 Some definitions :

Definition 2.1. Point Set

Any collection of points in the complex plane is called a point set and each point is called an element of the set.

Definition 2.2. Neighbourhood of a point

A neighbourhood of a point $z_0 \in C$ is the set of all point z such that $|z - z_0| < r$ where r is some positive number, i.e. the set of all points lying in the disc with centre z_0 and radius r.

Definition 2.3. Deleted neighbourhood of a point

A deleted neighbourhood of a point $z_0 \in C$ is a neighbourhood of z_0 in which the point z_0 itself is omitted, i.e. $0 < |z - z_0| < r$.

Definition 2.4. Neighbourhood of the point at infinity

The set of all points z such that $|z| > k$ where k is any positive real number is called a neighbourhood of the point of infinity.

Definition 2.5. Limit Point.

A point z_0 is called a limit point of a set S in the complex plane if every deleted neighbourhood of z_0 contains at least one point of S. A limit point may or may not belong to the set.

We consider the set of points defined by $|z| < r$. Evidently all points on the circle $|z| = r$ are the limit points of this set and they do not belong to the set. Again all points within the circle $|z| = r$ are also limit points of the set defined by $|z| < r$ and they belong to the set.

Definition 2.6. Interior, Exterior and Boundary Points

A point $z_0 \in S$ is called an interior point of the set S if $\exists$ a neighbourhood of z_0 contained entirely within S.

The point z_0 will be an exterior point of S if $\exists$ a neighbourhood of z_0, which contains no point of S.

Moreover, the point z_0 is called a boundary point of a set S if every neighbourhood of z_0 contains at least one point of S and at least one point not of S. The collection of all boundary points of a set S is called the boundary of S.

Definition 2.7. Open and closed set

A set S in the complex plane is said to be open if it consists only of interior points.

A set S is called closed if its complement is open. Equivalently a set S is said to be closed if every limit point of S belongs to S. or if S has no limit point.

The open dise $|z - z_0| < r$ is an open set and the closed disc $|z - z_0| \leq r$ is a closed set. There exist sets which are neither open nor closed : the set consisting of the point $z = 1$ and all points for which $|z| < 1$ is a set which is neither open nor closed.

Definition 2.8. Bounded, Unbounded and Compact Sets.

A set of points S is said to be bounded if there exists a positive number M such that $|z| \leq M \ \forall \ z \in S$. If there exists no such number M, the set is said to be unbounded.

A set which is bounded and closed is called compact.

Definition 2.9. Derived set and closure of a set

The set of all limit points of a set S is called the derived set of S and is denoted by S'.

The union of a set S and its derived set S' is called the closure of S and is denoted by $\bar{S}$ or cl(S). Thus

$$\bar{S} \text{ or } \mathrm{cl}(S) = S \cup S'.$$

Definition 2.10 Connected Set

A set S is called connected if any two of its points can be join by a polygon all of whose points belong to the set.

Definition 2.11 Open and closed Domain or Regions

An open connected set is called a domain or an open region. If however the boundary points are also included then it is called a closed domain.

Definition 2.12 Jordan Curve

The equation $z = z(t) = x(t) + iy(t)$ where $x(t)$ and $y(t)$ are real continuous function of real variable t, defined in the interval $a \leq t \leq b$, determines a set of points in the complex plane which we call a continuous arc. The equaiton

$$z = z(t) = x(t) + iy(t)$$

determines a simple arc if $t_1 \neq t_2 \Rightarrow z(t_1) \neq z(t_2)$.

The equation $z = z(t) = x(t) + iy(t)$ determines a simple closed curve if $t_1 < t_2$ and $z(t_1) = z(t_2) \Rightarrow t_1 = a$ and $t_2 = b$.

Simple arcs and simple closed curves are often called Jordan arcs or Jordan curves respectively.

A simple example of a Jordan arc is the polygonal arc which consists of a finite number of line segments joint end to end.

Jardan Curve Theorem

A Jordan curve divides the plane into two regions having the curve as common boundary. That region which is bounded is called the interior of the curve, while the other region is called the exterior of the curve.

Bolzano-Weierstrass Theorem

If a set is bounded and contains infinitely many points then it possessess at least one limit point.

Definition 2.13 Variables and Functions

A symbol, such as z, which can stand for any one of a set of complex numbers is called a complex variable.

If to each value which a complex variable z can assume there corresponds one or more values of a complex variable ω, we say that ω is a function of z and write $\omega = f(z)$. The variable z is sometimes called an independent variable, while ω is called a dependent variable. The value of a function at $z = a$ is often written as $f(a)$. Thus if $f(z) = z^4$, then $f(3i) = (3i)^4 = 81$.

Definition 2.14 Single and Multiple valued Functions

If only one value of ω corresponds to each value of z, we say that ω is a single-valued function of z or that $f(z)$ is single-valued. If more then one value of ω corresponds to each value of z, we say that ω is a multiple-valued or many-valued function of z.

A multiple-valued function can be considered as a collection of single-valued functions, each member of which is called a branch of the function. It is customary to consider one particular member as a principal branch of the multiple-valued function and the value of the function corresponding to this branch as the principal value.

Example 2.1. If $\omega = z^2$, then to each value of z there is only one value of ω. Hence $\omega = f(z) = z^2$ is a single-valued function of z.

Example 2.2. If $\omega = z^{1/2}$, then to each value of z there are two values of ω. Hence $\omega = f(z) = z^{1/2}$ is a multiple-valued function of z.

Definition 2.15 Inverse Function

If $\omega = f(z)$, then we can also consider z as a function of ω, written $z = g(\omega)$ $= f^{-1}(\omega)$. The function f^{-1} is often called the inverse function corresponding to f. Thus $\omega = f(z)$ and $\omega = f^{-1}(z)$ are inverse functions of each other.

2.3 The Elementary Functions :

1. Polynomial Functions

Polynomial functions are defined by

$$\omega = a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n = P(z)$$

where $a_0 \neq 0, a_1, \cdots, a_{n-1}, a_n$ are complex constants and n is a positive integer called the degree of the polynomial $P(z)$.

2. Rational Functions

Rational functions are defined by

$$\omega = \frac{P(z)}{Q(z)}$$

where $P(z)$ and $Q(z)$ are polynomials.

3. Exponential Functions

Exponential functions are defined by

$$\omega = e^z = e^{x+iy} = e^x(\cos y + i \sin y)$$

where $e = 2.71828$ ... is the natural base of logarithms.

If a is real and positive, we difine

$$a^z = e^{z \log a}$$

where $\log a$ is the natural logarithm of a.

Complex exponential functions have properties similar to those of real exponential functions. For example,

$$e^{z_1} . e^{z_2} = e^{z_1 + z_2}, \quad \frac{e^{z_1}}{e^{z_2}} = e^{z_1 - z_2}.$$

4. Trigonometric Functions

We define trigonometric functions $\sin z$, $\cos z$, etc., in terms of exponential functions as follows :

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} \qquad \cos z = \frac{e^{iz} + e^{-iz}}{2}$$

$$\sec z = \frac{1}{\cos z} = \frac{2}{e^{iz} + e^{-iz}} \qquad \operatorname{cosec} z = \frac{1}{\sin z} = \frac{2i}{e^{iz} - e^{-iz}}$$

$$\tan z = \frac{\sin z}{\cos z} = \frac{e^{iz} - e^{-iz}}{i(e^{iz} + e^{-iz})} \qquad \cot z = \frac{\cos z}{\sin z} = \frac{i(e^{iz} + e^{-iz})}{e^{iz} - e^{-iz}}$$

Many of the proporties familiar in the case of real trignometric functions also hold for the complex trigonometric functions. For example, we have

$\sin^2 z + \cos^2 z = 1$ $\quad 1 + \tan^2 z = \sec^2 z$ $\quad 1 + \cot^2 z = \operatorname{cosec}^2 z$

$\sin(-z) = -\sin z$ $\quad \cos(-z) = \cos z$ $\quad \tan(-z) = -\tan z$

$\sin(z_1 \pm z_2) = \sin z_1 \cos z_2 \pm \cos z_1 \sin z_2$

$\cos(z_1 \pm z_2) = \cos z_1 \cos z_2 \mp \sin z_1 \sin z_2$

$$\tan(z_1 \pm z_2) = \frac{\tan z_1 \pm \tan z_2}{1 \mp \tan z_1 \tan z_2}$$

5. Hyperbolic Functions

Hyperbolic functions are defined as follows :

$$\sin hz = \frac{e^z - e^{-z}}{2} \qquad \cos hz = \frac{e^z + e^{-z}}{2}$$

$$\sec hz = \frac{1}{\cos hz} = \frac{2}{e^z + e^{-z}} \qquad \operatorname{cosec} hz = \frac{1}{\sin hz} = \frac{2}{e^z - e^{-z}}$$

$$\tan hz = \frac{\sin hz}{\cos hz} = \frac{e^z - e^{-z}}{e^z + e^{-z}} \qquad \cot hz = \frac{\cos hz}{\sin hz} = \frac{e^z + e^{-z}}{e^z - e^{-z}}$$

The following properties hold :

$\cos h^2 z - \sin h^2 z = 1$ $\quad 1 - \tan h^2 z = \sec h^2 z$ $\quad \cot h^2 z - 1 = \operatorname{cosec} h^2 z$

$\sin h(-z) = -\sin hz$ $\quad \cos h(-z) = \cos hz$ $\quad \tan h(-z) = -\tan hz$

$\sin h(z_1 \pm z_2) = \sin hz_1 \cos hz_2 \pm \cos hz_1 \sin hz_2$

$\cos h(z_1 \pm z_2) = \cos hz_1 \cos hz_2 \pm \sin hz_1 \sin hz_2$

$$\tan h(z_1 \pm z_2) = \frac{\tan h\, z_1 \pm \tan h\, z_2}{1 \pm \tan h\, z_1 \tan h\, z_2}$$

The following relations exist between the trigonometric functions and the hyperbolic functions :

$\sin(iz) = i \sin h z$ $\quad \cos(iz) = \cos h z$ $\quad \tan(iz) = i \tan h z$

$\sin h(iz) = i \sin z$ $\quad \cos h(iz) = \cos z$ $\quad \tan h(iz) = i \tan z.$

6. Logarithmic Functions

If $z = e^{\omega}$, then we write $\omega = \log z$, called the natural logarithm of z. Thus the natural logarithmic functions is the inverse of the exponential function and can be defined by

$\omega = \log z = \log r + i(\theta + 2k\pi)$, $k = 0, \pm 1, \pm 2,$

where $z = re^{i\theta} = re^{i(\theta + 2k\pi)}$. Note that $\log z$ is a multiple-valued (in this case infinitely many-valued) function. The principal-value or principal branch of $\log z$ is sometimes defined as $\log r + i\theta$ where $0 \le \theta < 2\pi$. However, any other interval of length 2π can also be used, e.g. $-\pi < \theta \le \pi$. etc.

The logarithmic function can be defined for real bases other than e. Thus if $z = a^{\omega}$, then $\omega = \log_a z$ where $a > 0$ and $a \ne 0, 1$. In this case $z = e^{\omega \log a}$.

7. Inverse Trigonometric Functions

If $z = \sin \omega$, then $\omega = \sin^{-1} z$ is called the inverse sine of z or arc sine of z. Similarly we define other inverse trigonometirc functions $\cos^{-1}z$, $\tan^{-1}z$, etc. These functions, which are multiple-valued, can be expressed in terms of natural logarithms as follows. In all cases we omit an additive constant $2k\pi i$, $k = 0, \pm 1, \pm 2, \cdots$, in the logarithm.

$$\sin^{-1} z = \frac{1}{i}\log\left(iz + \sqrt{1-z^2}\right) \qquad \operatorname{cosec}^{-1} z = \frac{1}{i}\log\left(\frac{i+\sqrt{z^2-1}}{z}\right)$$

$$\cos^{-1} z = \frac{1}{i}\log\left(z + \sqrt{z^2-1}\right) \qquad \sec^{-1} z = \frac{1}{i}\log\left(\frac{1+\sqrt{1-z^2}}{z}\right)$$

$$\tan^{-1} z = \frac{1}{2i}\log\left(\frac{1+iz}{1-iz}\right) \qquad \cot^{-1} z = \frac{1}{2i}\log\left(\frac{z+i}{z-i}\right)$$

8. Inverse Hyperbolic Functions

If $z = \sin h\, \omega$ then $\omega = \sin h^{-1} z$ is called the inverse hyperbolic sine of z. Similarly we define other inverse hyperbolic functions $\cos h^{-1}z$, $\tan h^{-1}z$, etc. These functions, which are multiple-valued, can be expressed in terms of natural logarithms as follows. In all cases we omit an additive constant $2k\pi i$, $k=0, \pm 1, \pm 2, \ldots$, in the logarithm.

$$\sin h^{-1} z = \log\left(z + \sqrt{z^2+1}\right) \qquad \operatorname{cosec} h^{-1} z = \log\left(\frac{1+\sqrt{z^2+1}}{z}\right)$$

$$\cos h^{-1} z = \log\left(z + \sqrt{z^2-1}\right) \qquad \sec h^{-1} z = \log\left(\frac{1+\sqrt{1-z^2}}{z}\right)$$

$$\tan h^{-1} z = \frac{1}{2}\log\left(\frac{1+z}{1-z}\right) \qquad \cot h^{-1} z = \frac{1}{2}\log\left(\frac{z+1}{z-1}\right).$$

9. The functions z^α, where α may be complex, is defined as $e^{\alpha \log z}$. Similarly if $f(z)$ and $g(z)$ are two given functions of z, we can define $(f(z))^{g(z)} = e^{g(z)\log f(z)}$. In general such functions are multiple-valued.

10. Algebraic and Transcendental Functions

If ω is a solution of the polynomial equation

$$P_0(z)\omega^n + P_1(z)\omega^{n-1} + \cdots + P_{n-1}(z)\omega + P_n(z) = 0 \qquad \text{... (2.1)}$$

where $P_0(z) \neq 0$, $P_1(z)$, ..., $P_{n-1}(z)$, $P_n(z)$ are polynomials in z and n is a positive integer, then $\omega = f(z)$ is called an algebraic function of z.

Any function which cannot be expressed as a solution of (2.1) is called a transcendental function.

The function $\omega = z^{1/2}$ is a solution of the equation $\omega^2 - z = 0$ and so is an algebraic function of z. The logarithmic, trigonometric and hyperbolic functions and their corresponding inverses are examples of transcendental functions.

2.4 Limits :

Definition 2.16

Let $\omega = f(z)$ be defined in a domain D except perhaps at the point z_0 of D. A complex number l is said to be the limit of f as $z \to z_0$, symbolically $l = \lim_{z \to z_0} f(z)$, if for given $\varepsilon > 0$, $\exists$ a $\delta > 0$ such that

$$|f(z) - l| < \varepsilon \text{ whenever } 0 < |z - z_0| < \delta.$$

If no such number l exists we say that $\lim_{z \to z_0} f(z)$ does not exist. Note that z is allowed to approach z_0 in an arbitrary manner, not just from some particular direction. The limit is clearly independent of the path by which z approaches z_0.

Geometrically, if z_0 is a point in the complex plane, then $\lim_{z \to z_0} f(z) = l$ if the difference in absolute value between $f(z)$ and l can be made as small as we wish by choosing points z sufficiently close to z_0 (excluding $z = z_0$ itself).

Theorem 2.1. A necessary and sufficient condition that the function $f(z) = u + iv$ may tend to $l = \alpha + i\beta$ as $z = x + iy$ tends to $z_0 = a + ib$ is that

$$u(x,y) \to \alpha \text{ and } v(x,y) \to \beta \text{ as } (x,y) \to (a,b).$$

Proof. We first suppose that $\lim_{z \to z_0} f(z) = l$. Then for given $\varepsilon > 0$, $\exists\ \delta > 0$ such that

$$|f(z) - l| < \varepsilon \text{ whenever } 0 < |z - z_0| < \delta$$

i.e. $|u(x,y) + iv(x,y) - \alpha - i\beta| < \varepsilon$ whenever $0 < |(x+iy) - (a+ib)| < \delta$

i.e. $|(u(x,y) - \alpha) + i(v(x,y) - \beta)| < \varepsilon$ whenever $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$

i.e. $|u(x,y) - \alpha| < \varepsilon$ and $|v(x,y) - \beta| < \varepsilon$ whenever $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$
as $|Re(z)| \le |z|$ and $|Im(z)| \le |z|$.

This implies that

$$\lim_{(x,y) \to (a,b)} u(x,y) = \alpha \text{ and } \lim_{(x,y) \to (a,b)} v(x,y) = \beta.$$

This proves the necessary part.

Next we suppose that $\lim_{(x,y) \to (a,b)} u(x,y) = \alpha$ and $\lim_{(x,y) \to (a,b)} v(x,y) = \beta$. Then for given $\varepsilon\,(> 0)$, we can find a $\delta(> 0)$ such that

$$|u(x,y) - \alpha| < \frac{\varepsilon}{2} \text{ and } |v(x,y) - \beta| < \frac{\varepsilon}{2}$$

$$\text{whenever } 0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta.$$

So for $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$ i.e. for $0 < |z - z_0| < \delta$ we get

$$\begin{aligned} |f(z) - l| &= |(u(x,y) - \alpha) + i(v(x,y) - \beta)| \\ &\le |u(x,y) - \alpha| + |v(x,y) - \beta| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

This implies that $\lim_{z \to z_0} f(z) = l$. This proves the sufficient part and hence the theorem.

2.5 More definitions of limits :

(i) Let f be defined in a domain D except perhsps at the point $z_0 \in D$. The function f is said to tend to infinity as z tends to z_0 if for any real number $k\,(>0)$, however large, there is a $\delta(> 0)$ such that $|f(z)| > k$ whenever $0 < |z - z_0| < \delta$.

In symbol we write $\lim_{z \to z_0} f(z) = \infty$.

(ii) Let f be defined for $|z| > k$ where $k > 0$. Then the function f is said to tend to a finite limit l as z tends to infinity, symbolically $\lim_{z\to\infty} f(z) = l$ if for any $\varepsilon(> 0)$ $\exists$ a number $k_0(> 0)$ such that $|f(z) - l| < \varepsilon$ whenever $|z| > k_0$.

(iii) We say that $\lim_{z\to\infty} f(z) = \infty$ if for each number $k(> 0)$, $\exists$ a number $k_0(> 0)$ such that $|f(z)| > k$ whenever $|z| > k_0$.

Theorem 2.2

Suppose that $\lim_{z\to z_0} f(z) = l$ and $\lim_{z\to z_0} g(z) = m$. Then

(i) $\lim_{z\to z_0} [f(z) \pm g(z)] = l \pm m$;

(ii) $\lim_{z\to z_0} cf(z) = cl$;

(iii) $\lim_{z\to z_0} f(z)\, g(z) = lm$;

(iv) $\lim_{z\to z_0} \dfrac{f(z)}{g(z)} = \dfrac{l}{m}$ if $m \neq 0$.

Example 2.3. Using the definition of limit, show that

$$\lim_{z\to z_0} (az^2 + bz + c) = az_0^2 + bz_0 + c,$$

where a, b, c are complex constants.

Solution : To verify this limit we have to find a $\delta > 0$, for given $\varepsilon > 0$ such that

$$0 < |z - z_0| < \delta \Rightarrow |(az^2 + bz + c) - (az_0^2 + bz_0 + c)| < \varepsilon.$$

Now,
$$\begin{aligned}|(az^2 + bz + c) - (az_0^2 + bz_0 + c)| &= |a(z^2 - z_0^2) + b(z - z_0)|\\ &= |(z - z_0)\,(a(z + z_0) + b)|\\ &\leq |z - z_0|\,(|a|\,|z + z_0| + |b|) \quad \ldots (2.2)\end{aligned}$$

From (2.2), we note that $|(az^2 + bz + c) - (az_0^2 + bz_0 + c)| < \varepsilon$ holds true if we assume

$$|z - z_0|\,(|a|\,|z + z_0| + |b|) < \varepsilon.$$

Restricting $|z - z_0| < 1$, we observe

$$\begin{aligned}&|a|\,|z + z_0| + |b| = |a|\,|z - z_0 + 2z_0| + |b|\\ &\leq |a|\,(|z - z_0| + 2|z_0|) + |b|\\ &< |a|\,(1 + 2|z_0|) + |b|.\end{aligned}$$

If we choose

$$\delta = \min\left\{1, \frac{\varepsilon}{|a|(1+2|z_0|)+|b|}\right\}.$$

then (2.1) holds.

2.6 Continuity :

Let $f(z)$ be a complex ffunction defined in some neighbourhood of z_0 (including the point). The function is said to be continuous at z_0 if, for every $\varepsilon > 0$ there corresponds a $\delta > 0$ such that

$$|f(z) - f(z_0)| < \varepsilon \text{ whenever } |z - z_0| < \delta.$$

Symbolically, we write

$$\lim_{z \to z_0} f(z) = f(z_0).$$

This means that for continuity at a point, the limiting value and functional value at the point have the same value.

A function $f(z)$ is contipuous on a set S if it is contimuous at every point of S. If a function is not continuous at z_0, then we say that the function is discontinuous at z_0 or z_0 is a point of discontinuity.

Remark 2.1

If the function $f(z)$ is continuous, so are $|f(z)|$, $f(\bar{z})$ and $\overline{f(z)}$.

Remark 2.2

Suppose that f and g are continuous functions at the point z_0. Then the following functions are continuous at z_0 :

(i) The sum $f(z) + g(z)$;

(ii) The difference $f(z) - g(z)$;

(iii) Their product $f(z)\, g(z)$;

(iv) Their quotient $f(z) \mid g(z)$, provided $g(z) \neq 0$.

Example 2.4. Test the continuity of the function

Test the continuity of the function

$$f(z) = \frac{z^3 + (1+i)z^2 + (2+i)z + 2}{z - i} \text{ at } z = i.$$

Solution : The function is undefined at $z = i$, but

$$\lim_{z \to i} f(z) = \lim_{z \to i} \frac{z^3 + (1+i)z^2 + (2+i)z + 2}{z - i}$$

$$= \lim_{z \to i} \frac{(z-i)(z^2 + 2iz + z + 2i)}{z - i} = -3 + 3i.$$

Hence if we define

$$f(z) = \begin{cases} \dfrac{z^3 + (1+i)z^2 + (2+i)z + 2}{z - i}, & z \neq i \\ -3 + 3i, & z = i \end{cases}$$

then the function $f(z)$ is continuous at $z = i$.

Example 2.5. Prove that the function arg : $\mathbb{C} \setminus \{0\} \to (-\pi, \pi]$ is not a continuous function.

Solution : Let $z_n = -1 + (1)^n \dfrac{1}{n}$; obviously, $(z_n) \subset \mathbb{C} - \{0\}$ and $z_n \to -1$ as $n \to \infty$;

but $\arg z_{2k} = \pi -$ arc $\tan \dfrac{1}{2k}$ and $\arg z_{2k+1} = -\pi +$ arc $\tan \dfrac{1}{2k+1}$.

Thus $\{\arg z_n\}$ is not a convergent sequence and hence 'arg' as a function is not continuous.

Theorem 2.3

A continuous function of a continuous function is continuous.

Proof. Let $f(z)$ be a continuous function at z_0. Then $f(z)$ is defined in some neighbouthood of z_0. Suppose that $g(w)$ is a function which is defined on f-image of this neighbourhood. Given that $g(w)$ is continuous at $\omega_0 = f(z_0)$. Then, for given $\varepsilon > 0$, $\exists$ a $\gamma > 0$ such that

$$|g(f(z)) - g(f(z_0))| < \varepsilon \text{ whenever } |f(z) - f(z_0)| < \gamma. \qquad ...(2.3)$$

Now, $f(z)$ is continuous at z_0. Hence, for this $\gamma > 0$, $\exists$ a $\delta > 0$ such that

$$|f(z) - f(z_0)| < \gamma \text{ whenever } |z - z_0| < \delta. \qquad ...(2.4)$$

combining (2.3) and (2.4) we have

$$|(gf)(z) - (gf)(z_0)| < \varepsilon \text{ whenever } |z - z_0| < \delta.$$

Hence the composite function gf is continuous at z_0.

Theorem 2.4

If $f(z)$ is continuous in a region, then the real and imaginary parts of $f(z)$ are also continuous in the region.

Proof. The proof of the theorem is for the readers.

Theorem 2.5

If a function is continuous in a closed region, it is bounded in the region.

Proof. The proof of the theorem is left as an exercise.

Theorem 2.6

If a function $f(z)$ is continuous on a bounded and closed set $S \subset C$, then minimum and maximum of $|f(z)|$ exist on S.

Proof. Given that $f(z) = u(x, y) + iv(x,y)$ is continuous on S. This implies that the component function $u(x, y)$ and $v(x, y)$ are continuous on S. Hence

$$|f(z)| = \sqrt{(u(x,y))^2 + (v(x,y))^2}$$

is a real valued continuous function on the bounded and closed set S. Hence by real calculus, $|f(z)|$ attains its maximum and minimum on S. This completes the proof.

Note 2.1 The examine the continuity of a function $f(z)$ at $z = \infty$, replace z by $\frac{1}{\zeta}$ and examine the continuity of

$$g(\zeta) = f\left(\frac{1}{\zeta}\right) \text{ at } \zeta = 0.$$

2.7 Uniform Continuity :

A function $f(z)$ is said to be uniformly continuous on a set S if, for given $\varepsilon > 0 \ \exists$ a $\delta > 0$ such that $|f(z_1) - f(z_2)| < \varepsilon$ whenever $|z_1 - z_2| < \delta \ \forall \ z_1, z_2, \in S$.

Here, the choice of δ is independent of z_1 and z_2 in S.

Theorem 2.7

Let $f(z)$ be a continuous function on a closed and bounded set S in the complex plane. Then it is uniformly continuous on S.

Proof. Suppose that f is not uniformly continuous on S. Then $\exists$ an $\varepsilon > 0$ and two sequences $\{p_n\}$ and $\{q_n\}$ corresponding to p and q in S such that for every n,

$$|p_n - q_n| < \frac{1}{n}, \ |f(p_n) - f(q_n)| \geq \varepsilon. \quad \text{...(2.5)}$$

Since S is closed and bounded, every sequence $\{p_n\}$ contains a subsequence $\{p_{n_k}\}$ converging to $z_0 \in S$.

Let $\{q_{n_k}\}$ be the corresponding subsequence of $\{q_n\}$. Then, $q_{n_k} \to z_0$ as $k \to \infty$. This is obvicus by the triangle inequality

$$|q_{n_k} - z_0| < |q_{n_k} - p_{n_k}| + |p_{n_k} - z_0|.$$

Hence for subsequences $\{p_{n_k}\}$ and $\{q_{n_k}\}$, (2.5) implies

$$|p_{n_k} - q_{n_k}| < \frac{1}{n_k}, \ |f(p_{n_k}) - f(q_{n_k})| \geq \varepsilon \qquad \text{... (2.6)}$$

for every k. Since $f(z)$ is continuous at z_0, we have

$$f(q_{n_k}) \to f(z_0), \ f(p_{n_k}) \to f(z_0) \text{ as } k \to \infty.$$

This contradicts (2.6). Therefore $f(z)$ is uniformly continuous.

Example 2.6. Show that $f(z) = z^2$ is uniformly continuous in the region $|z| < 1$, but the function $g(z) = \frac{1}{z}$ is not uniformly continuous in this region.

Solution : Let z and z_0 be any two points in $|z| < 1$ such that $|z - z_0| < \delta$. Then

$$|z^2 - z_0^2| = |(z - z_0)(z + z_0)| \leq (|z| + |z_0|)\,|z - z_0| < 2|z - z_0|.$$

Thus if we choose $\delta = \frac{\varepsilon}{2}$, we obtain $|z^2 - z_0^2| < \varepsilon$. This shows that $f(z) = z^2$ is uniformly continuous in $|z| < 1$.

For the second case, if possible, we assume that $g(z) = \frac{1}{z}$ is uniformly continuous in $|z| < 1$. Then for given $\varepsilon > 0$ we can find a $\delta > 0$, say between 0 and 1, such that $|f(z) - f(z_0)| < \varepsilon$ whenever $|z - z_0| < \delta$ for all z and z_0 in the given region. Fix $z = \delta$ and $z_0 = \delta/(1+\varepsilon)$. Clearly z and z_0 are in $|z| < 1$ and

$$|z - z_0| = \left|\delta - \frac{\delta}{1+\varepsilon}\right| = \frac{\varepsilon}{1+\varepsilon}\delta < \delta.$$

However, $\left|\frac{1}{z} - \frac{1}{z_0}\right| = \left|\frac{1}{\delta} - \frac{1+\varepsilon}{\delta}\right| = \frac{\varepsilon}{\delta} > \varepsilon$ since $0 < \delta < 1$.

Thus we reach at a contradiction and therefore the function $g(z) = \frac{1}{z}$ cannot be uniformly continuous in the region.

Exercise - 2

1. Using the definition of limit, verify that

$$\lim_{z \to 2+3i} [x + i(x + y^2)] = 2 + 11i.$$

2. Using the definition of limit, verify that

$$\lim_{z \to i} (z^2 + 2) = 1.$$

3. If $\lim_{z \to z_0} f(z) = l$, Prove that the limit is unique.
4. If $|z_n| \to |z_0|$ and Arg $z_n \to$ Arg z_0, prove that $z_n \to z_0$.
5. If a function f is continuous in a domain D, prove that $|f(z)|$ is also continuous in the domain.
6. Is the function

$$\frac{z^2 + (2-i)z - 2i}{z - i}$$

continuous at $z = i$? If not, can it be made continuous by redefining at $z = i$?

7. Prove that $f(z) = \frac{z^3+1}{z^3+9}$ is continuous and bounded in the region $|z| \leq 2$.
8. Prove that the function $f(z) = \frac{1}{z^2}$ is not uniformly continuous in the region $|z| \leq 1$, but it is uniformly continuous in the region $\frac{1}{2} \leq |z| \leq 1$.

Unit : 3 ❑ Analytic Functions

3.1 Complex Differentiation :

The differentiation of a function of a complex variable is defined in the same way as in the case of a function of a real variable.

In contrast with real analysis, here differentiability at a point and differentiability in the neighbourhood of a point are not the same. The latter is more powerful condition, which gives rise to a different class of functions called analytic functions. Actually, to study the theory of functions of a complex variable (or several variables), is to learn the behaviour and properties of such functions.

Definition 3.1.

Let $f(z)$ be a single-valued function defined in a domain D of the complex plane C. If $z_0 \in D$ and if

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \qquad \ldots(3.1)$$

exists, we denote this limit by $f'(z_0)$ and call it the derivative of $f(z)$ at the point z_0. If $f'(z_0)$ exists then $f(z)$ is said to be differentiable at z_0. Equivalently we can write

$$f'(z_0) = \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h} = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}.$$

If $f(z)$ is differentiable at each point of D, we say that $f(z)$ is differentiable in D.

We stress once again that the limit (3.1) exists means that the limit exists and is same along whatever path z approaches z_0.

Theorem 3.1

If $f(z)$ is differentiable at z_0, then it is continuous at z_0.

Proof. Since $f(z)$ is differentiable at z_0, we get

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

Now $\lim_{z \to z_0} [f(z) - f(z_0)] = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}(z - z_0)$

$$= \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \cdot \lim_{z \to z_0} (z - z_0)$$

$$= f'(z_0) \times 0 = 0.$$

i.e. $\lim_{z \to z_0} f(z) = f(z_0)$.

This proves that $f(z)$ is continuous at z_0.

The following example shows that the converse of the above theorem is not necessarily true.

Example 3.1. Show that the function $f(z) = \bar{z}$ is continuous at a point $z = z_0$, but the derivative does not exist at z_0.

Solution : The function $f(z) = \bar{z}$ is clearly continuous at $z = z_0$.

By definition, $$\frac{d}{dz} f(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

if this limit exists independent of the manner in which $\Delta z = \Delta x + i\Delta y$ approaches zero. Then

$$\frac{d}{dz}(\bar{z}) = \lim_{\Delta z \to 0} \frac{\overline{z+\Delta z} - \bar{z}}{\Delta z} = \lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z} = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y}$$

If $\Delta y = 0$, the required limit is $\lim\limits_{\Delta x \to 0} \dfrac{\Delta x}{\Delta x} = 1$.

If $\Delta x = 0$, the required limit is $\lim\limits_{\Delta y \to 0} -\dfrac{i\Delta y}{i\Delta y} = -1$.

Since the limit depends on the manner in which $\Delta z \to 0$, the derivative does not exist.

3.2 Geometric Interpretation of the Derivative :

Let z_0 [Fig. 3.1] be a point P in the z-plane and let ω_0 [Fig. 3.2] be its image P' in the ω-plane under the transformation $\omega = f(z)$.

We suppose that $f(z)$ is single-valued. Then the point z_0 maps into only one point ω_0.

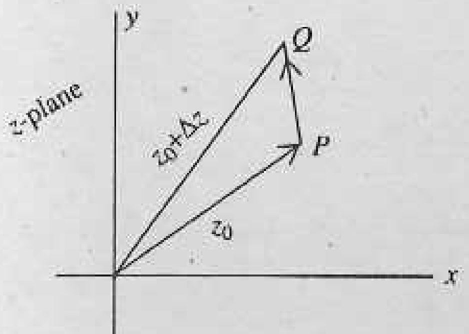


Fig. 3.1

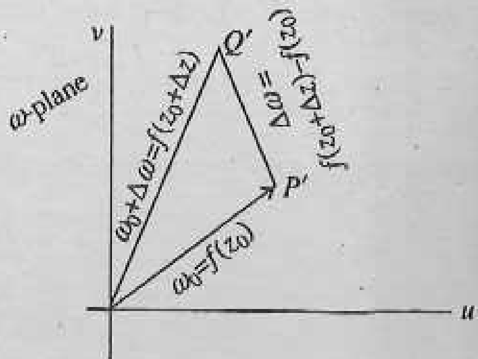


Fig. 3.2

If we give z_0 an increment Δz we obtain the point Q [Fig. 3.1]. This point has image Q' in the ω-plane. Thus from Fig. 3.2 we see that $P'Q'$ represents the complex number $\Delta\omega = f(z_0 + \Delta z) - f(z_0)$. It follows that the derivative at z_0 (if it exists) is given by

$$\lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \lim_{Q \to P} \frac{Q'P'}{QP}$$

i.e. the limit of the ratio $Q'P'$ to QP as point Q approaches point P. The above interpretation clearly holds when z_0 is replaced by any point z.

Differentials

Let $\Delta z = dz$ be an increment given to z. Then

$$\Delta\omega = f(z + \Delta z) - f(z)$$

is called the increment in $\omega = f(z)$. If $f(z)$ is continuous and has a continuous first derivative in a region, then

$$\Delta\omega = f'(z)\,\Delta z + \varepsilon\Delta z = f'(z)\,dz + \varepsilon dz$$

where $\varepsilon \to 0$ as $\Delta z \to 0$. The expression

$$d\omega = f'(z)\,dz$$

is called the differential of ω or $f(z)$. Note that $\Delta\omega \neq d\omega$ in general. We call dz the differential of z. Hence we write

$$\frac{d\omega}{dz} = f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} = \lim_{\Delta z \to 0} \frac{\Delta\omega}{\Delta z}. \qquad ...(3.2)$$

It is emphasized that dz and $d\omega$ are not the limits of Δz and $\Delta\omega$ as $\Delta z \to 0$, since these limits are zero whereas dz and $d\omega$ are not necessarily zero. Instead, given dz we determine $d\omega$ from (3.2), i.e. $d\omega$ is a dependent variable determined from the independent variable dz for a given z.

It is usedul to think of $\frac{d}{dz}$ as being an operator which when operating on $\omega = f(z)$ leads to $\frac{d\omega}{dz} = f'(z)$.

3.3 Rules for Differentiation :

It $f(z)$, $g(z)$ and $h(z)$ are analytic functions of z, the following differentiation rules (identical with those of elementary calculus) are valid.

(i) $\frac{d}{dz}\{f(z) \pm g(z)\} = \frac{d}{dz} f(z) \pm \frac{d}{dz} g(z) = f'(z) \pm g'(z)$.

(ii) $\frac{d}{dz}\{cf(z)\} = c\frac{d}{dz}f(z) = cf'(z)$,

where c is a constant.

(iii) $\frac{d}{dz}\{f(z)g(z)\} = f(z)\frac{d}{dz}g(z) + g(z)\frac{d}{dz}f(z)$

$$= f(z)g'(z) + g(z)f'(z).$$

(iv) $\frac{d}{dz}\left\{\frac{f(z)}{g(z)}\right\} = \frac{g(z)\frac{d}{dz}f(z) - f(z)\frac{d}{dz}g(z)}{[g(z)]^2}$

$$= \frac{g(z)f'(z) - f(z)g'(z)}{[g(z)]^2}, \text{ if } g(z) \neq 0.$$

(v) If $\omega = f(\zeta)$ where $\zeta = g(z)$ then

$$\frac{d\omega}{dz} = \frac{d\omega}{d\zeta}\cdot\frac{d\zeta}{dz} = f'(\zeta)\frac{d\zeta}{dz} = f'\{g(z)\}g'(z). \quad ...(3.3)$$

Similarly, if $\omega = f(\zeta)$ where $\zeta = g(\eta)$ and $\eta = h(z)$, then

$$\frac{d\omega}{dz} = \frac{d\omega}{d\zeta}\cdot\frac{d\zeta}{d\eta}\cdot\frac{d\eta}{dz}. \quad ...(3.4)$$

The results (3.3) and (3.4) are often called chain rules for differentiation of composite functions.

(vi) If $\omega = f(z)$, then $z = f^{-1}(\omega)$; and $\frac{d\omega}{dz}$ and $\frac{dz}{d\omega}$ are related by $\frac{d\omega}{dz} = \frac{1}{\frac{dz}{d\omega}}$.

(vii) If $z = f(t)$ and $\omega = g(t)$ where t is a parameter, then

$$\frac{d\omega}{dz} = \frac{d\omega/dt}{dz/dt} = \frac{g'(t)}{f'(t)}.$$

Similar rules can be formulated for differentials.

3.4 Analytic Function :

A function $f(z)$ defined in a domain D is said to be an analytic function in D if $f(z)$ has a derivative at each point of D. The terms regular and holomorphic are also used in state of analytic.

A function $f(z)$ is said to be analytic at a point z_0 if it is analytic in a neighbourhood of z_0 i.e. if there exist a neighbourhood of z_0 at all points of which $f'(z)$ exists.

If $f(z)$ is not analytic at a point z_0, then z_0 is called a singular point or a singularity of $f(z)$.

Example 3.2. Show that the function $f(z) = z\bar{z}$ is differentiable only at origin.

Solution : $\frac{d}{dz}f(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z)-f(z)}{\Delta z}$

$$= \lim_{\Delta z \to 0} \frac{(z+\Delta z)(\overline{z+\Delta z})-z\bar{z}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} z\frac{\overline{\Delta z}}{\Delta z} + \lim_{\Delta z \to 0} \bar{z} + \lim_{\Delta z \to 0} \overline{\Delta z}.$$

Since $\lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z} = 1$, when Δz approaches zero along real axis and $\lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z} = -1$ when Δz approches zero along imaginary axis, $\lim_{\Delta z \to 0} z\frac{\overline{\Delta z}}{\Delta z}$ does not exist.

Therefore the given function is differentiable only at the origin.

Example 3.3. If $\omega = f(z) = z^3 - 2z^2$, find (a) $\Delta\omega$, (b) $d\omega$, (c) $\Delta\omega - d\omega$.

Solution :

(a) $\Delta\omega = f(z + \Delta z) - f(z)$

$= \{(z + \Delta z)^3 - 2(z + \Delta z)^2\} - \{z^3 - 2z^2\}$

$= (3z^2 - 4z)\,\Delta z + (3z - 2)\,(\Delta z)^2 + (\Delta z)^3.$

(b) $d\omega$ = principal part of $\Delta\omega = (3z^2 - 4z)\,\Delta z = (3z^2 - 4z)dz$

since by definition $\Delta z = dz$. Note that

$f'(z) = 3z^2 - 4z$ and $d\omega = (3z^2 - 4z)\,dz$, i.e. $\frac{d\omega}{dz} = 3z^2 - 4z.$

(c) From (a) and (b) we obtain

$\Delta\omega - d\omega = (3z - 2)\,(\Delta z)^2 + (\Delta z)^3 = \varepsilon\Delta z,$

where $\varepsilon = (3z - 2)\,\Delta z + (\Delta z)^2.$

Note that $\varepsilon \to 0$ as $\Delta z \to 0$, i.e. $\frac{\Delta\omega - d\omega}{\Delta z} \to 0$ as $\Delta z \to 0$.

It follows that $\Delta\omega - d\omega$ is an infinitesimal of higher order than Δz.

Example 3.4. If $f(z) = \frac{xy^2(x+iy)}{x^2+y^4}$, $z \neq 0$

$= 0,\ z = 0,$

prove that $\frac{f(z)-f(0)}{z} \to 0$ as $z \to 0$ along any straight line but $f'(0)$ does not exist.

Solution : For $y = mx$, $\lim_{z\to 0} \frac{f(z)-f(0)}{z} = \lim_{(x,y)\to(0,0)} \frac{\frac{xy^2(x+iy)}{x^2+y^4}}{x+iy}$

$$= \lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^4}$$

$$= \lim_{x\to 0} \frac{m^2 x}{1+m^4x^2} = 0.$$

For $x = y^2$, $\lim_{z\to 0} \frac{f(z)-f(0)}{z} = \lim_{(x,y)\to(0,0)} \frac{xy^2(x+iy)}{(x^2+y^4)(x+iy)}$

$$= \lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4} = \frac{1}{2}.$$

This shows that $f'(0)$ does not exist.

Cauchy-Riemann Equations (C–R equation)

Theorem 3.2

A necessary condition for $\omega = f(z) = u(x,y) + iv(x,y)$ to be differentiable at the point $z_0 = x_0 + iy_0$ is that

$$u_x(x_0, y_0) = v_y(x_0, y_0) \text{ and } u_y(x_0, y_0) = -v_x(x_0, y_0).$$

Proof. Suppose that $f'(z_0)$ exists. Then

$$f'(z_0) = \lim_{z\to 0} \frac{f(z)-f(z_0)}{z-z_0}$$

$$= \lim_{(x,y)\to(x_0,y_0)} \frac{\{u(x,y)-u(x_0,y_0)\}+i\{v(x,y)-v(x_0,y_0)\}}{(x-x_0)+i(y-y_0)}. \quad ...(3.5)$$

Since $f'(z_0)$ exists, (3.5) must exists for all modes of approach of the point (x,y) to (x_0, y_0) and all the limiting values must be same.

Let $z \to z_0$ along a line parallel to real axis. Then $y = y_0$ and $x \to x_0$.

So from (3.5) we obtain

$$f'(z_0) = \lim_{x\to x_0} \frac{\{u(x,y_0)-u(x_0,y_0)\}+i\{v(x,y_0)-v(x_0,y_0)\}}{x-x_0}$$

$$= \lim_{x\to x_0} \frac{u(x,y_0)-u(x_0,y_0)}{x-x_0} + i\lim_{x\to x_0} \frac{v(x,y_0)-v(x_0,y_0)}{x-x_0}$$

$$= u_x(x_0, y_0) + iv_x(x_0, y_0). \quad ...(3.6)$$

Now letting $z \to z_0$ along a line parallel to the imaginary axis i.e. $x = x_0$ and $y \to y_0$ we obtain from (3.5)

$$f'(z_0) = \lim_{y \to y_0} \frac{u(x_0, y) - u(x_0, y_0)}{i(y - y_0)} + i \lim_{y \to y_0} \frac{v(x_0, y) - v(x_0, y_0)}{i(y - y_0)}$$

$$= -i\, u_y(x_0, y_0) + v_y(x_0, y_0). \qquad ...(3.7)$$

Comparing (3.6) and (3.7) and equating the real and imaginary parts we obtain

$$u_x(x_0, y_0) = v_y(x_0, y_0) \text{ and } u_y(x_0, y_0) = -v_x(x_0, y_0).$$

The proves the theorem.

Note 3.1. The differential equations $u_x = v_y$ and $u_y = -v_x$ are known as Cauchy-Riemann equations.

Example 3.5. Find the nature of C–R equations for the function $f(z) = |z|^2$.

Solution: $f(z) = |z|^2 = x^2 + y^2$.

Hence $u(x,y) = x^2 + y^2$, $v(x,y) = 0$.

Therefore $u_x = 2x$, $u_y = 2y$, $v_x = 0 = v_y$.

Thus C–R equations are not satisfied unless $x = y = 0$, and hence $f'(z)$ does not exist at any point $z \neq 0$.

Example 3.6. If $u - v = (x - y)(x^2 + 4xy + y^2)$ and $f(z) = u + iv$ is an analytic function of $z = x + iy$, find $f(z)$ in terms of z.

Solution : Now $f(z) = u + iv$ so that $i f(z) = iu - v$.

$$\therefore \quad (1 + i) f(z) = (u - v) + i(u + v) = U + iV, \text{ say}$$

Here $U = u - v = (x - y)(x^2 + 4xy + y^2)$.

$$\text{So } \frac{\partial U}{\partial x} = \frac{\partial u}{\partial x} - \frac{\partial v}{\partial x}$$

$$= x^2 + 4xy + y^2 + (x - y)(2x + 4y)$$

$$= 3x^2 + 6xy - 3y^2$$

$$\text{and } \frac{\partial U}{\partial y} = \frac{\partial u}{\partial y} - \frac{\partial v}{\partial y}$$

$$= -(x^2 + 4xy + y^2) + (x - y)(4x + 2y)$$

$$= 3x^2 - 6xy - 3y^2.$$

Let $\frac{\partial V}{\partial x} = \phi_1(x,y)$ and $\frac{\partial V}{\partial y} = \phi_2(x,y)$.

$$\text{Then } (1 + i) f(z) = \int [\phi_1(z,0) - i\phi_2(z,0)]\, dz + c$$

$$= \int (3z^2 - 3iz^2)\, dz + c = (1 - i) z^3 + c.$$

$$\therefore \quad f(z) = -\frac{1-i}{1+i}z^3 + \frac{c}{1+i}$$

or, $f(z) = -iz^3 + \beta$, where $\beta = \frac{c}{1+i}$.

Example. 3.7. Let $f(z) = \dfrac{x^3 - y^3}{x^2 + y^2} + i\,\dfrac{x^3 + y^3}{x^2 + y^2}$, $z \neq 0$

$= 0$, $z = 0$.

Show that though C–R equation are satisfied at (0, 0), $f'(0)$ does not exist.

Solution : Here $u(x, y) = \dfrac{x^3 - y^3}{x^2 + y^2}$ if $(x,y) \neq (0,0)$

$= 0$ if $(x,y) = (0,0)$,

and $v(x,y) = \dfrac{x^3 + y^3}{x^2 + y^2}$ if $(x,y) \neq (0,0)$

$= 0$ if $(x,y) = (0,0)$.

$$u_x(0,0) = \lim_{x\to 0} \frac{u(x,0) - u(0,0)}{x} = \lim_{x\to 0} \frac{x}{x} = 1.$$

$$u_y(0,0) = \lim_{y\to 0} \frac{u(0,y) - u(0,0)}{y} = \lim_{y\to 0} \frac{-y}{y} = -1.$$

$$v_x(0,0) = \lim_{x\to 0} \frac{v(x,0) - v(0,0)}{x} = 1.$$

$$v_y(0,0) = \lim_{y\to 0} \frac{v(0,y) - v(0,0)}{y} = 1.$$

Since $u_x = v_y$ and $u_y = -v_x$, C–R equations are satisfied at origin.

Now $f'(0) = \lim\limits_{z\to 0} \dfrac{f(z) - f(0)}{z}$

$$= \lim_{(x,y)\to(0,0)} \frac{(x^3 - y^3) + i(x^3 + y^3)}{(x^2 + y^2)(x + iy)}.$$

On $y = mx$,

$$f'(0) = \lim_{x\to 0} \frac{(x^3 - m^3x^3) + i(x^3 + m^3x^3)}{(x^2 + m^2x^2)(x + imx)}$$

$$= \frac{(1 - m^3) + i(1 + m^3)}{(1 + m^2)(1 + im)}.$$

Since the value of the limits depends on m, $f'(0)$ does not exist.

Example 3.8. Let $f(z) = e^{-z^{-4}}, z \neq 0$
$= 0, z = 0.$

Show that though C–R equation are satisfied at (0,0), $f'(0)$ does not exist.

Solution : $f(z) = e^{-z^{-4}}, z \neq 0$
$= 0, z = 0.$

$$\text{Now } -z^{-4} = \frac{-1}{(x+iy)^4} = -\frac{(x-iy)^4}{(x^2+y^2)^4}$$

$$= -\frac{(x^4+y^4-6x^2y^2)}{(x^2+y^2)^4} + i\frac{4xy(x^2-y^2)}{(x^2+y^2)^4}$$

$$= A + iB.$$

So, $u(x,y) = e^A \cos B$, $v(x,y) = e^A \sin B$ when $(x,y) \neq (0,0)$.

$$\text{Now } u_x(0,0) = \lim_{x\to 0} \frac{u(x,0)-u(0,0)}{x-0} = \lim_{x\to 0} \frac{e^{-1/x^4}-0}{x}$$

$$= \lim_{x\to 0} \frac{1}{xe^{1/x^4}} = \lim_{x\to 0} \frac{1}{x\left[1+\frac{1}{x^4}+\frac{1}{2!}\cdot\frac{1}{x^3}+\cdots\right]}$$

$$= 0.$$

$$u_y(0,0) = \lim_{y\to 0} \frac{u(0,y)-u(0,0)}{y-0}$$

$$= \lim_{y\to 0} \frac{1}{ye^{1/y^4}} = 0, \text{ as above.}$$

$$v_x(0,0) = \lim_{x\to 0} \frac{v(x,0)-v(0,0)}{x-0} = 0.$$

$$v_y(0,0) = \lim_{y\to 0} \frac{v(0,y)-v(0,0)}{y} = 0.$$

Since $u_x = v_y$ and $u_y = -v_x$, C–R equations are satisfied at origin.

$$\text{Now } f'(0) = \lim_{z\to 0} \frac{f(z)-f(0)}{z} = \lim_{z\to 0} \frac{e^{-z^{-4}}}{z}.$$

We choose $z \to 0$ along the path $z = re^{\frac{i\pi}{4}}$. Then
$z^{-4} = r^{-4} e^{-i\pi} = -r^{-4}.$

$$\therefore \quad f'(0) = \lim_{z\to 0} \frac{e^{-z^{-4}}}{z} = \lim_{r\to 0} \frac{e^{\frac{1}{r^4}}}{re^{\frac{i\pi}{4}}}$$

$$= \lim_{r\to 0} \frac{1}{re^{\frac{i\pi}{4}}}\left[1+\frac{1}{r^4}+\frac{1}{2!}\frac{1}{r^8}+\cdots\right] = \infty$$

This shows that $f'(0)$ does not exist.

Examples 3.7 and 3.8 shows that the validity of C–R equations at a point is not sufficient to ensure the existance of the derivative at that point.

Sufficient Condition for Analyticity.

Theorem 3.3

A single valued continuous function $\omega = f(z) = u(x,y) + iv(x,y)$ is differentiable in a damain D if the four partial derivatives u_x, u_y, v_x, v_y exist, are continuous and satisfy C–R equations at each point of D.

Proof. We are to show that $f'(z) = \lim\limits_{\Delta z\to 0} \dfrac{\Delta\omega}{\Delta z}$ exits at each point of D. Let $z = x+iy$ be any point of D. Since u_x, u_y, v_x, v_y exists and continuous at (x, y),

$u(x,y)$ and $v(x,y)$ are differentiable at (x, y).

Therefore,

$$\Delta u = u(x + \Delta x,. y + \Delta y) - u(x,y)$$
$$= u_x\,\Delta x + u_y\,\Delta y + \varepsilon_1\Delta x + \varepsilon_2\Delta y,$$

where $\varepsilon_1, \varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0,0)$.

$$\Delta v = v(x + \Delta x, y + \Delta y) - v(x,y)$$
$$= v_x\,\Delta x + v_y\,\Delta y + \eta_1\,\Delta x + \eta_2\,\Delta y,$$

where $\eta_1, \eta_2 \to 0$ as $(\Delta x, \Delta y) \to (0,0)$.

Now $\Delta\omega = \Delta u + i\Delta v$

$$= u_x(\Delta x + i\Delta y) + v_x(i\Delta x - \Delta y) + \Delta x(\varepsilon_1 + i\eta_1) + \Delta y(\varepsilon_2 + i\,\eta_2)$$
$$= (u_x + iv_x)\,\Delta z + (\varepsilon_1 + i\,\eta_1)\Delta x + (\varepsilon_2 + i\,\eta_2)\Delta y. \qquad ...(3.8)$$

[$\because$ C–R equations are satisfied]

Form (3.8) we obtain

$$\frac{\Delta\omega}{\Delta z} = u_x + iv_x + (\varepsilon_1 + i\,\eta_1)\frac{\Delta x}{\Delta z} + (\varepsilon_2 + i\,\eta_2)\frac{\Delta y}{\Delta z}. \qquad ...(3.9)$$

Now $\left|(\varepsilon_1 + i\eta_1)\frac{\Delta x}{\Delta z}\right| = |(\varepsilon_1 + i\eta_1)|\left|\frac{\Delta x}{\Delta z}\right|$

$\leq |\varepsilon_1| + |\eta_1| \to 0$ as $(\Delta x, \Delta y) \to (0,0)$ i.e. as $\Delta z \to 0$.

Similarly $|(\varepsilon_2 + i\eta_2)|\left|\frac{\Delta y}{\Delta z}\right| \to 0$ as $\Delta z \to 0$.

Therefore proceding to the limit as $\Delta z \to 0$ we get from (3.9)

$$\lim_{\Delta z \to 0} \frac{\Delta \omega}{\Delta z} = u_x + iv_x$$

i.e. $f'(z)$ exist and is equal to $u_x + iv_x$. Since z is any point of D we thus conclude that f is differentiable in D.

This proves the theorem.

Example 3.9. Show that the function $f(z) = \bar{z}$ is non-analytic everywhere.

Solution : See Example 3.1.

Example 3.10. For what values of z do the functions ω defined by the following equations cease to be analytic?

(i) $z = \log \rho + i\phi$ where $\omega = \rho(\cos\phi + i\sin\phi)$,

(ii) $z = \sinh u \cos v + i\cosh u \sin v$, where $\omega = u + iv$.

Solution :

(i) We have

$$\frac{dz}{d\omega} = (\cos\phi - i\sin\phi)\frac{\partial z}{\partial \rho} = (\cos\phi - i\sin\phi)\cdot\frac{1}{\rho}$$

$$\text{or, } \frac{d\omega}{dz} = \frac{\rho}{\cos\phi - i\sin\phi} = \rho(\cos\phi + i\sin\phi) = \omega. \qquad \text{...(*)}$$

Therfore in order that ω may be an analytic function of z, we conclude from (*) that ω should be finite. Now ω will be finite so long as ρ is finite i.e. so long as z is finite. Hence ω is an analytic function of z in any finite domain.

(ii) We know that $\frac{dz}{d\omega} = \frac{\partial z}{\partial u} = \cosh u \cos v + i \sinh u \sin v$.

Now $z^2 = \sinh^2 u \cos^2 v - \cosh^2 u \sin^2 v + 2i \sinh u \cosh u \sin v \cos v$

$= (\cosh^2 u - 1)\cos^2 v - (1 + \sinh^2 u)\sin^2 v + 2i \sinh u \cosh u \sin v \cos v$

$= (\cosh u \cos v + i \sinh u \sin v)^2 - 1 = \left(\frac{dz}{d\omega}\right)^2 - 1.$

Thus $\frac{dz}{d\omega} = \pm\sqrt{(z^2+1)}$ or, $\frac{d\omega}{dz} = \pm\frac{1}{\sqrt{z^2+1}}$.

Hence ω will not be analytic when $z^2 + 1 = 0$, i.e., when $z = \pm i$.

Harmonic Function.

Definition 3.2.

A function $u(x,y)$ of two real variables x and y is said to be harmonic on a domain D if the partial derivatives

$$\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial x \partial y}, \frac{d^2 u}{\partial y^2}$$

exist and are continuous on D and if at any point of D, $u(x,y)$ satisfies the partial differential equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$

known as Laplace's equation.

Definition 3.3.

Let $u(x,y)$ and $v(x,y)$ be two harmonic functions on a domain D satisfying the C–R equations

$$u_x = v_y,\ u_y = -v_x$$

in D. Then $u(x,y)$ and $v(x,y)$ are said to be conjugate harmonic functions on D and each of the functions $u(x,y)$, $v(x,y)$ are said to be harmonic conjugate of the other.

There is an intimate relation between harmonic functions and analytic functions as shown in the following theorem.

Theorem 3.4. (Statement only)

A necesary and sufficient condition for a function $f(z) = u(x,y) + iv(x,y)$ to be analytic on a domain D is that its real part $u(x,y)$ and imaginary part $v(x,y)$ be conjugate harmonic functions on D.

Example 3.11. The function $e^z = e^{x+iy} = e^x(\cos y + i\sin y)$

$$= e^x\cos y + ie^x\sin y$$

is analytic in the whole complex plane and hence its real and imaginary part

$$u(x,y) = e^x\cos y,\ v(x,y) = e^x\sin y$$

are harmonic in the whole complex plane.

Example 3.12. Prove that the function $u = e^{-x}(x\sin y - y\cos y)$ is harmonic and find its harmonic conjugate and the corresponding analytic function.

Solution : Here $u = e^{-x}(x\sin y - y\cos y)$.

$$\frac{\partial u}{\partial x} = e^{-x}\sin y - xe^{-x}\sin y + ye^{-x}\cos y.$$

$$\frac{\partial^2 u}{\partial x^2} = 2e^{-x}\sin y + xe^{-x}\sin y - ye^{-x}\cos y. \qquad ...(3.10)$$

$$\frac{\partial u}{\partial y} = xe^{-x}\cos y + ye^{-x}\sin y - e^{-x}\cos y.$$

$$\frac{\partial^2 u}{\partial y^2} = -xe^{-x}\sin y + 2e^{-x}\sin y + ye^{-x}\cos y. \qquad ...(3.11)$$

Adding (3.10) and (3.11) we obtain

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Hence u is harmonic.

Let v be the harmonic conjugate of u. Then from the Cauchy-Riemann equations,

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = e^{-x}\sin y - xe^{-x}\sin y + ye^{-x}\cos y. \qquad ...(3.12)$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = e^{-x}\cos y - xe^{-x}\cos y - ye^{-x}\sin y. \qquad ...(3.13)$$

Integrating (3.12) w.r.t. y, keeping x constant we obtain

$$v = -e^{-x}\cos y + xe^{-x}\cos y + e^{-x}(y\sin y + \cos y) + F(x)$$

$$= ye^{-x}\sin y + xe^{-x}\cos y + F(x) \qquad ...(3.14)$$

where $F(x)$ is an orbitrary real function of x.

Substituting (3.14) into (3.13) we obtain

$$-ye^{-x}\sin y + e^{-x}\cos y - xe^{-x}\cos y + F'(x)$$

$$= e^{-x}\cos y - xe^{-x}\cos y - ye^{-x}\sin y$$

i.e. $F'(x) = 0$

i.e. $F(x) = c$, a constant.

Then from (3.14) we get

$$v = e^{-x}(y\sin y + x\cos y) + c.$$

Thus $f(z) = u + iv$

$$= e^{-x}(x\sin y - y\cos y) + ie^{-x}(y\sin y + x\cos y) + ic.$$

Example 3.13. If $u = (x-1)^3 - 3xy^2 + 3y^2$, determine v so that $u+iv$ is a regular function of $x + iy$.

Solution : Here $\frac{\partial u}{\partial x} = 3(x-1)^2 - 3y^2$ and $\frac{\partial u}{\partial y} = -6xy + 6y$.

By Cauchy-Riemann equations, we have

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = 6xy - 6y.$$

Integrating with respect to x, we get that

$$v = 3x^2y - 6yx + f(y) \qquad \text{...(3.15)}$$

$$\therefore \quad \frac{\partial v}{\partial y} = 3x^2 - 6x + f'(y). \qquad \text{...(3.16)}$$

Also $\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = 3(x-1)^2 - 3y^2. \qquad \text{...(3.17)}$

From (3.16) and (3.17), we obtain that

$$3x^2 - 6x + f'(y) = 3x^2 - 6x + 3 - 3y^2$$

or, $f'(y) = 3 - 3y^2$.

Integrating, $f(y) = 3y - y^3 + c$.

Substituting this value of $f(y)$ in (3.15), we get that

$$v = 3x^2y - 6yx + 3y - y^3 + c.$$

Example 3.14. If $f(z)$ is analytic, prove that

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) |f(z)|^2 = 4|f'(z)|^2.$$

Solution : Let $\phi(x,y) = |f(z)|^2 = u^2 + v^2$

where $f(z) = u + iv$. Since $f(z)$ is analytic, u and v are harmonic conjugate functions and hence

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \text{...(3.18)}$$

and $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0. \qquad \text{...(3.19)}$

Now $\frac{\partial \phi}{\partial x} = 2(uu_x + vv_x)$.

$$\frac{\partial^2 \phi}{\partial x^2} = 2(uu_{xx} + u_x{}^2 + v_x{}^2 + vv_{xx}).$$

$$\frac{\partial \phi}{\partial y} = 2(uu_y + vv_y).$$

$$\frac{\partial^2 \phi}{\partial y^2} = 2(uu_{yy} + u_y^2 + v_y^2 + vv_{yy}).$$

So, $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) |f(z)|^2 = 2(u_x^2 + v_x^2 + u_y^2 + v_y^2)$, using (3.18) and (3.19)

$= 4(u_x^2 + v_x^2)$ [$\because$ u, v satisfy C–R equation]

$= 4|f'(z)|^2$.

Example 3.15. Show that the function $f(z) = xy + iy$ is everywhere continuous but is not analytic.

Solution : Here $u = xy$ and $v = y$. Continuity of $f(z)$ follows from continuity of u and v.

Now $\frac{\partial u}{\partial x} = y$, $\frac{\partial u}{\partial y} = x$, $\frac{\partial v}{\partial x} = 0$, $\frac{\partial v}{\partial y} = 1$.

Since $\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} \neq -\frac{\partial v}{\partial x}$,

$f(z)$ is not an analytic function.

Example 3.16. Let $f(z) = u + iv$ be analytic in a domain D and $|f(z)|$ is equal to constant in D. Then show that $f(z)$ is constant in D.

Solution : $|f(z)|$ = constant = c, say

i.e. $u^2 + v^2 = c^2$.

Differentiating w.r.t. x and y we obtain

$uu_x + vv_x = 0$ ...(3.20)

and $uu_y + vv_y = 0$. ...(3.21)

(3.21) implies $-uv_x + vu_x = 0$ [$\because$ $u_x = v_y$ & $u_y = -v_x$] ...(3.22)

From (3.20) and (3.22) we obtain

$(u^2 + v^2)u_x = 0$.

If $u^2 + v^2 = 0$, then $u = 0 = v$, that means $f(z) = 0 = a$ constant function. Hence $u_x = 0$. Similarly from (3.20) and (3.22) we obtain $v_x = 0$.

Hence $u_x = v_x = u_y = v_y = 0$.

Thus $du = u_x d_x + u_y d_y = 0$

i.e. u = constant.

Similarly, v = constant and so $f(z)$ is constant.

Example 3.17. Prove that in polar form the Cauchy-Riemann equation can be written as

$$\frac{\partial u}{\partial r}=\frac{1}{r}\frac{\partial v}{\partial \theta},\ \frac{\partial v}{\partial r}=-\frac{1}{r}\frac{\partial u}{\partial \theta}.$$

Solution : We have $x = r\cos\theta$, $y = r\sin\theta$. Then

$$r = \sqrt{x^2+y^2} \text{ and } \theta = \tan^{-1}\left(\frac{y}{x}\right).$$

Now $$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial r}\frac{\partial r}{\partial x}+\frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x}$$

$$=\frac{\partial u}{\partial r}\left(\frac{x}{\sqrt{x^2+y^2}}\right)+\frac{\partial u}{\partial \theta}\left(\frac{-y}{x^2+y^2}\right)$$

$$=\frac{\partial u}{\partial r}\cos\theta-\frac{1}{r}\frac{\partial u}{\partial \theta}\sin\theta. \quad ...(3.23)$$

$$\frac{\partial u}{\partial y}=\frac{\partial u}{\partial r}\frac{\partial r}{\partial y}+\frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial y}$$

$$=\frac{\partial u}{\partial r}\left(\frac{y}{\sqrt{x^2+y^2}}\right)+\frac{\partial u}{\partial \theta}\left(\frac{x}{x^2+y^2}\right)$$

$$=\frac{\partial u}{\partial r}\sin\theta+\frac{1}{r}\frac{\partial u}{\partial \theta}\cos\theta. \quad ...(3.24)$$

Similarly, $$\frac{\partial v}{\partial x}=\frac{\partial v}{\partial r}\frac{\partial r}{\partial x}+\frac{\partial v}{\partial \theta}\frac{\partial \theta}{\partial x}$$

$$=\frac{\partial v}{\partial r}\cos\theta-\frac{1}{r}\frac{\partial v}{\partial \theta}\sin\theta. \quad ...(3.25)$$

$$\frac{\partial v}{\partial y}=\frac{\partial v}{\partial r}\frac{\partial r}{\partial y}+\frac{\partial v}{\partial \theta}\frac{\partial \theta}{\partial y}$$

$$=\frac{\partial v}{\partial r}\sin\theta+\frac{1}{r}\frac{\partial v}{\partial \theta}\cos\theta. \quad ...(3.26)$$

From the Cauchy-Riemann equation $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$ we have, using (3.23) and (3.26),

$$\left(\frac{\partial u}{\partial r}-\frac{1}{r}\frac{\partial v}{\partial \theta}\right)\cos\theta-\left(\frac{\partial v}{\partial r}+\frac{1}{r}\frac{\partial u}{\partial \theta}\right)\sin\theta=0. \quad ...(3.27)$$

From the Cauchy-Riemann equation $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ we have, using (3.24) and (3.25),

$$\left(\frac{\partial u}{\partial r} - \frac{1}{r}\frac{\partial v}{\partial \theta}\right)\sin\theta + \left(\frac{\partial v}{\partial r} + \frac{1}{r}\frac{\partial u}{\partial \theta}\right)\cos\theta = 0. \qquad ...(3.28)$$

Multiplying (3.27) by $\cos\theta$, (3.28) by $\sin\theta$ and adding we obtain

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}.$$

Again multiplying (3.27) by $-\sin\theta$, (3.28) by $\cos\theta$ and adding we obtain

$$\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}.$$

Exercise - 3

1. Show that the function $f(z) = x^2 + iy^3$ is not analytic anywhere.
2. Verify that the Cauchy-Riemann equations are satisfied for the functions e^{z^2} and $\cos 2z$.
3. If $\omega = iz^2 - 4z + 3i$, find (a) $\Delta\omega$, (b) $d\omega$, (c) $\Delta\omega - d\omega$ at the point $z = 2i$.
4. If $f(z) = \dfrac{x^2y^3(xy^2 - i)}{x^3 + y^6}$, $z \neq 0$

   $= 0$, $z = 0$,

   then prove that $\dfrac{f(z) - f(0)}{z} \to 0$ as $z \to 0$ along any radius vector, but not as $z \to 0$ in any other manner.
5. Let $f(z) = \sqrt{|xy|}$. Show that $f'(0)$ does not exist but the C–R equations are satisfied at the origin.
6. Prove that the function $u = 2x(1 - y)$ is harmonic. Find its harmonic conjugate and the corresponding analytic function.
7. Prove that the function $u = \frac{1}{2}\log(x^2 + y^2)$ is harmonic. Find its harmonic conjugate and corresponding analytic function $f(z)$ in terms of z.
8. Prove that $f(z) = |z|^4$ is differentiable but not analytic at $z = 0$.

9. If $f'(z) = 0$ in a region R, Prove that $f(z)$ must be a constant in R.

10. Let $f = u + iv$ be analytic in a domain D. Show that f is constant in D if any one of the following conditions hold :

    (i) Real $\{f(z)\}$ = constant in D

    (ii) Im $\{f(z)\}$ = constant in D

    (iii) arg $f(z)$ = constant in D.

11. If u and v are harmonic in a domain D, Prove that

$$\left(\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}\right)+i\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)$$

    is analytic in D.

12. Give an example of a function which is continuous everywhere on C but is differentiable nowhere on C.

13. Give an example of a function which is differentiable only at a single point.

14. Let $\omega = f(z) = \frac{1+z}{1-z}$. What can you say about the differentiability of the function?

Unit : 4 ❑ Complex Integration

4.1 Introduction :

In order to develop the subject of the theory of function of a complex variable further, it is necessary to consider the definition of the integral of a function of a complex variable along a plane curve. In the theory of real variables, the integration is considered from two points of view; viz. the indefinite integration as an operation inverse to that of differentiation and the definite integration as the limit of a sum.

Historically, Euler was the first to obtain the value of a definite integral by replacing a real variable by a complex variable, where as P.S. Laplace (1749–1827) is believed to have been the first to use a line integral in the complex plane.

It is interesting to note that the concept of indefinite integral as the process of inverse differentiation in case of a function of a real variable is extended to a function of a complex variable if the complex function $f(z)$ is analytic. It means that if $f(z)$ is an analytic function of a complex variable z, and if $\int f(z)dz = F(z)$, then the differential of $F(z)$ is equal to $f(z)$; i.e., $F'(z) = f(z)$.

However, the concept of definite integral of a function of a real variable does not extended out, rightly to the domain of complex variables. For example, in the case of real variable, the path of integration of $\int_a^b f(x)\,dx$ is always along the real axis from $x = a$ to $x = b$. But in the case of a complex function $f(z)$, the path of the definite integral $\int_a^b f(z)dz$ may be along any curve joining the points $z = a$ and $z = b$; so that its value depends upon the path (curve of integration). However, we shall see that this variation in the value of definite integral will disappear in some special circumstances. For instance, the variation in values can be made to disappear if the different paths (curves) joining $z = a$ to $z = b$ are regular paths (curves).

Now we will state some basic definitions.

4.2 Some Basic Definitions :

Rectifiable Arcs.

Let : $\Gamma : z = z(t)$, $a \le t \le b$, be any Jordan arc. By a partition of $[a, b]$ we mean a set of points $P = \{t_0, t_1, t_2, \cdots, t_n\}$ satisfying $a = t_0 < t_1 < t_2 < \cdots < t_n = b$.

The collection of all possible partitions of $[a,b]$ will be denoted by P$[a,b]$. Let $P = \{a = t_0, t_1, t_2, \cdots, t_n = b\}$ be a partition of $[a, b]$. We write $z_r = z(t_r)$ to get the points $z_0, z_1, z_2, \cdots, z_n$ on Γ

Construct the sum

$$S_P = \sum_{r=1}^{n} |z_r - z_{r-1}|.$$

Clearly S_P denotes the length of the polygon inscribed which is obtained by drawing straight lines from z_0 to z_1, z_1 to z_2 and so on. Taking into account all possible partitions of $[a,b]$ we get aggregate $\{S_P\}$. The curve Γ is said to be rectifiable if the set $\{S_P\}$ is bounded for all partitions P of $[a,b]$. If the curve Γ is rectifiable, then the *l.u.b* of the set $\{S_P\}$ is defined to be the length of the curve Γ. If the set $\{S_P\}$ is unbounded, then Γ is called non-rectifiable.

Regular Curves (Arcs)

A simple curve (arc) defined by

$$z = z(t) = x(t) + iy(t),\ a \le t \le b,$$

is called a regular curve (arc) if the derivatives $\dot{x}(t)$ and $\dot{y}(t)$ exists, continuous and do not vanish simultaneously over $[a,b]$.

A regular arc is rectifiable and its length L is given by

$$L = \int_a^b |\dot{z}(t)|\, dt.$$

Contour.

A simple curve is called a contour if it consists of a finite number of regular arcs. A contour is rectifiable.

Closed Contour

A simple closed curve is called a closed contour if it consists of a finite number of regular arcs.

$z(t) = \cos t + i\sin t = e^{it}$, $0 \le t \le 2\pi$ is an example of a closed contour.

Simply and Multiply Connected Regions or Domains

A domain or a region R is called simply connected if any simple closed curve which lies in R can be shrunk to a point without leaving R. Alternatively a region R is said to be simply connected if every closed curve lying within it encloses only points of the region.

A region R which is not simply connected is called multiply connected.

For example, suppose R is the region defined by $|z| < 2$ shown in Fig. 4.1. If Γ is any simple closed curve lying in R, we see that it can be shrunk to a point which lies in R, and thus does not leave R, so that R is simply connected.

On the otherhand if R is the region defined by $1 < |z| < 2$, shown in Fig. 4.2, then there is a simple closed curve Γ lying in R which cannot possibly be shrunk to a point without leaving R, so that R is multiply connected.

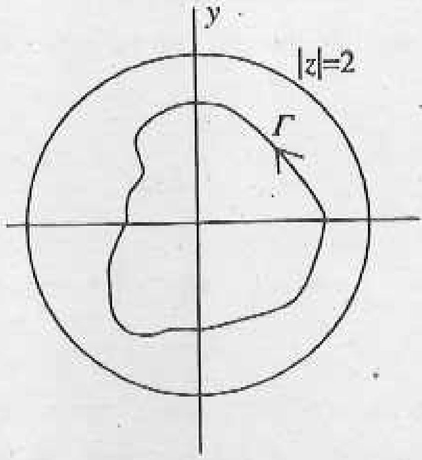


Fig. 4.1

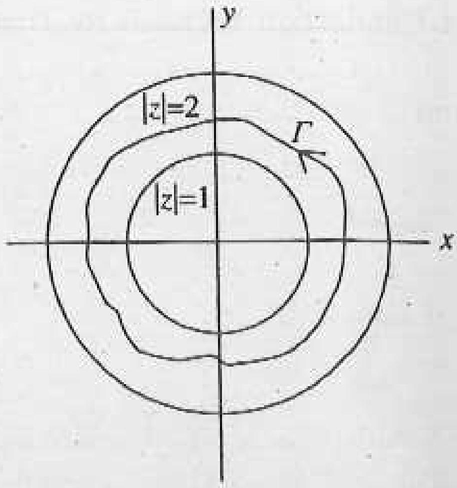


Fig. 4.2

4.3 Complex Line Integral :

Let $f(z)$ be continuous at all points of a rectifiable curve Γ. Divide Γ into n parts by means of points $z_1, z_2, \cdots, z_{n-1}$, chosen arbitrarily, and call $a = z_0$, $b = z_n$.

On each arc joining z_{k-1} to $z_k (k = 1, 2, \dots, n)$ choose a point ξ_k. Construct the sum

$$S_n = \sum_{k=1}^{n} f(\xi_k)(z_k - z_{k-1}).$$

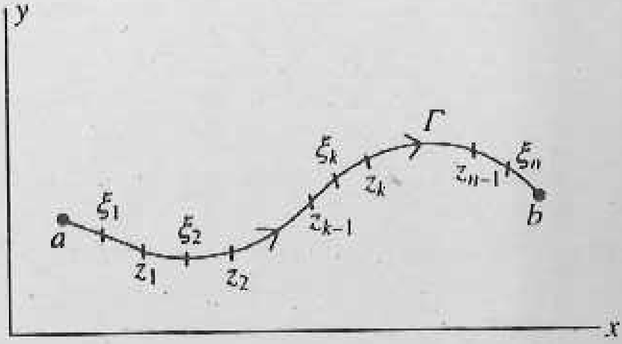


Fig. 4.3

Let the number of subdivisions n increase in such a way that the largest of the chord lengths $|z_k - z_{k-1}|$ approaches zero. Then S_n approaches a limit which does not depend on the mode of subdivision and we denote this limit by

$$\int_a^b f(z)dz \text{ or } \int_{\Gamma} f(z)dz$$

called the complex line integral or briefly line integral of $f(z)$ along curve Γ, or the definite integral of $f(z)$ from a to b along curve Γ. In such case $f(z)$ is said to be integrable along Γ. Note that if $f(z)$ is analytic at all points of a region R and if Γ is a curve lying in R, then $f(z)$ is certainly integrable along Γ.

4.4 Connection between Real and Complex Line Integrals :

If $f(z) = u(x,y) + iv(x,y) = u + iv$, the complex line integral can be expressed in terms of real line integral as

$$\int_{\Gamma} f(z)dz = \int_{\Gamma} (u+iv)\,(dx+idy)$$

$$= \int_{\Gamma} udx - vdy + i\int_{\Gamma} vdx + udy.$$

Example 4.1. If Γ is a rectifiable arc joining the points a and b, prove that

$$\int_{\Gamma} zdz = \frac{1}{2}(b^2 - a^2).$$

Solution : Here $f(z) = z$ and so the integral exist as the integrand is continuous on Γ. Divide Γ into smaller arcs by the points $a = z_0, z_1, z_2, \ldots, z_{k-1}, z_k, \ldots, z_n = b$ and formed the sum

$$S = \sum_{k=1}^{n} f(\xi_k)(z_k - z_{k-1}) = \sum_{k=1}^{n} \xi_k\,(z_k - z_{k-1}),$$

where ξ_k is a point on Γ between z_{k-1} and z_k.

We take $\xi_k = z_k$ and $\xi_k = z_{k-1}$ and obtain the sums

$$S_1 = \sum_{k=1}^{n} z_k\,(z_k - z_{k-1}) \text{ and } S_2 = \sum_{k=1}^{n} z_{k-1}(z_k - z_{k-1}).$$

Since $\int_{\Gamma} f(z)dz$ exist, taking $\| P \| = \max | z_k - z_{k-1} |$, $\lim\limits_{\|P\|\to 0} S$, $\lim\limits_{\|P\|\to 0} S_1$ and $\lim\limits_{\|P\|\to 0} S_2$ all exists and tends to the same limit $\int_{\Gamma} f(z)dz$.

Therefore

$$2\int_{\Gamma} f(z)dz = 2\lim_{\|P\|\to 0} S = \lim_{\|P\|\to 0} S_1 + \lim_{\|P\|\to 0} S_2 = \lim_{\|P\|\to 0} (S_1 + S_2)$$

$$= \lim_{\|P\|\to 0} \sum_{k=1}^{n} (z_k{}^2 - z_{k-1}^2) = z_n{}^2 - z_0{}^2 = b^2 - a^2.$$

i.e. $\int_{\Gamma} f(z)dz = \frac{1}{2}(b^2 - a^2).$

4.5 Some Elementary Properties of Integrals :

If $f(z)$ and $g(z)$ are continuous functions over a rectifiable curve Γ then

(i) $\int_{\Gamma}\{f(z)\pm g(z)\}dz=\int_{\Gamma}f(z)dz\pm\int_{\Gamma}g(z)dz.$

(ii) $\int_{\Gamma}kf(z)dz=k\int_{\Gamma}f(z)dz$, k is a constant.

(iii) If $-\Gamma$ denotes the arc Γ described in the opposite sense then

$$\int_{-\Gamma}f(z)dz=-\int_{\Gamma}f(z)dz.$$

(iv) If Γ is a rectifiable arc consisting of a finite number of rectifiable arcs Γ_1, Γ_2, ..., Γ_n i.e. $\Gamma=\Gamma_1+\Gamma_2+\ldots+\Gamma_n$, then

$$\int_{\Gamma}f(z)dz=\sum_{k=1}^{n}\int_{\Gamma_k}f(z)dz.$$

Change of variables

Let $z=g(\zeta)$ be a continuous function of a complex variable $\zeta=u+iv$. Suppose that curve Γ_1 in the z-plane corresponds to curve Γ_2 in the ζ-plane and that the derivative $g'(\zeta)$ is continuous on Γ_2. Then

$$\int_{\Gamma_1}f(z)dz=\int_{\Gamma_2}f\{g(\zeta)\}\,g'(\zeta)d\zeta.$$

These conditions are certainly satisfied if g is analytic in a region containing curve Γ_2.

4.6 Convention Regarding Traversal of a closed Path :

The boundary C of a region is said to be traversed in the positive sense or direction if an observer travelling in this direction (and perpendicular to the plane) has the region to the left. This convention leads to the directions indicated by the arrows in Figures 4.1 and 4.2. We use the special symbol

$$\oint_C f(z)dz$$

to denote integration of $f(z)$ around the boundary C in the positive sense. In case of a circle the positive direction is the counterclockwise direction. The integral around C is often called a contour integral.

An inequality for Complex Integrals

Theorem 4.1 (M–L Formula)

If a function $f(z)$ is continuous on a contour Γ of length L and if there exists a positive number M such that $|f(z)| \leq M \ \forall z$ on Γ, then

$$\left| \int_{\Gamma} f(z)dz \right| \leq ML.$$

Proof. By definition we have

$$\int_{\Gamma} f(z)dz = \lim_{n \to \infty} \sum_{k=1}^{n} f(\xi_k)\,(z_k - z_{k-1}) \qquad ...(4.1)$$

Now $$\left| \sum_{k=1}^{n} f(\xi_k)\,(z_k - z_{k-1}) \right| \leq \sum_{k=1}^{n} |f(\xi_k)|\,|z_k - z_{k-1}|$$

$$\leq M \sum_{k=1}^{n} |z_k - z_{k-1}| \leq ML \qquad ...(4.2)$$

where we have used the facts that $|f(z)| \leq M \ \forall z$ on Γ and that $\sum_{k=1}^{n} |z_k - z_{k-1}|$ represents the sum of all chord lengths joining points z_{k-1} and z_k, $k = 1, 2, \cdots, n$, and that this sum is not greater than the length of Γ. Taking the limit of both sides of (4.2) and using (4.1) we obtain the required result.

Example 4.2. Evalute $\int_C \bar{z}\,dz$ where C is the upper half of the circle $|z| = 1$ from $z = -1$ to $z = 1$.

Solution : Let $z = e^{i\theta}$, $\pi \geq \theta \geq 0$

Then $dz = ie^{i\theta}d\theta$.

Therefore, $\int_C \bar{z}\,dz = \int_{\pi}^{0} e^{-i\theta}\,.ie^{i\theta}d\theta = -\pi i.$

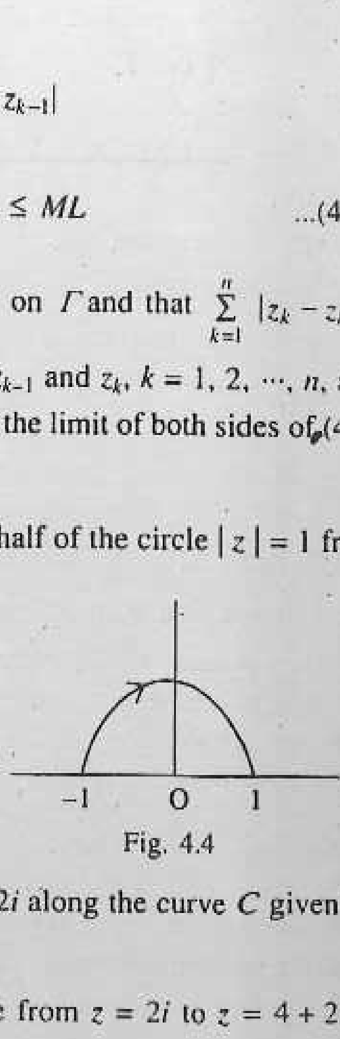


Fig. 4.4

Example 4.3. Evaluate $\int_C \bar{z}\,dz$ from $z = 0$ to $z = 4+2i$ along the curve C given by

(i) $z = t^2 + it$

(ii) the line from $z = 0$ to $z = 2i$ and then the line from $z = 2i$ to $z = 4 + 2i$.

Solution :

(i) The given integral equals

$$\int_C (x-iy)(dx+idy) = \int_C xdx+ydy+i\int_C xdy-ydx$$

The parametric equations of C are $x = t^2$, $y = t$ from $t = 0$ to $t = 2$. Then the line integral equals

$$\int_{t=0}^{2} (t^2)(2tdt)+(t)(dt)+i\int_{t=0}^{2} (t^2)(dt)-(t)(2tdt)$$

$$= \int_{t=0}^{2} (2t^3+t)dt - i\int_{t=0}^{2} t^2\,dt = 10 - \frac{8i}{3}.$$

(ii) The line from $z = 0$ to $z = 2i$ is same as the line from (0,0) to (0,2) for which $x = 0$ and the line integral equals

$$\int_{y=0}^{2} ydy = 2.$$

Again the line from $z = 2i$ to $z = 4 + 2i$ is the same as the line from (0,2) to (4,2) for which $y = 2$ and the line integral equals

$$\int_{x=0}^{4} xdx + i\int_{x=0}^{4} (-2)dx = 8 - 8i.$$

Then the required value $= 2 + 8 - 8i = 10 - 8i$.

Example 4.4. Evaluate $\int_{\Gamma} \frac{1}{(z-1)^3}dz$, where L is the directed line segment from $z = 1 + i$ to $z = 3 + 2i$.

Solution : Consider the path of integration joining the points (1, 1) and (3, 2) as a curve made of

(i) a line parallel to real axis from the point (1,1) to the point (3,1) and

(ii) a line parallel to the imaginary axis from the point (3,1) to the point (3,2).

For (i) we have $z = x + i$, $ay = 0$ and x goes from 1 to 3. Hence

$$\int_{(1,1)}^{(3,1)} \frac{dz}{(z-1)^3} = \int_1^3 \frac{dx}{(x-1+i)^3} = -\frac{1}{2}\left[\frac{1}{(x-1+i)^2}\right]_1^3$$

$$= \frac{1}{2i^2} - \frac{1}{2(2+i)^2} = -\frac{2(1+i)}{3+4i}.$$

For (ii) we have $z = 3 + iy$, $dx = 0$ and y goes from 1 to 2. Therefore

$$\int_{(3,1)}^{(3,2)} \frac{dz}{(z-1)^3} = i\int_1^2 \frac{dy}{(2+iy)^3} = \frac{1}{2}\left[\frac{1}{(2+i)^2} - \frac{1}{(2+2i)^2}\right]$$

$$= \frac{24-7i}{400}.$$

Thus $$\int_{1+i}^{3+2i} \frac{dz}{(z-1)^3} = -\frac{2(1+i)}{3+4i} + \frac{24-7i}{400} = \frac{i-8}{16}.$$

Example 4.5. Evaluate the integral $\int_{-2+i}^{5+3i} z^3 dz$.

Solution : Since $f(z) = z^3$ is an analytic function for all finite values of z, so its integration along a curve joining two fixed points will be the same, whatever be the path.

Here we have to integrate z^3 between two points (–2, 1) and (5, 3). Let us consider the path of integration joining these points be along the curve made up of

(i) a line parallel to the real axis from the point (–2, 1) to the point (5, 1). On this line : $z = x + i$, $dz = dx$ and x goes from –2 to 5.

(ii) followed by a line parallel to the imaginary axis from the point (5,1) to the point (5, 3). On this line :

$z = 5 + iy$, $dz = idy$ and y goes from 1 to 3.

Hence $\int_{-2+i}^{5+3i} z^3 dz = \int_{-2}^{5}(x+i)^3 dx + \int_1^3 (5+iy)^3 i dy$ along the chosen path

$$= \left[\frac{1}{4}(x+i)^4\right]_{-2}^{5} + \left[\frac{1}{4}(5+iy)^4\right]_1^3$$

$$= \frac{1}{4}[(5+i)^4 - (-2+i)^4] + \frac{1}{4}[(5+3i)^4 - (5+i)^4].$$

Example 4.6. Let $f(z) = \frac{1}{z^2}$ and Γ be the straight line joining the points i and $i + 2$. Show that

$$\left|\int_\Gamma f(z)dz\right| \le 2$$

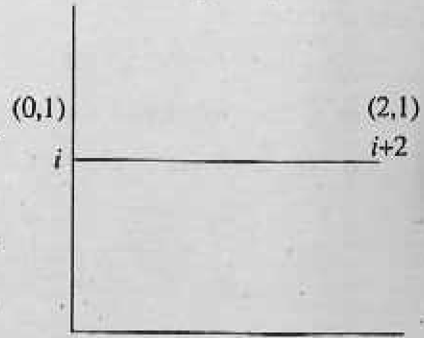


Fig. 4.5

Solution : On Γ, $|f(z)| = \frac{1}{|z|^2} = \frac{1}{x^2+y^2}$

$$= \frac{1}{1+x^2} \le 1 = M.$$

Here $L = 2$.

Hence by using ML formula we obtain

$$\left|\int_{\Gamma} f(z)\,dz\right| \le ML = 2.$$

4.7 Cauchy's Fundamental Theorem :

It is one of the most important results in the theory of functions of a complex variables. It has a far riching implication in the sense that almost every thing to follow will depend in one way or another on it.

Theorem 4.2

If $f(z)$ is analytic in a region R and on its boundary C, then

$$\oint_C f(z)\,dz = 0.$$

Note. This fundamental theorem is valid for both simply and multiply connected regions. It was first proved by use of Green's theorem with the added restriction that $f'(z)$ be continuous in R. However, Goursat gave a proof which removed this restriction. For this reason the theorem is sometimes called the Cauchy-Goursat theorem.

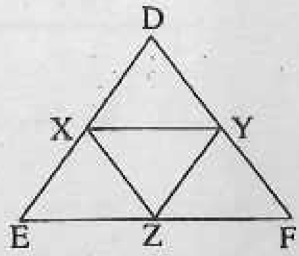


Fig. 4.6

Proof. We first prove the theorem taking C to be a triangle. Consider any triangle DEF, denoted briefly by Δ. Joint the midpoints X, Y and Z of sides DE, DF and EF respectively to form four triangles indicated briefly by Δ_I, Δ_{II}, Δ_{III} and Δ_{IV}.

If $f(z)$ is analytic inside and on triangle DEF we have, omitting the integrand on the right,

$$\oint_{DEFD} f(z)dz = \int_{YDX} + \int_{XEZ} + \int_{ZFY}$$

$$= \left\{\int_{YDX} + \int_{XY}\right\} + \left\{\int_{XEZ} + \int_{ZX}\right\} + \left\{\int_{ZFY} + \int_{YZ}\right\} + \left\{\int_{YX} + \int_{XZ} + \int_{ZY}\right\}$$

$$= \int_{YDXY} + \int_{XEZX} + \int_{ZFYZ} + \int_{YXZY}$$

$$= \oint_{\Delta_I} f(z)\,dz + \oint_{\Delta_{II}} f(z)\,dz + \oint_{\Delta_{III}} f(z)\,dz + \oint_{\Delta_{IV}} f(z)\,dz$$

Then

$$\left|\oint_{\Delta} f(z)dz\right| \le \left|\oint_{\Delta_I} f(z)dz\right| + \left|\oint_{\Delta_{II}} f(z)dz\right| + \left|\oint_{\Delta_{III}} f(z)dz\right| + \left|\oint_{\Delta_{IV}} f(z)dz\right| \quad ...(4.3)$$

Let Δ_1 be the triangle corresponding to that term on the right of (4.3) having largest value. Then

$$\left|\oint_{\Delta} f(z)dz\right| \le 4\left|\oint_{\Delta_1} f(z)dz\right|$$

By joining midpoints of the sides of triangle Δ_1, we obtain similarly a triangle Δ_2 such that

$$\left|\oint_{\Delta_1} f(z)dz\right| \le 4\left|\oint_{\Delta_2} f(z)dz\right|$$

so that $$\left|\oint_{\Delta} f(z)dz\right| \le 4^2\left|\oint_{\Delta_2} f(z)dz\right|$$

After n steps we obtain a triangle Δ_n such that

$$\left|\oint_{\Delta} f(z)dz\right| \le 4^n\left|\oint_{\Delta_n} f(z)dz\right| \quad ...(4.4)$$

Now $\Delta, \Delta_1, \Delta_2, ...$ is a sequence of triangles each of which contained in the preceding (i.e. a sequence of nested triangles) and there exists a point z_0 which lies in each triangle of the sequence. Since $f(z)$ is analytic at z_0, we have

$$f(z) = f(z_0) + f'(z_0)\,(z - z_0) + \eta(z - z_0) \quad ...(4.5)$$

where for any $\varepsilon > 0$ we can find δ such that $|\eta| < \varepsilon$ whenever $|z - z_0| < \delta$. Thus by integration of both sides of (4.5) and using the results

$$\oint_{\Delta_n} dz = 0 \text{ and } \oint_{\Delta_n} z\,dz = 0 \text{ we get } \oint_{\Delta_n} f(z)dz = \oint_{\Delta_n} \eta(z - z_0)dz \quad ...(4.6)$$

Now if P be the perimeter of Δ, then the perimeter of Δ_n is $P_n = P/2^n$. If z is any point on Δ_n, then as seen from Fig. 4.7 we must have $|z - z_0| < P/2^n < \delta$.

Hence from (4.6) and using ML-formula we obtain

$$\left|\oint_{\Delta_n} f(z)\,dz\right| = \left|\oint_{\Delta_n} \eta\,(z-z_0)\,dz\right| \le \varepsilon\cdot\frac{P}{2^n}\cdot\frac{P}{2^n} = \frac{\varepsilon P^2}{4^n}.$$

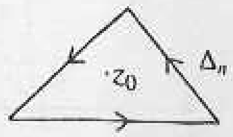


Fig. 4.7

Then (4.4) becomes

$$\left|\oint_{\Delta} f(z)\,dz\right| \le 4^n\cdot\frac{\varepsilon P^2}{4^n} = \varepsilon P^2.$$

Since ε can be made arbitrarily small it follows that,

$$\oint_{\Delta} f(z)\,dz = 0.$$

- Now we assume C as any closed Polygon.

We consider a simple closed polygon ABCDEFA such as indicated in Fig. 4.8. By constructing the lines BF, CF and DF the polygon is subdivided into triangles. So by above

$$\oint_{ABCDEFA} f(z)dz = \oint_{ABFA} f(z)dz + \oint_{BCFB} f(z)dz + \oint_{CDFC} f(z)dz + \oint_{DEFD} f(z)dz = 0$$

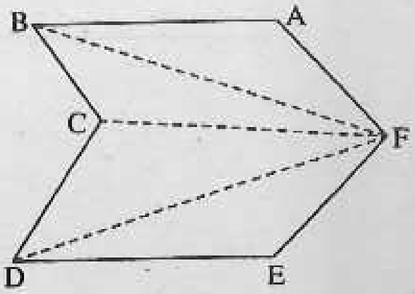


Fig. 4.8

where we suppose that $f(z)$ is analytic inside and on the polygon.

Note 4.1. It should be noted that we have proved the result for simple polygons whose sides do not cross. A proof can also be given for any polygon which intersects itself. For the proof of this theorem for any simple closed curve as well as for any multiply-connected regions, the readers can see any reference books.

4.8 Consequences of Cauchy's Fundamental Theorem :

Theorem 4.3

Let f be analytic in a simply connected region R and let α and β be any two points in R. Then $\int_\alpha^\beta f(z)\,dz$ is independent of the path in R joining α and β.

Proof. Let us join the points $A(\alpha)$ and $B(\beta)$ by two curves C_1 and C_2. Then we get a simple closed rectifiable curve $ACBDA$. Since f is analytic in R, by Cauchy's fundamental theorem we get

$$\int_{ACBDA} f(z)\,dz = 0$$

$$\Rightarrow \int_{ACB} f(z)dz + \int_{BDA} f(z)dz = 0$$

$$\Rightarrow \oint_{C_2} f(z)dz - \oint_{C_1} f(z)dz = 0$$

$$\Rightarrow \oint_{C_1} f(z)dz = \oint_{C_2} f(z)dz.$$

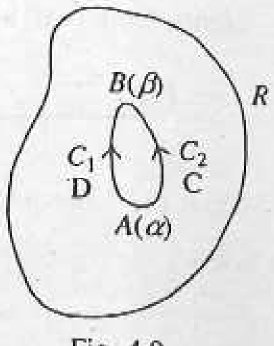


Fig. 4.9

So $\int_\alpha^\beta f(z)\,dz$ is independent of the path in R joining α and β.

Theorem 4.4

Let $f(z)$ be analytic in a simply connected region R and let a and z be points in R. Then $F(z) = \int_a^z f(\omega)d\omega$ is analytic in R and $F'(z) = f(z)$.

Proof. We have

$$\frac{F(z+\Delta z)-F(z)}{\Delta z} - f(z) = \frac{1}{\Delta z}\left[\int_a^{z+\Delta z} f(\omega)d\omega - \int_a^{z} f(\omega)d\omega\right] - f(z)$$

$$= \frac{1}{\Delta z}\int_z^{z+\Delta z} [f(\omega)-f(z)]d\omega. \qquad ...(4.7)$$

By Cauchy's fundamental theorem, the last integral is independent of the path joining z and $z + \Delta z$ so long as the path is in R. In particular we can choose as path the straight line segment joining z and $z + \Delta z$ provided we choose $|\Delta z|$ small enough so that this path lies in R.

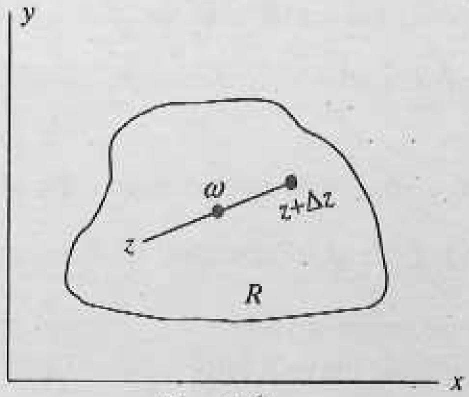


Fig. 4.10

Now by the continuity of $f(z)$ we have for all points ω on this straight line path $|f(\omega) - f(z)| < \varepsilon$ whenever $|\omega - z| < \delta$, which will certainly be true if $|\Delta z| < \delta$. Again we have

$$\left| \int_z^{z+\Delta z} [f(\omega) - f(z)]\, d\omega \right| < \varepsilon \Delta z.$$

So from (4.7) we obtain

$$\left| \frac{F(z+\Delta z) - F(z)}{\Delta z} - f(z) \right| = \frac{1}{|\Delta z|} \left| \int_z^{z+\Delta z} [f(\omega) - f(z)]\, d\omega \right| < \varepsilon \text{ for } |\Delta z| < \delta.$$

This, however, amounts to saying that

$$\lim_{\Delta z \to 0} \frac{F(z+\Delta z) - F(z)}{\Delta z} = f(z), \text{ i.e. } F(z) \text{ is analytic and } F'(z) = f(z).$$

Theorem 4.5

Let C_1 and C_2 be two simple closed curves, C_2 lieing holly within C_1. If f is analytic in the close annulus determine by C_1 and C_2, then

$$\oint_{C_1} f(z)\, dz = \oint_{C_2} f(z)\, dz$$

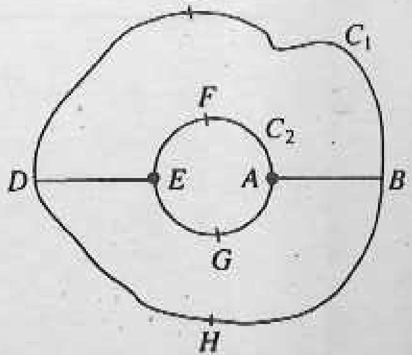


Fig. 4.11

Proof. We introduce two cuts AB and DE joining C_1 and C_2. Since f is analytic in the annular region determined by C_1 and C_2, we have

$$\int_{ABCDEFA} f(z)\, dz = 0 \text{ and } \int_{EDHBAGE} f(z)\, dz = 0.$$

From first one we obtain

$$\int_{AB} f(z)\, dz + \int_{BCD} f(z)\, dz + \int_{DE} f(z)\, dz + \int_{EFA} f(z)\, dz = 0. \qquad \text{...(4.8)}$$

From second one we obtain

$$\int_{ED} f(z)\, dz + \int_{DHB} f(z)\, dz + \int_{BA} f(z)\, dz + \int_{AGE} f(z)\, dz = 0. \qquad \text{...(4.9)}$$

From (4.8) and (4.9) we obtain

$$\int_{BCD} f(z)\, dz + \int_{DHB} f(z)\, dz + \int_{EFA} f(z)\, dz + \int_{AGE} f(z)\, dz = 0$$

i.e. $\oint_{C_1} f(z) dz + \oint_{-C_2} f(z)\, dz = 0$

i.e. $\oint_{C_1} f(z)dz = \oint_{C_2} f(z)\,dz.$

This proves the theorem.

The following theorem is a generalization of Theorem 4.5

Theorem 4.6

If $C_1, C_2, \cdots, C_n$ are simple closed curves no two of which have common point and if C is any simple closed curve which contains $C_1, C_2, \cdots, C_n$ in its interior then

$$\oint_C f(z)dz = \sum_{k=1}^{n} \oint_{C_k} f(z)\,dz.$$

Proof. The proof of the theorem is left out as an exercise.

Example 4.7. Evaluate $\oint_C \frac{dz}{z-a}$ where C is any simple closed curve and $z = a$ is

(i) outside C, (ii) inside C.

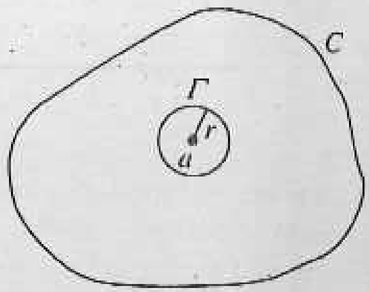


Fig. 4.12

Solution :

(i) If a is outside C, then $f(z) = \frac{1}{z-a}$ is analytic everywhere inside and on C. Hence by Cauchy's fundamental theorem

$$\oint_C \frac{dz}{z-a} = 0.$$

(ii) Suppose a is inside C and let Γ be a circle of radius r with centre at $z = a$ so that Γ is inside C. Since $f(z)$ is analytic in the closed annulus bounded by C and Γ, we have

$$\oint_C f(z)dz = \oint_\Gamma f(z)\,dz.$$

Now on Γ, $z - a = re^{i\theta}$, $0 \le \theta \le 2\pi$. Hence

$$\oint_C \frac{dz}{z-a} = \oint_\Gamma \frac{dz}{z-a} = \int_0^{2\pi} \frac{ire^{i\theta}d\theta}{re^{i\theta}} = 2\pi i,$$

which is the required value.

4.9 Cauchy's Integral Formula :

Theorem 4.7

Let f be analytic within and on a simple closed contour C and let α be any point inside C. Then

$$f(\alpha)=\frac{1}{2\pi i}\oint_C \frac{f(z)}{z-\alpha}dz$$

Proof. Let Γ denote the circle $|z-\alpha|=r$, r is taken so small that Γ lies entirely within C. The function $\frac{f(z)}{z-\alpha}$ is clearly analytic in the closed annulus bounded by C and Γ.

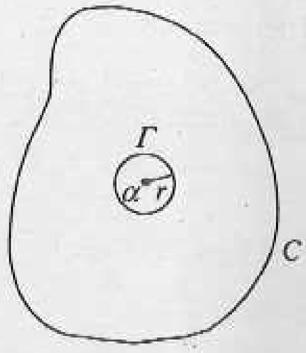


Fig. 4.13

Hence

$$\oint_C \frac{f(z)}{z-\alpha}dz = \oint_\Gamma \frac{f(z)}{z-\alpha}dz$$

$$= \oint_\Gamma \frac{f(z)-f(\alpha)}{z-\alpha}dz + \oint_\Gamma \frac{f(\alpha)}{z-\alpha}dz$$

$$= \oint_\Gamma \frac{f(z)-f(\alpha)}{z-\alpha}dz + 2\pi i.f(\alpha).$$

Thus $\left|\frac{1}{2\pi i}\oint_C \frac{f(z)}{z-\alpha}dz - f(\alpha)\right| = \left|\frac{1}{2\pi i}\oint_\Gamma \frac{f(z)-f(\alpha)}{z-\alpha}dz\right|$ ...(4.10)

Since f is continuous at $z=\alpha$, given $\varepsilon\,(>0)$, there exist a $\delta\,(>0)$ such that $|f(z)-f(\alpha)|<\varepsilon$ whenever $|z-\alpha|<\delta$. We choose $r<\delta$, so that $|f(z)-f(\alpha)|<\varepsilon\ \forall z$ on Γ.

Hence on Γ

$$\left|\frac{f(z)-f(\alpha)}{z-\alpha}\right| < \frac{\varepsilon}{r}.$$

Thus from (4.10) using ML-formula we obtain

$$\left|\frac{1}{2\pi i}\oint_C \frac{f(z)}{z-\alpha}dz - f(\alpha)\right| \le \frac{1}{2\pi}\cdot\frac{\varepsilon}{r}\cdot 2\pi r = \varepsilon\,.$$

Since $\varepsilon\,(>0)$ is orbitrary, it follows that

$$\frac{1}{2\pi i}\oint_C \frac{f(z)}{z-\alpha}dz - f(\alpha)=0$$

i.e. $f(\alpha) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-\alpha}dz\,.$

This proves the theorem.

Note 4.2. Cauchy's Integral Formula is remarkable in the sense that it expresses the value of an analytic function at a point within a closed contour in terms of its values on the contour.

Gauss' Mean Value Theorem

Theorem 4.8

If $f(z)$ is analytic within and on a circle C with centre at α and radius r, then $f(\alpha)$ is the mean of the values of $f(z)$ on C, i.e.,

$$f(\alpha) = \frac{1}{2\pi}\int_0^{2\pi} f(\alpha + re^{i\theta})\,d\theta$$

Proof. By Cauchy's integral formula,

$$f(\alpha) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-\alpha}\,dz \qquad \text{...(4.11)}$$

Here the equation of C is $|z - \alpha| = r$ or $z = \alpha + re^{i\theta}$.

Thus (4.11) becomes

$$f(\alpha) = \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(\alpha + re^{i\theta})\,ire^{i\theta}}{re^{i\theta}}\,d\theta = \frac{1}{2\pi}\int_0^{2\pi} f(\alpha + re^{i\theta})\,d\theta$$

which is the required result.

Cauchy's Integral Formula for multiply connected region

Theorem 4.9

If $f(z)$ is analytic in the closed annulus bounded by two closed contours C_1 and C_2 (C_2 lying entirely within C_1) and if α is any point in this annular region then

$$f(\alpha) = \frac{1}{2\pi i}\oint_{C_1} \frac{f(z)}{z-\alpha}\,dz - \frac{1}{2\pi i}\oint_{C_2} \frac{f(z)}{z-\alpha}\,dz$$

Proof. Let $\Gamma : |z - \alpha| = r$, where r is chosen so small that Γ lies entirely within the annulus. The function $\frac{f(z)}{z-\alpha}$ is analytic in the region bounded by C_1, C_2 and Γ. Therefore

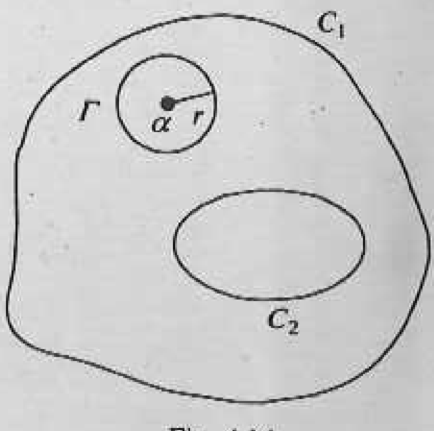


Fig. 4.14

$$\frac{1}{2\pi i}\oint_{C_1} \frac{f(z)}{z-\alpha}\,dz - \frac{1}{2\pi i}\oint_{C_2} \frac{f(z)}{z-\alpha}\,dz$$

$$= \frac{1}{2\pi i}\oint_{\Gamma} \frac{f(z)}{z-\alpha}\,dz \qquad \text{...(4.12)}$$

Now $\frac{1}{2\pi i}\oint_{\Gamma}\frac{f(z)}{z-\alpha}dz = \frac{1}{2\pi i}\oint_{\Gamma}\frac{f(z)-f(\alpha)}{z-\alpha}dz + f(\alpha)$ $\left[\because \oint_{\Gamma}\frac{dz}{z-\alpha} = 2\pi i\right]$

So from (4.12) we obtain

$$\left|\frac{1}{2\pi i}\oint_{C_1}\frac{f(z)}{z-\alpha}dz - \frac{1}{2\pi i}\oint_{C_2}\frac{f(z)}{z-\alpha}dz - f(\alpha)\right| = \left|\frac{1}{2\pi i}\oint_{\Gamma}\frac{f(z)-f(\alpha)}{z-\alpha}dz\right| \quad ...(4.13)$$

Since $f(z)$ is continuous at $z = \alpha$, for $\varepsilon > 0$, $\exists$ a $\delta > 0$ such that $|f(z) - f(\alpha)| < \varepsilon$ whenever $|z - \alpha| < \delta$. We take $r < \delta$ so that $|f(z) - f(\alpha)| < \varepsilon \;\forall\; z$ on Γ. Hence

$$\left|\frac{f(z)-f(\alpha)}{z-\alpha}\right| < \frac{\varepsilon}{r}.$$

So using ML-formula we obtain from (4.13)

$$\left|\frac{1}{2\pi i}\oint_{C_1}\frac{f(z)}{z-\alpha}dz - \frac{1}{2\pi i}\oint_{C_2}\frac{f(z)}{z-\alpha}dz - f(\alpha)\right| \leq \frac{1}{2\pi}\cdot\frac{\varepsilon}{r}\cdot 2\pi r = \varepsilon.$$

Since $\varepsilon > 0$ is orbitrary, we have

$$f(\alpha) = \frac{1}{2\pi i}\oint_{C_1}\frac{f(z)}{z-\alpha}dz - \frac{1}{2\pi i}\oint_{C_2}\frac{f(z)}{z-\alpha}dz.$$

This completes the proof.

Cauchy's integral Formula for Derivatives

Theorem 4.10

Let f be analytic within and on a simple closed contour C. If α is any point interior to C, then

$$f'(\alpha) = \frac{1}{2\pi i}\oint_{C}\frac{f(z)}{(z-\alpha)^2}dz.$$

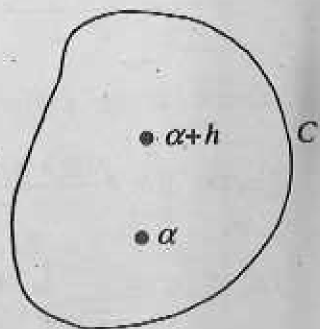


Fig. 4.15

Proof. Let d be the lower bound of the distances of the point α from the contour C. If h denote a complex number such that $|h| < d$, then the point $\alpha + h$ also lies within C. Therefore by Cauchy's Integral Formula,

$$f(\alpha) = \frac{1}{2\pi i}\oint_{C}\frac{f(z)}{z-\alpha}dz$$

and $\quad f(\alpha + h) = \dfrac{1}{2\pi i}\oint\limits_C \dfrac{f(z)}{z-\alpha-h}dz.$

Now $\quad \dfrac{f(\alpha+h)-f(\alpha)}{h} = \dfrac{1}{2\pi i}\oint\limits_C \dfrac{f(z)}{(z-\alpha)(z-\alpha-h)}dz.$

Hence $\left|\dfrac{f(\alpha+h)-f(\alpha)}{h} - \dfrac{1}{2\pi i}\oint\limits_C \dfrac{f(z)}{(z-\alpha)^2}dz\right|$

$$= \left|\frac{1}{2\pi i}\oint\limits_C f(z)\left[\frac{1}{(z-\alpha)(z-\alpha-h)} - \frac{1}{(z-\alpha)^2}\right]dz\right|$$

$$= \frac{|h|}{2\pi}\left|\oint\limits_C \frac{f(z)}{(z-\alpha)^2(z-\alpha-h)}dz\right| \qquad \text{...(4.14)}$$

Since f is continuous on C, it is bounded on C. So there exist a positive number M such that $|f(z)| \le M$ on C. Also by the definition of d it follows that for all z on C

$$|z-\alpha|^2 \ge d^2$$

and $|z-\alpha-h| \ge |z-\alpha| - |h| \ge d - |h|$.

So on C,

$$\left|\frac{f(z)}{(z-\alpha)^2(z-\alpha-h)}\right| \le \frac{M}{d^2(d-|h|)}.$$

Hence using ML-formula we get from (4.14)

$$\left|\frac{f(\alpha+h)-f(\alpha)}{h} - \frac{1}{2\pi i}\oint\limits_C \frac{f(z)}{(z-\alpha)^2}dz\right| \le \frac{|h|}{2\pi}\cdot\frac{Ml}{d^2(d-|h|} \to 0 \text{ as } h \to 0,$$

where l is the length of C.

Hence $\lim\limits_{h\to 0} \dfrac{f(\alpha+h)-f(\alpha)}{h} = \dfrac{1}{2\pi i}\oint\limits_C \dfrac{f(z)}{(z-\alpha)^2}dz.$

i.e. $f'(\alpha) = \dfrac{1}{2\pi i}\oint\limits_C \dfrac{f(z)}{(z-\alpha)^2}dz.$

This proves the theorem.

Theorem 4.11

Let $f(z)$ be analytic within and on a simple closed contour C. Then for any point α interior to C

$$f^{(n)}(\alpha) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-\alpha)^{n+1}}\,dz,\ n = 0, 1, 2, \ldots \qquad ..(4.15)$$

Proof. We prove the theorem by mathematical induction. First we note that (4.15) is true for $n = 0, 1$. We suppose that (4.15) is true for $n = m$. Then

$$\frac{f^{(m)}(\alpha+h)-f^{(m)}(\alpha)}{h} = \frac{1}{h}\frac{m!}{2\pi i}\left[\oint_C \frac{f(z)}{(z-\alpha-h)^{m+1}}\,dz - \oint_C \frac{f(z)}{(z-\alpha)^{m+1}}\,dz\right]$$

$$= \frac{1}{h}\frac{m!}{2\pi i}\oint_C\left[(z-\alpha-h)^{-(m+1)} - (z-\alpha)^{-(m+1)}\right]f(z)\,dz$$

$$= \frac{1}{h}\frac{m!}{2\pi i}\oint_C \frac{1}{(z-\alpha)^{m+1}}\left\{\left(1-\frac{h}{z-\alpha}\right)^{-(m+1)} - 1\right\}f(z)\,dz$$

$$= \frac{1}{h}\frac{m!}{2\pi i}\oint_C \frac{1}{(z-\alpha)^{m+1}}\left[1+(m+1)\frac{h}{z-\alpha} + \frac{(m+1)(m+2)}{2!}\frac{h^2}{(z-\alpha)^2}+\cdots-1\right]f(z)\,dz$$

Taking limit as $h \to 0$ we get

$$\lim_{h\to 0}\frac{f^{(m)}(\alpha+h)-f^{(m)}(\alpha)}{h} = \frac{(m+1)!}{2\pi i}\oint_C \frac{f(z)}{(z-\alpha)^{m+2}}\,dz$$

i.e. $$f^{(m+1)}(\alpha) = \frac{(m+1)!}{2\pi i}\oint_C \frac{f(z)}{(z-\alpha)^{(m+1)+1}}\,dz.$$

Therefore by mathematical induction (4.15) is true for $n = 0, 1, 2, \cdots$.

This proves the theorem.

Theorem 4.12

Let $f(z)$ be analytic in a domain D. Then all the derivatives of $f(z)$ exist and are analytic functions in D.

Proof. Let $z_0 \in D$ and let C be a circle with centre at z_0 contained in D. Then for $n = 0, 1, 2, \cdots$

$$f^{(n)}(\alpha) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-\alpha)^{n+1}}\,dz$$

for all α interior to C.

Thus $f(z)$ has a derivative of all order in a neighbourhood of z_0. Since z_0 is any point in D, the theorem is proved.

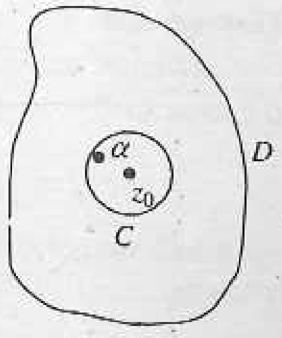


Fig. 4.16 .

Note 4.3. The situation is completely different in the case of a real function. There exist functions which have a first derivative but no second derivative. As for example we consider the function

$$f(x) = \begin{cases} x^2 \sin\frac{1}{x}, & x \neq 0 \\ 0 & x = 0 \end{cases},$$

where x is real. Then $f(x)$ has a first derivative in $[0, 1]$ but no second derivative in $[0, 1]$.

4.10 Winding Number or Index of a Curve :

Suppose that Γ is a closed curve in C. Let a be a given point in $C\setminus\Gamma$. Then, there is a useful formula that measures how often Γ winds around a. For example if Γ: $\Gamma(t) = \{z : z - a = re^{i\theta}, 0 \leq \theta \leq 2k\pi\}$, then Γ encircles the point a k times (counterclockwise). Further,

$$\int_\Gamma \frac{dz}{z-a} = \int_0^{2k\pi} \frac{ire^{i\theta}}{re^{i\theta}} d\theta = 2k\pi i$$

i.e. $$\frac{1}{2\pi i}\int_\Gamma \frac{dz}{z-a} = k.$$

From this we also observe that if Γ encircles the point a k-times in the clockwise direction, then

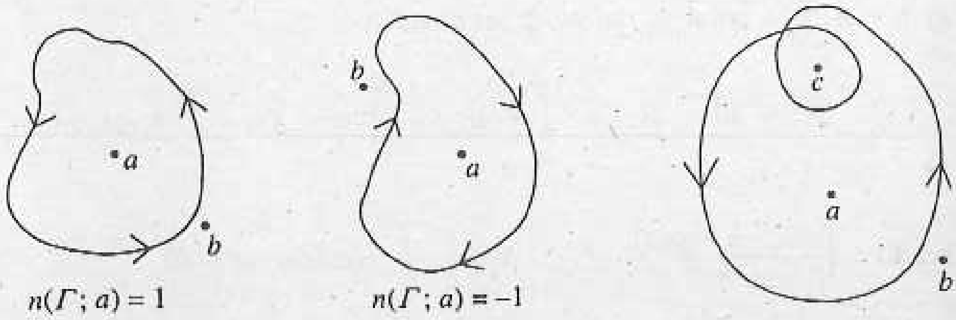
$$\frac{1}{2\pi i}\int_\Gamma \frac{dz}{z-a} = -k.$$

In either case, $\frac{1}{2\pi i}\int_\Gamma \frac{dz}{z-a}$ is an integer. Here is the analytic definition of the winding number of a, which captures the intuitive notion of "the number of times Γ wraps around a in the counterclockwise direction" (see Fig. 4.17).

Definition

Let Γ be a closed contour in C that avoids a point $a \in C$. The index or winding number of Γ about a, denoted by $n(\Gamma; a)$ is given by the integral

$$n(\Gamma : a) = \frac{1}{2\pi i}\int_\Gamma \frac{dz}{z-a}.$$



$n(\Gamma; a) = 1$
$n(\Gamma; b) = 0$

$n(\Gamma; a) = -1$
$n(\Gamma; b) = 0$

$n(\Gamma; a) = 1$
$n(\Gamma; b) = 0$
$n(\Gamma; c) = 2$

Fig. 4.17

Some Properties of the index $n(\Gamma : a)$

(i) For every closed contour Γ in C and $a \in C \setminus \Gamma$, $n(\Gamma ; a)$ is an integer.

(ii) If Γ is a closed contour in C, then the mapping $a \to n(\Gamma; a)$ is a continuous function of a at any point $a \notin \Gamma$.

(iii) We have $n(\Gamma; a) = 0$ in the unbounded component of the closed contour Γ.

(iv) If Γ consists of finitely many closed contours $\Gamma_1, \Gamma_2, \cdots, \Gamma_k$ in C, then for every $a \notin \Gamma_i$ $(i = 1, 2, \cdots, k)$,

$$n(\Gamma; a) = n(\Gamma_1 ; a) + n(\Gamma_2 ; a) + \cdots + n(\Gamma_k ; a).$$

Morera's Theorem

Theorem 4.13

If $f(z)$ is continuous in a simply connected domain D and if

$$\int_{\Gamma} f(z)dz = 0$$

for every closed curve Γ in D, then $f(z)$ is analytic in D.

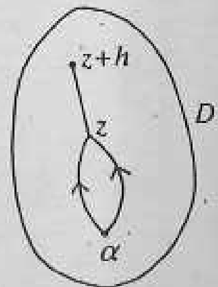


Fig. 4.18

Proof. Let α be a fixed point and z be a variable point in D. Then the value of the integral $\int_{\alpha}^{z} f(\omega)\, d\omega$ is independent of the path so long as the path lies in D. We define

$$\phi(z) = \int_{\alpha}^{z} f(\omega)\, d\omega.$$

Let $z + h$ be a point in the neighbourhood of z. Now

$$\phi(z + h) - \phi(z) = \int_{\alpha}^{z+h} f(\omega)\, d\omega - \int_{\alpha}^{z} f(\omega)\, d\omega = \int_{z}^{z+h} f(\omega)\, d\omega.$$

Therefore $$\left|\frac{\phi(z+h)-\phi(z)}{h} - f(z)\right| = \left|\frac{1}{h}\int_{z}^{z+h} f(\omega)d\omega - f(z)\right|$$

$$\leq \frac{1}{|h|}\int |f(\omega)-f(z)||d\omega| \qquad ...(4.16)$$

Since $f(z)$ is continuous at z, for $\varepsilon > 0$, $\exists$ a $\delta > 0$ such that $|f(\omega) - f(z)| < \varepsilon$ whenever $|\omega - z| < \delta$. We choose $|h| < \delta$. Then for every point ω on the straight line joining z to $z + h$, we have $|f(\omega) - f(z)| < \varepsilon$.

Hence from (4.16) we obtain by using ML-formula

$$\left|\frac{\phi(z+h)-\phi(z)}{h} - f(z)\right| \leq \frac{1}{|h|}\cdot\varepsilon\cdot|h| = \varepsilon.$$

This gives

$$\lim_{h\to 0}\frac{\phi(z+h)-\phi(z)}{h} = f(z)$$

i.e. $\phi'(z) = f(z)$.

Therefore $\phi(z)$ is analytic in D. Since $f(z)$ is the derivative of an analytic function, it follows that $f(z)$ itself is analytic in D. This proves the theorem.

Note 4.4. Let $f(z) = \begin{cases} \dfrac{\cos z}{z^2} & \text{if } z \in \Delta \setminus \{0\} \\ 0 & \text{if } z = 0 \end{cases}$

where Δ denotes the circle $|z| = 1$. Then

$$\int_{\Delta} f(z)\, dz = 0,$$

However $f(z)$ is not analytic in Δ, since $f(z)$ is not continuous at $z = 0$. Here Morera's theorem is not applicable since the continuity requirement is not satisfied.

4.11 Cauchy's inequality :

If $f(z)$ is analytic within and on a circle C with centre α and radius r and if $|f(z)| \leq M$ on C, M being a positive constant, then

$$|f^{(n)}(\alpha)| \leq \frac{M.n!}{r^n} \text{ for } n = 0, 1, 2, \cdots .$$

Proof. Since $f(z)$ is analytic within and on C and α is an interior point of C we have

$$f^{(n)}(\alpha) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-\alpha)^{n+1}}dz,\ n = 0, 1, 2, \cdots$$

On C, $|z - \alpha| = r$. Therefore by ML-formula we get

$$|f^{(n)}(\alpha)| = \frac{n!}{2\pi}\left|\oint_C \frac{f(z)}{(z-\alpha)^{n+1}}dz\right| \leq \frac{n!}{2\pi}\cdot\frac{M}{r^{n+1}}\cdot 2\pi r = \frac{M.n!}{r^n},\ n = 0, 1, 2, \cdots$$

This proves the theorem.

Liouville's Theorem

Theorem 4.14

Suppose that for all z in the entire complex plane, $f(z)$ is analytic and bounded. Then $f(z)$ must be a constant.

Proof. Let α be any point in the open complex plane and C denote the circle $|z - \alpha| = r$. Let $|f(z)| < M$ for some constant M for all z on C, no matter how large the radius r is. The function f is analytic within and on C and α is a point within C. So by Cauchy integral formula for derivatives we get

$$f'(\alpha) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-\alpha)^2}dz.$$

Therefore by using ML-formula we obtain

$$|f'(\alpha)| = \frac{1}{2\pi}\left|\oint_C \frac{f(z)}{(z-\alpha)^2}dz\right| \leq \frac{1}{2\pi}\cdot\frac{M}{r^2}\cdot 2\pi r = \frac{M}{r}.$$

Letting $r \to \infty$ we have $f'(\alpha) = 0$. Since α is any point it follows that $f'(z) = 0$ $\forall\ z$. Hence $f(z)$ is a constant.

This proves the theorem.

Fundamental Theorem of Classical Algebra

Theorem 4.15

If $f(z)$ is a polynomial of degree $n(\geq 1)$ with real or complex coefficient then the equation $f(z) = 0$ has at least one root.

Proof. Let $f(z) = a_n z^n + a_{n-1}z^{n-1} + \ldots + a_1 z + a_0$, $a_n \neq 0$. be a polynomial of degree n. If possible, suppose that $f(z) = 0$ has no root. Since $f(z)$ is a polynomial, it is analytic for all z. Also since $f(z) \neq 0$ for any z, $\phi(z) = \dfrac{1}{f(z)}$ is analytic for all z.

Now for $z \neq 0$,

$$f(z) = z^n\left(a_n + \frac{a_{n-1}}{z} + \cdots + \frac{a_1}{z^{n-1}} + \frac{a_0}{z^n}\right)$$

and so

$$|f(z)| = |z|^n \left|a_n + \frac{a_{n-1}}{z} + \cdots + \frac{a_1}{z^{n-1}} + \frac{a_0}{z^n}\right|$$

$$\geq |z|^n\left[|a_n| - \frac{|a_{n-1}|}{|z|} - \cdots - \frac{|a_1|}{|z|^{n-1}} - \frac{|a_0|}{|z|^n}\right] \to \infty \text{ as } |z| \to \infty.$$

Therefore $|\phi(z)| \to 0$ as $|z| \to \infty$. Hence $\phi(z)$ is bounded. Thus $\phi(z)$, being a bounded analytic function, is constant by Liouville's theorem. So $f(z)$ is also constant, a contradiction. Therefore $f(z) = 0$ has at least one root. This proves the theorem.

Example 4.8.

Evaluate $\displaystyle\oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz$, where C is a circle $|z| = 3$.

Solution :

$$\oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz = \oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{z-2} dz - \oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{z-1} dz$$

$$= A - B, \text{ say.}$$

Since $\sin \pi z^2 + \cos \pi z^2$ is analytic inside C and $z = 1$, $z = 2$ lies inside C, by Cauchy's integral formula we have

$$A = 2\pi i\ (\sin 4\pi + \cos 4\pi) = 2\pi i \text{ and } B = 2\pi i\ (\sin \pi + \cos \pi) = -2\pi i.$$

So the required integral has the value $2\pi i - (-2\pi i) = 4\pi i$.

Example 4.9. Evaluate $\displaystyle\oint_C \frac{e^{2z}}{(z+1)^4} dz$, where C is the circle $|z| = 3$.

Solution : Let $f(z) = e^{2z}$ and $\alpha = -1$ in the formula

$$f^{(n)}(\alpha) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-\alpha)^{n+1}}\,dz \qquad \text{...(4.17)}$$

If $n = 3$, then $f'''(z) = 8e^{2z}$ and $f'''(-1) = 8e^{-2}$.

Hence (4.17) becomes

$$8e^{-2} = \frac{3!}{2\pi i}\oint_C \frac{e^{2z}}{(z+1)^4}\,dz$$

i.e. $$\oint_C \frac{e^{2z}}{(z+1)^4}\,dz = \frac{8\pi i e^{-2}}{3}.$$

Example 4.10

Evaluate $\oint_C \frac{dz}{e^z(z-1)^2}$, where C is the circle $|z| = 4$ traversed once counterclockwise.

Solution : Let $f(z) = \frac{1}{e^z}$. Then $f(z)$ is analytic within and on the circle $|z| = 4$. Taking $\alpha = 1$, by Cauchy's integral formula for derivative we get

$$f'(1) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-1)^2}\,dz$$

i.e. $$\oint_C \frac{f(z)}{(z-1)^2}\,dz = 2\pi i.\, f'(1) = -\frac{2\pi i}{e}.$$

Example 4.11. Evaluate, using Cauchy's integral formula

(i) $\int_C \frac{e^z}{(z+1)^2}\,dz$ where C is the circle $|z-1| = 3$.

(ii) $\int_C \frac{\log z}{(z-1)^3}\,dz$ where C is the circle $|z-1| = \frac{1}{2}$.

Solution : (i) The centre of the circle $C : |z-1| = 3$ is $z = 1$ and its radius is 3. Clearly, $\frac{e^z}{(z+1)^2}$ is not analytic at $z = -1$. However $z = -1$ lies within the circle C.

$\therefore$ By Cauchy's integral formula for the derivative, we have

$$f'(-1) = \frac{1}{2\pi i}\int_C \frac{e^z}{(z+1)^2}dz \text{, where } f(z) = e^z$$

i.e. $$e^{-1} = \frac{1}{2\pi i}\int_C \frac{e^z}{(z+1)^2}dz$$

i.e. $$\int_C \frac{e^z}{(z+1)^2}dz = 2\pi i e^{-1}.$$

(ii) Here, the centre of the circle $C : |z - 1| = \frac{1}{2}$ is $z = 1$ and its radius is $\frac{1}{2}$. Clearly, $\frac{\log z}{(z-1)^3}$ is not analytic at $z = 1$. However $z = 1$ lies within the circle C.

$\therefore$ By Cauchy's integral formula for the double derivative, we have

$$f''(1) = \frac{2!}{2\pi i}\int_C \frac{\log z}{(z-1)^3}dz \text{ where } f(z) = \log z$$

i.e. $$\left[\frac{d^2}{dz^2}\log z\right]_{z=1} = \frac{1}{\pi i}\int_C \frac{\log z}{(z-1)^3}dz$$

i.e. $$\left[-\frac{1}{z^2}\right]_{z=1} = \frac{1}{\pi i}\int_C \frac{\log z}{(z-1)^3}dz$$

i.e. $$\int_C \frac{\log z}{(z-1)^3}dz = -\pi i.$$

Example 4.12. If C is a closed contour containing the origin inside it, prove that

$$\frac{a^n}{n!} = \frac{1}{2\pi i}\int_C \frac{e^{az}}{z^{n+1}}dz.$$

Solution : By Cauchy's generalised integral formula

We get that

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{n+1}}dz$$

$$\therefore \quad f^{(n)}(0) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{z^{n+1}}dz$$

Let us take $f(z) = e^{az}$ so that $f^{(n)}(z_0) = a^n e^{az}$.

Thus we obtain that

$$f^{(n)}(0) = a^n = \frac{n!}{2\pi i} \int_C \frac{e^{az}}{z^{n+1}} dz$$

or, $$\frac{a^n}{n!} = \frac{1}{2\pi i} \int_C \frac{e^{az}}{z^{n+1}} dz.$$

Exercise - 4

1. Evaluate $\int_i^{2-i} (3xy + iy^2)\, dz$ (a) along the straight line joining $z = i$ and $z = 2-i$, (b) along the curve $x = 2t-2,\ y = 1+t-t^2$.

2. Evaluate $\int_C \bar{z}^2 dz + z^2 d\bar{z}$ along the curve C defined by $z^2 + 2z\bar{z} + \bar{z}^2 = (2-2i)z + (2+2i)\bar{z}$ from the point $z = 1$ to $z = 2 + 2i$.

3. Evaluate $\int_C \frac{dz}{z-2}$ around (a) the circle $|z - z| = 4$, (b) the circle $|z - 1| = 5$.

4. Evaluate $\oint_C \frac{z dz}{(9-z^2)(z+i)}$, where C is the circle $|z| = 2$.

5. Evaluate $\int_C \frac{\cos z}{z^3} dz$, where C is a positively oriented closed curve around the origin.

6. By evaluating $\int_C e^z dz$ around the circle $|z| = 1$,

   Show that $\int_0^{2\pi} e^{\cos\theta} \cos(\theta + \sin\theta) d\theta = 0 = \int_0^{2\pi} e^{\cos\theta} \sin(\theta + \sin\theta) d\theta$.

7. Show directly that $\int_{3+4i}^{4-3i} (6z^2 + 8iz)\, dz$ has the same value along the following paths C joining the points $3 + 4i$ and $4 - 3i$: (a) a straight line, (b) the straight lines form $3 + 4i$ to $4 + 4i$ and then from $4 + 4i$ to $4 - 3i$, (c) the circle $|z| = 5$. Determine this value.

8. Use the definition of an integral to prove that along any arbitrary path joining points a and b.

   (i) $\int_a^b dz = b - a$, (ii) $\int_a^b k dz = k(b-a)$.

9. Find the value of (a) $\oint\limits_C \frac{\sin^6 z}{z-\pi/6}dz$, (b) $\oint\limits_C \frac{\sin^6 z}{(z-\pi/6)^3}dz$ if C is the circle $|z| = 1$.

10. Evaluate $\oint\limits_C \frac{e^{3z}}{z-\pi i}dz$ if C is (a) the circle $|z-1| = 4$, (b) the ellipse $|z-2|+|z+2| = 6$.

11. Evaluate $\oint\limits_{|z|=1} \frac{dz}{z+2}$ and deduce that $\int_0^{\pi} \frac{1+2\cos\theta}{5+4\cos\theta}\,d\theta = 0$.

12. Evaluate the integral $\int_0^{1+i} z^2 dz$.

13. Use Cauchy's integral formula to calculate :

    $\int\limits_C \frac{e^z}{(z^2+\pi^2)^2}dz$, where C is $|z| = 4$.

14. Evaluate the integral $\int_0^{1+i} (x - y + ix^2)dz$

    (a) along the straight line from $z = 0$ to $z = 1 + i$

    (b) along the real axis from $z = 0$ to $z = 1$ and then along a line parallel to the imaginary axis from $z = 1$ to $z = 1 + i$.

Unit : 5 ❑ Infinite Series : Taylor's and Laurent's Series

5.1 Sequence of Functions :

Let $f_1(z), f_2(z), \ldots, f_n(z), \ldots$, denoted briefly by $\{f_n(z)\}$, be a sequence of single-valued functions of z defined in some region of the z-plane. We call $f(z)$ the limit of $f_n(z)$ as $n \to \infty$, and write $\lim_{n\to\infty} f_n(z) = f(z)$, if given any positive number ε we can find a number N (depending in general on both ε and z) such that

$$|f_n(z) - f(z)| < \varepsilon \ \forall \ n > N.$$

In such case we say that the sequence converges or is convergent to $f(z)$.

If a seqence converges for all values of z in a region R, we call R the region of convergence of the sequence. A sequence which is not convergent at some point z is called divergent at z.

5.2 Series of Functions :

From the sequence of functions $\{f_n(z)\}$ let us form a new sequence $\{S_n(z)\}$ defined by

$$S_1(z) = f_1(z)$$
$$S_2(z) = f_1(z) + f_2(z)$$
$$\vdots \qquad \vdots$$
$$S_n(z) = f_1(z) + f_2(z) + \cdots + f_n(z)$$

where $S_n(z)$, called the nth partial sum, is the sum of the first n terms of the sequence $\{f_n(z)\}$. $\sum_{n=1}^{\infty} f_n(z)$ is called an infinite series. If $\lim_{n\to\infty} S_n(z) = S(z)$, the series is called convergent and $S(z)$ is its sum; otherwise the series is called divergent.

If a series converges for all values of z in a region R, we call R the region of convergence of the series.

5.3 Absolute Convergence :

A series $\sum_{n=1}^{\infty} f_n(z)$ is called absolutely convergent if the series of absolute values, i.e. $\sum_{n=1}^{\infty} |f_n(z)|$, converges.

If $\sum_{n=1}^{\infty} f_n(z)$ converges but $\sum_{n=1}^{\infty} |f_n(z)|$ does not converge, we call $\sum_{n=1}^{\infty} f_n(z)$ conditionally convergent.

5.4 Uniform Convergence of Sequence and Series :

In the definition of limit of a sequence of functions it was pointed out that the number N depends in general on ε and the particular value of z. It may happen, however, that we can find a number N such that $|f_n(z) - f(z)| < \varepsilon \ \forall \ n > N$, where the same number N holds for all z in a region R (i.e. N depends only on ε and not on the particular value of z in the region). In such case we say that $f_n(z)$ converges uniformly, or is uniformly convergent to $f(z)$ for all z in R.

Similarly if the sequence of partial sums $\{S_n(z)\}$ converges uniformly to $S(z)$ in a region, we say that the infinite series $\sum_{n=1}^{\infty} f_n(z)$ converges uniformly, or is uniformly convergent, to $S(z)$ in the region.

If we call $R_n(z) = f_{n+1}(z) + f_{n+2}(z) + \ldots = S(z) - S_n(z)$ the remainder of the infinite series $\sum_{n=1}^{\infty} f_n(z)$ after n terms, we can say that the series is uniformly convergent to $S(z)$ in R if given $\varepsilon > 0$ we can find a number N such that for all z in R,

$$|R_n(z)| = |S(z) - S_n(z)| < \varepsilon \ \forall \ n > N.$$

5.5 Power Series :

A series of the form

$$a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots = \sum_{n=0}^{\infty} a_n(z - z_0)^n \qquad \ldots(5.1)$$

is called a power series in $z - z_0$.

Clearly the power series (5.1) converges for $z = z_0$, and this may indeed be the only point for which it converges. In general, however, the series converges for other points as well. In such case we can show that there exists a positive number R such that (5.1) converges for $|z - z_0| < R$ and diverges for $|z - z_0| > R$, while for $|z - z_0| = R$ it may or may not converge.

Geometrically if Γ is a circle of radius R with centre at $z = z_0$, then the series (5.1) converges at all points inside Γ and diverges at all points outside Γ, while it may or may not converge on Γ. We can consider the special cases $R = 0$ and $R = \infty$ respectively to be the cases where (5.1) converges only at $z = z_0$ or converges for all finite values of z. R is often called the radius of convergence of (5.1) and the corresponding circle is called the circle of convergence.

5.6 Some Important Theorems :

For reference purposes we list here some important theorems involving sequences and series. Many of these will be familiar from their analogs for real variables.

Theorem 5.1

If a sequence has a limit, the limit is unique.

Theorem 5.2

Let $f_n = a_n + ib_n$, $n = 1, 2, 3, \cdots$, where a_n and b_n are real. Then a necessary and sufficient condition that $\{f_n\}$ converge is that $\{a_n\}$ and $\{b_n\}$ converge.

Theorem 5.3

A necessary and sufficient condition that $\{f_n(z)\}$ converges is that given any $\varepsilon > 0$, we can find a number N such that $|f_p - f_q| < \varepsilon \ \forall\ p > N, q > N$.

This result, which has the advantage that the limit itself is not present, is called Cauchy's convergence criterion.

Theorem 5.4

A necessary condition that $\sum f_n(z)$ converge is that $\lim_{n\to\infty} f_n(z) = 0$. However, the condition is not sufficient.

Theorem 5.5

Multiplication of each term of a series by a constant different from zero does not affect the convergence or divergence. Removal (or addition) of a finite number of terms from (or to) a series does not affect the convergence or divergence.

Theorem 5.6

If $\sum_{n=1}^{\infty} |f_n(z)|$ converges, then $\sum_{n=1}^{\infty} f_n(z)$ converges.

Theorem 5.7

The terms of an absolutely convergent series can be rearranged in any order and all such rearranged series converge to the same sum. Also the sum, difference and product of absolutely convergent series is absolutely convergent.

Theorem 5.8 (Comparison Tests)

(a) If $\sum |g_n|$ converges and $|f_n| \le |g_n|$, then $\sum f_n$ converges absolutely.

(b) If $\sum |g_n|$ diverges and $|f_n| \ge |g_n|$, then $\sum |f_n|$ diverges but $\sum f_n$ may or may not converge.

Theorem 5.9 (Ratio Test)

If $\lim_{n\to\infty} \left| \frac{f_{n+1}}{f_n} \right| = L$, then $\sum f_n$ converges (absolutely) if $L < 1$ and diverges if $L > 1$.

If $L = 1$ the test fails.

Theorem 5.10 (Cauchy's Root Test)

If $\lim\limits_{n\to\infty} |f_n|^{1/n} = L$, then $\sum f_n$ converges (absolutely) if $L < 1$ and diverges if $L > 1$. If $L = 1$, the test fails.

Theorem 5.11 (Raabe's Test)

If $\lim\limits_{n\to\infty} n\left(1-\left|\dfrac{f_{n+1}}{f_n}\right|\right) = L$, then $\sum f_n$ converges (absolutely) if $L > 1$ and diverges if $L < 1$. If $L = 1$, the test fails.

Theorem 5.12 (Gauss' Test)

If $\left|\dfrac{f_{n+1}}{f_n}\right| = 1-\dfrac{L}{n}+\dfrac{c_n}{n^2}$ where $|c_n| < M \;\forall\; n > N$, then $\sum f_n$ converges (absolutely) if $L > 1$ and diverges or converges conditionally if $L \le 1$.

Theorem 5.13 (Principle of uniform convergence)

A sequence of functions $\{f_n(z)\}$, defined on a bounded closed domain R, converges uniformly in R iff given any $\varepsilon > 0$, $\exists$ a + *ve* integer N depending only on ε such that $|f_{n+p}(z) - f_n(z)| < \varepsilon \;\forall\; n \ge N$ and $p = +$ *ve* integer, $z \in R$.

Theorem 5.14

A convergent series $f(z) = \sum\limits_{n=1}^{\infty} f_n(z)$ $(z \in R)$ is uniformly convergent on R iff given $\varepsilon > 0$, $\exists$ a positive integer $N = N(\varepsilon)$ such that

$$|S_n(z) - f(z)| < \varepsilon \text{ for } n \ge N \text{ and } \forall\; z \in R.$$

Proof. If the series $\sum\limits_{n=1}^{\infty} f_n(z)$ is uniformly convergent on R, then for given $\varepsilon > 0$, $\exists$ a positive integer $N = N(\varepsilon)$ such that

$$|S_{n+p}(z) - S_n(z)| < \frac{\varepsilon}{2} \text{ for } n \ge N,\; p = 1, 2, 3, \cdots \text{ and for all } z \in R.$$

Now taking limit as $p \to \infty$ we get

$$|f(z) - S_n(z)| < \frac{\varepsilon}{2} < \varepsilon \text{ for } n \ge N \;\&\; \text{for all } z \in R.$$

Thus the condition is satisfied.

Conversely, if for given $\varepsilon > 0$, $\exists$ a positive integer $N = N(\varepsilon)$ such that

$$|S_n(z) - f(z)| < \frac{\varepsilon}{2} \text{ for } n \ge N \;\&\; \forall\; z \in R.$$

Then we get for $n \ge N$, $p = 1, 2, \ldots, n$ & $\forall\; z \in R$

$$|S_{n+p}(z) - S_n(z)| \le |S_{n+p}(z) - f(z)| + |S_n(z) - f(z)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Hence $\sum_{n=1}^{\infty} f_n(z)$ is uniformly convergent on R.

Theorem 5.15 (Weierstrass M-test)

If in a region R, $|f_n(z)| \le M_n$, $n = 1, 2, 3, \cdots$, where M_n are positive constants such that $\sum_{n=1}^{\infty} M_n$ converges, then $\sum f_n(z)$ is uniformly (and absolutely) convergent in R.

Proof. Since $\sum_{n=1}^{\infty} M_n$ convervges, for given $\varepsilon > 0$, $\exists$ a positive integer $N_1 = N_1(\varepsilon)$ such that

$$|M_{n+p} + M_{n+p-1} + \cdots + M_{n+1}| < \varepsilon \text{ for } n \ge N_1,\ p = 1, 2, 3, \cdots$$

Then for $n \ge N_1$, $p = 1, 2, 3, \ldots$, and $\forall\ z \in R$ we get

$$\begin{aligned} |S_{n+p}(z) - S_n(z)| &= |f_{n+p}(z) + f_{n+p-1}(z) + \cdots + f_{n+1}(z)| \\ &\le |f_{n+p}(z)| + |f_{n+p-1}(z)| + \cdots + |f_{n+1}(z)| \\ &\le M_{n+p} + M_{n+p-1} + \cdots + M_{n+1} < \varepsilon. \end{aligned}$$

So $\sum_{n=1}^{\infty} f_n(z)$ is uniformly convergent in R. The absolute convergence follows at once from the comparison test.

Theorem 5.16

The sum of a uniformly convergent series of continuous functions is continuous, i.e. if $f_n(z)$ is continuous in R and $f(z) = \sum_{n=1}^{\infty} f_n(z)$ is uniformly convergent in R, then $f(z)$ is continuous in R.

Proof. Let z and z_0 are any two points in R. Then

$$|f(z) - f(z_0)| \le |f(z) - S_n(z)| + |S_n(z) - S_n(z_0)| + |S_n(z) - f(z_0)| \quad \ldots(5.2)$$

Since $\sum_{n=1}^{\infty} f_n(z)$ converges uniformly to $f(z)$ in R, for given $\varepsilon > 0$, $\exists$ a positive integer $N = N(\varepsilon)$ such that

$$|S_n(z) - f(z)| < \frac{\varepsilon}{3} \text{ for } n \ge N \ \&\ \forall\ z \in R.$$

If n_0 is any integer $\ge N$, it follows form (5.2) that

$$|f(z) - f(z_0)| < |S_{n_0}(z) - S_{n_0}(z_0)| + \frac{2\varepsilon}{3} \quad \ldots (5.3)$$

Since $S_{n_0}(z)$, being the sum of a finite number of continuous functions at z_0, is continuous at $z_0 \in R$, we can choose $\delta = \delta(\varepsilon) > 0$ such that

$$|S_{n_0}(z) - S_{n_0}(z_0)| < \frac{\varepsilon}{3} \text{ for } |z - z_0| < \delta.$$

Hence from (5.3) we get for $|z - z_0| < \delta$,

$$|f(z) - f(z_0)| < \varepsilon.$$

Thus $f(z)$ is continuous at z_0. Since z_0 is arbitrary it follows that $f(z)$ is continuous in R.

Theorem 5.17 (Term by Term Integration)

If $\{f_n(z)\}$ are continuous in R, $f(z) = \sum_{n=1}^{\infty} f_n(z)$ is uniformly convergent in R and C is a curve in R, then

$$\int_C f(z)\,dz = \sum_{n=1}^{\infty} \int_C f_n(z)\,dz.$$

Proof. Since each term of $\sum_{n=1}^{\infty} f_n(z)$ is continuous on C, $f(z)$ is also continuous on C. Since $\sum_{n=1}^{\infty} f_n(z)$ converges uniformly to $f(z)$ on C, given $\varepsilon > 0$, $\exists$ a positive interger $N = N(\varepsilon)$ such that

$|S_n(z) - f(z)| < \frac{\varepsilon}{l}$ for $n \geq N$ and $\forall$ z on C, l is the length of C. Then for $n \geq N$, *we* obtain by ML-formula

$$\left| \sum_{k=1}^{\infty} \int_C f_n(z)\,dz - \int_C f(z)\,dz \right| = \left| \int_C \sum_{k=1}^{\infty} f_k(z)\,dz - \int_C f(z)dz \right|$$

$$= \left| \int_C (S_n(z) - f(z))\,dz \right| \leq \frac{\varepsilon}{l} \cdot l = \varepsilon.$$

This shows that $\lim_{n\to\infty} \sum_{k=1}^{n} \int_C f_k(z)\,dz = \int_C f(z)\,dz$

i.e. $\sum_{n=1}^{\infty} \int_C f_n(z)\,dz = \int_C f(z)\,dz.$

This proves the theorem.

Weierstrass theorem on uniform convergence on compact sets

Theorem 5.18 (Statement only)

A series $f(z) = \sum_{n=1}^{\infty} f_n(z)$ which is convergent on a domain G, is uniformly convergent on every compact subset of G iff given any point $z_0 \in G$, $\exists$ a neighbourhood $N(z_0) \subset G$ in which the series is uniformly convergent.

Weierstrass Theorem on Uniformly Convergent Series of Analytic Functions

Theorem 5.19 (Statement only)

If the series $f(z) = \sum_{n=1}^{\infty} f_n(z)$ is uniformly convergent on every compact subset of a domain G and if every term $f_n(z)$ is analytic on G then the sum $f(z)$ of the series is also analytic on G. Moreover, the series can be differentiated term by term any number of times i.e.

$$f^{(k)}(z) = \sum_{n=1}^{\infty} f_n^{(k)}(z),\ k = 1, 2, \cdots \ \&\ \forall\ z \in G$$

and each differentiated series is uniformly convergent on every compact subset of G.

Cauchy-Hadamard Theorem

Theorem 5.20

For a power series $\sum_{n=1}^{\infty} a_n(z-z_0)^n$, let $R = \frac{1}{\Lambda}$ where $\Lambda = \lim_{n\to\infty} \sup |a_n|^{1/n}$ and let Γ be the circle $|z - z_0| = R$ with interior $I(\Gamma)$ and exterior $E(\Gamma)$. Then there are three possibilities :

(i) If $R = 0$, then the series diverges for all $z \neq z_0$.

(ii) If $0 < R < \infty$, then the series converges absolutely for all $z \in I(\Gamma)$ and diverges for all $z \in E(\Gamma)$.

(iii) If $R = \infty$, then the series is absolutely convergent for all z.

Proof. We examine each of three possibilities separately. It is to be noted that in each case the series is absolutely convergent for $z = z_0$.

(i) Let $R = 0$. Then $\Lambda = \infty$. So,

$$\Lambda = \lim_{n\to\infty} \sup |a_n|^{1/n} > \frac{1}{|z-z_0|} \quad \text{for any } z \neq z_0.$$

So for any $z \neq z_0$ we can find an infinite sequence $\{n_k\}$ of positive integers such that

$$\left|a_{n_k}(z-z_0)^{n_k}\right| > 1 \text{ for } k = 1, 2, 3, \cdots.$$

This contradicts the necessary condition $\lim_{n\to\infty} a_n(z-z_0)^n = 0$ for the convergence of the given power series for any $z \neq z_0$. Hence the series diverges for all $z \neq z_0$.

(ii) Let $0 < R < \infty$. Then $0 < \Lambda < \infty$. If $z \in I(\Gamma)$, then we can write $|z - z_0| = \frac{\theta^2}{\Lambda}$ where $0 < \theta < 1$. Then

$$\Lambda < \frac{\Lambda}{\theta} = \frac{\theta}{|z - z_0|}$$

i.e. $\limsup_{n\to\infty} |a_n|^{1/n} < \frac{\theta}{|z-z_0|}$

i.e. $|a_n(z - z_0)^n| < \theta^n$

for all large values of n.

Since the geometric series $\sum_{n=0}^{\infty} \theta^n$, $(0 < \theta < 1)$ is covergent, by comparison test, the given power series converges absolutely.

We now assume that $z \in E(\Gamma)$. Then

$$|z - z_0| > R = \frac{1}{\Lambda}$$

i.e. $\Lambda = \limsup_{n\to\infty} |a_n|^{1/n} > \frac{1}{|z-z_0|}$.

Then proceeing in a similar way as the case $R = 0$ we can say that the power series diverges.

(iii) Let $R = \infty$. Then $\Lambda = 0$. Hence

$$\Lambda < \frac{\theta}{|z-z_0|} \text{ for any } z \neq z_0, 0 < \theta < 1.$$

Then for all large values of n we have

$$|a_n(z - z_0)^n| < \theta^n.$$

Since the geometric series $\sum_{n=0}^{\infty} \theta^n$ is convergent, the given power series converges absolutely for any finite z. This proves the theorem.

Theorem 5.21

Let $\Gamma: |z - z_0| = R$ be the circle of convergence of the power series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$. Then the series is uniformly convergent on every compact subset of $I(\Gamma)$.

Proof. Since any compact subset of $I(\Gamma)$ is contained in some closed disc $|z - z_0| \leq r < R$, if r is sufficiently closed to R, we need only to prove that the series is uniformly convergent on every such closed disc.

Thus given a closed disc $|z - z_0| \leq r < R$, let ζ be a point such that $r < |\zeta - z_0| = \rho < R$.

Since the given power series converges absolutely for $z = \zeta$, the series

$$\sum_{n=0}^{\infty} |a_n|\,|\zeta - z_0|^n = \sum_{n=0}^{\infty} |a_n|\,\rho^n$$

converges. Therefore by Weierstrass M-Test the series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ converges uniformly on the disc $|z - z_0| \le r$ because $|a_n(z - z_0)^n| < |a_n|\rho^n \ \forall\ z$ in the disc.

This proves the theorem.

Remark 5.1.

The power series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ need not converge uniformly on $I(\Gamma)$ itself. In fact, the geometric series $\sum_{n=0}^{\infty} z^n$, which is a power series with radius of convergence 1, is not uniformly convergent on the unit disc $|z| < 1$ but only on every compact subset of the unit disc.

Example 5.1. Prove that $\frac{1}{1^p} + \frac{1}{2^p} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^p}$ converges for any constant $p > 1$.

Solution : We have

$$\frac{1}{1^p} = \frac{1}{1^{p-1}}$$

$$\frac{1}{2^p} + \frac{1}{3^p} \le \frac{1}{2^p} + \frac{1}{2^p} = \frac{1}{2^{p-1}}$$

$$\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p} \le \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p} = \frac{1}{4^{p-1}}$$

etc., where we consider 1, 2, 4, 8, ... terms of the series. It follows that the sum of any finite number of terms of the given series is less than the geometric series

$$\frac{1}{1^{p-1}} + \frac{1}{2^{p-1}} + \frac{1}{4^{p-1}} + \frac{1}{8^{p-1}} + \cdots = \frac{1}{1 - \frac{1}{2^{p-1}}}$$

which converges for $p > 1$. Thus the given series, [illegible] called the p series, converges.

Example 5.2. Find the region of convergence of the series $\sum_{n=1}^{\infty} \frac{(z+2)^{n-1}}{(n+1)^3 4^n}$.

Solution : Let $u_n = \frac{(z+2)^{n-1}}{(n+1)^3 4^n}$. Then $u_{n+1} = \frac{(z+2)^n}{(n+2)^3 4^{n+1}}$.

Hence excluding $z = -2$ for which the given series converges, we have

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(z+2)}{4}\frac{(n+1)^3}{(n+2)^3}\right| = \frac{|z+2|}{4}.$$

Then the series converges (absolutely) for $\frac{|z+2|}{4} < 1.$

i.e. $|z+2| < 4$. The point $z = -2$ is included in $|z+2| < 4$.

If $|z+2| = 4$, the ratio test fails. In this case

$$\left|\frac{(z+2)^{n-1}}{(n+1)^3 4^n}\right| = \frac{1}{4(n+1)^3} \le \frac{1}{n^3}$$

and since $\sum \frac{1}{n^3}$ converges, the given series converges absolutely.

It follows that the given series converges (absolutely) for $|z+2| \le 4$. Geometrically this is the set of all points inside and on the circle of radius 4 with centre at $z = -2$, called the circle of convergence. The radius of convergence is equal to 4.

Example 5.3. Find the region of convergence of the series

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}z^{2n-1}}{(2n-1)!}$$

Solution : If $u_n = \frac{(-1)^{n-1}z^{2n-1}}{(2n-1)!}$, then $u_{n+1} = \frac{(-1)^n z^{2n+1}}{(2n+1)!}$.

Hence excluding $z = 0$ for which the given series converges, we have

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{z^2(2n-1)!}{(2n+1)!}\right| = \lim_{n\to\infty}\frac{|z|^2}{(2n+1)2n} = 0$$

for all finite z. Thus the series converges (absolutely) for all z, and we say that the series converges for $|z| < \infty$. We can equivalently say that the circle of convergence is infinite or that the radius of convergence is infinite.

Example 5.4. Find the radii of convergence of the following power series :

(i) $\sum_n (3+4i)^n z^n$, (ii) $\sum_n \frac{(-1)^n}{n}(z-2i)^n$

Soloution :

(i) Here $a_n = (3+4i)^n$, so that $\lim_{n\to\infty}|a_n|^{1/n} = \lim_{n\to\infty}|3+4i| = 5$.

Hence $R = \frac{1}{5}$.

(ii) Here the centre of the circle of convergence is at $z = 2i$ and the radius of convergence R is given by

$$\frac{1}{R}=\lim_{n\to\infty}|a_n|^{1/n}=\lim_{n\to\infty}\left|\frac{(-1)^n}{n}\right|^{1/n}=\lim_{n\to\infty}\frac{1}{n^{1/n}}=1 \text{ since } \lim_{n\to\infty}n^{1/n}=1.$$

Hence $R = 1$.

Example 5.5: Prove that $1+\frac{a.b}{1.c}z+\frac{a(a+1)b(b+1)}{1.2.c(c+1)}z^2+\cdots$ has unit radius of convergence.

Solution : Here $a_n = \dfrac{a(a+1)(a+2)\cdots(a+n-1)\,b(b+1)\cdots(b+n-1)}{1.2.3\cdots n.c(c+1)\cdots(c+n-1)}$

and $a_{n+1} = \dfrac{a(a+1)\cdots(a+n-1)(a+n)\,b(b+1)\cdots(b+n-1)(b+n)}{1.2.3\cdots n(n+1)c.(c+1)\cdots(c+n-1)(c+n)}$.

$$\therefore\ R = \lim_{n\to\infty}\frac{a_n}{a_{n+1}} = \lim_{n\to\infty}\frac{(n+1)(c+n)}{(a+n)(b+n)}=1$$

Example 5.6. Show that the radius of convergence of the series

$$\frac{1}{2}z+\frac{1.3}{2.5}z^2+\frac{1.3.5}{2.5.8}z^3+\cdots \text{ is } \frac{3}{2}.$$

Solution : Here $a_n = \dfrac{1.3.5\cdots(2n-1)}{2.5.8\cdots(3n-1)}$

and $a_{n+1} = \dfrac{1.3.5\cdots(2n-1)(2n+1)}{2.5.8\cdots(3n-1)(3n+2)}$

Hence $R = \lim_{n\to\infty}\dfrac{a_n}{a_{n+1}} = \lim_{n\to\infty}\dfrac{3n+2}{2n+1}=\dfrac{3}{2}$.

Example 5.7. Find the domains of convergence of the following series :

$$\text{(i) } \sum n^2\left(\frac{z^2+1}{1+i}\right)^n,\quad \text{(ii) } \sum_{n=1}^{\infty}\frac{1.3.5\cdots(2n-1)}{n!}\left(\frac{1-z}{z}\right)^n$$

Solution :

(i) Put $z^2 = \xi$.

Then the series in ξ is $\sum n^2\cdot\dfrac{(\xi+1)^n}{(1+i)^n}$.

Now $a_n = \dfrac{n^2}{(1+i)^n}$ and $a_{n+1} = \dfrac{(n+1)^2}{(1+i)^{n+1}}$

Hence $R = \lim_{n\to\infty}\left|\dfrac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\dfrac{n^2}{(n+1)^2}|1+i|=|1+i|=\sqrt{2}$.

Thus the radius of the circle of convergence for ξ is $\sqrt{2}$ and centre is -1, so that the domain of convergence is given by

$$|\xi+1| < \sqrt{2} \text{ or, } |z^2+1| < \sqrt{2}.$$

(ii) Put $\frac{1}{z} = \xi$, so that the series becomes

$$\sum_{n=1}^{\infty} \frac{1.3.5\cdots(2n-1)}{n!}(\xi-1)^n.$$

Hence $a_n = \dfrac{1.3.5\cdots(2n-1)}{n!}$ and $a_{n+1} = \dfrac{1.3.5\cdots(2n-1)(2n+1)}{(n+1)!}$

Thus $R = \lim_{n\to\infty}\left|\dfrac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\left|\dfrac{n+1}{2n+1}\right| = \dfrac{1}{2}$.

Therefore the domain of convergence is given by

$$|\xi-1| < \frac{1}{2}$$

i.e. $\left|\dfrac{1}{z}-1\right| < \dfrac{1}{2}$ i.e. $|1-z|^2 < \dfrac{1}{4}|z|^2$

i.e. $(1-z)(1-\bar{z}) < \dfrac{1}{4}z\bar{z}$

i.e. $3z\bar{z} - 4(z+\bar{z}) + 4 < 0$

i.e. $z\bar{z} - \dfrac{4}{3}(z+\bar{z}) + \dfrac{4}{3} < 0$

i.e. $z\bar{z} - \dfrac{4}{3}(z+\bar{z}) + \dfrac{16}{9} < \dfrac{4}{9}$

i.e. $\left(z-\dfrac{4}{3}\right)\left(\bar{z}-\dfrac{4}{3}\right) < \dfrac{4}{9}$

i.e. $\left|z-\dfrac{4}{3}\right|^2 < \dfrac{4}{9}$

i.e. $\left|z-\dfrac{4}{3}\right| < \dfrac{2}{3}$.

Example 5.8. Test the uniform convergence in the indicated region

$$\sum_{n=1}^{\infty} \frac{\cos nz}{n^3}, \; |z| \le 1.$$

Solution : If $z = x + iy$, we have

$$\frac{\cos nz}{n^3} = \frac{e^{inz}+e^{-inz}}{2n^3} = \frac{e^{inx-ny}+e^{-inx+ny}}{2n^3}$$

$$= \frac{e^{-ny}(\cos nx + i\sin nx)}{2n^3} + \frac{e^{ny}(\cos nx - i\sin nx)}{2n^3}$$

The series $\sum_{n=1}^{\infty} \frac{e^{-ny}(\cos nx + i\sin nx)}{2n^3}$ and $\sum_{n=1}^{\infty} \frac{e^{ny}(\cos nx - i\sin nx)}{2n^3}$ cannot converge for $y < 0$ and $y > 0$ respectively [since in these cases the nth term does not approach zero]. Hence the series does not converge for all z such that $|z| \leq 1$, and so cannot possibly be uniformly convergent in this region.

The series does converge for $y = 0$, i.e. if z is real. In this case $z = x$ and the series becomes $\sum_{n=1}^{\infty} \frac{\cos nx}{n^3}$. Since $\left|\frac{\cos nx}{n^3}\right| \leq \frac{1}{n^3}$ and $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges, it follows from the weierstrass M-test (with $M_n = \frac{1}{n^3}$) that the given series converges uniformly in any interval on the real axis.

Example 5.9. Prove that both the power series $\sum_{n=0}^{\infty} a_n z^n$ and the corresponding series of derivatives $\sum_{n=0}^{\infty} n a_n z^{n-1}$ have the same radius of convergence.

Solution : From the definition of radius of convergence

$$\frac{1}{R} = \lim_{n\to\infty} |a_n|^{1/n} \quad \text{and} \quad \frac{1}{R'} = \lim_{n\to\infty} |na_n|^{1/n}.$$

In order to prove $R = R'$ we have to establish that $\lim_{n\to\infty} n^{1/n} = 1$.
Now by Cauchy theorem on limits we have

$$\lim_{n\to\infty} a_n^{1/n} = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}.$$

Therefore $\lim_{n\to\infty} n^{1/n} = \lim_{n\to\infty} \frac{n+1}{n} = 1$. Hence $R = R'$.

Example 5.10. For what values of z does the series $\sum_{n=1}^{\infty} \frac{1}{(z^2+1)^n}$ converges and find its sum.

Solution : Clearly $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} \left|\frac{1}{z^2+1}\right| = \frac{1}{|z^2+1|}$.

Hence the series converges if $\frac{1}{|z^2+1|} < 1$ or $|z^2+1| > 1$.

Let $S_n(z)$ be the sum of first n terms of the given series.

Then $S_n(z) = \frac{1}{z^2+1} + \frac{1}{(z^2+1)^2} + \cdots n$ terms

$$= \frac{1}{z^2+1} \cdot \frac{1 - \frac{1}{(z^2+1)^n}}{1 - \frac{1}{z^2+1}} = \frac{1}{z^2}\left[1 - \frac{1}{(z^2+1)^n}\right].$$

If $S(z)$ be the sum of the given series then

$$S(z) = \lim_{n\to\infty} S_n(z) = \lim_{n\to\infty} \frac{1}{z^2}\left[1 - \frac{1}{(z^2+1)^n}\right] = \frac{1}{z^2} \text{ as } |z^2+1| > 1.$$

Example 5.11. For the following series, find a number R such that the series converges for $|z| < R$ and diverges for $|z| > R$.

$$\sum_{n=0}^{\infty} \{\cos in\} z^n$$

Solution : $\frac{1}{R} = \lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} |\cos in|^{1/n}$

$$= \lim_{n\to\infty} \left|\frac{e^{-n}+e^n}{2}\right|^{1/n} \le \lim_{n\to\infty} \left(\frac{|e^{-n}|+|e^n|}{2}\right)^{1/n}$$

$$\le \frac{1}{2} \lim_{n\to\infty} \left(|e^{-n}|^{1/n} + |e^n|^{1/n}\right)$$

$$\le \frac{1}{2}\left(e + \frac{1}{e}\right) = \frac{e^2+1}{2e}.$$

i.e. $R = \frac{2e}{e^2+1}$.

Taylor's Theorem

Theorem 5.22

Let f be analytic in the interior of a circle C with centre α and radius r. Then at each point z interior to C

$$f(z) = \sum_{n=0}^{\infty} a_n (z-\alpha)^n \text{ where } a_n = \frac{f^{(n)}(\alpha)}{n!}.$$

Proof. Let z_0 be an arbitrary but fixed point within C and let $|z_0 - \alpha| = R$. We now choose a positive number ρ such that $R < \rho < r$. Let C_1 denote the circle $|z - \alpha| = \rho$. Then C_1 lies entirely within C and z_0 is an interior point of C_1. Hence by Cauchy's Integral formula we get

$$f(z_0) = \frac{1}{2\pi i}\oint_{C_1} \frac{f(z)}{z-z_0}\,dz$$

$$= \frac{1}{2\pi i}\oint_{C_1} \frac{f(z)}{(z-\alpha)-(z_0-\alpha)}\,dz$$

$$= \frac{1}{2\pi i}\oint_{C_1} \frac{f(z)}{(z-\alpha)(1-t)}\,dz,\ t = \frac{z_0-\alpha}{z-\alpha}$$

$$= \frac{1}{2\pi i}\oint_{C_1} \frac{f(z)}{z-\alpha}\cdot\frac{1-t^n+t^n}{1-t}\,dz$$

$$= \sum_{k=0}^{n-1} \frac{1}{2\pi i}\oint_{C_1} \frac{f(z)}{z-\alpha}t^k dz + \frac{1}{2\pi i}\oint_{C_1} \frac{f(z)}{z-\alpha}\cdot\frac{t^n}{1-t}\,dz$$

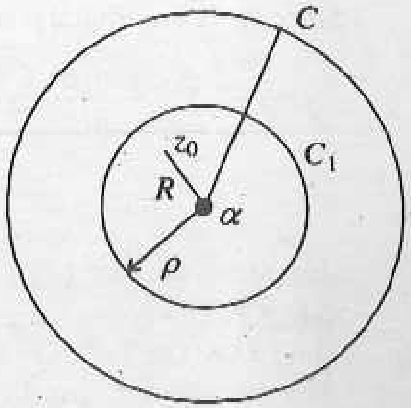


Fig. 5.1

$$= \sum_{k=0}^{n-1} \frac{1}{2\pi i}\oint_{C_1} \frac{f(z)}{(z-\alpha)^{k+1}}(z_0-\alpha)^k\,dz + \frac{1}{2\pi i}\oint_{C_1} \frac{f(z)}{(z-\alpha)^n}\cdot\frac{(z_0-\alpha)^n}{z-z_0}\,dz$$

$$= \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k\,!}(z_0-\alpha)^k + R_n \qquad \text{...(5.4)}$$

where $f^{(k)}(\alpha) = \dfrac{k\,!}{2\pi i}\displaystyle\oint_{C_1} \frac{f(z)}{(z-\alpha)^{k+1}}\,dz$

and $R_n = \dfrac{1}{2\pi i}\displaystyle\oint_{C_1} \frac{f(z)}{(z-\alpha)^n}\cdot\frac{(z_0-\alpha)^n}{z-z_0}\,dz.$

Since f is analytic within and on C_1, f is bounded on C_1. Hence there exist a positive number M such that $|f(z)| \le M \;\forall\; z \in C_1$. Also

$$|z-z_0| = |(z-\alpha)-(z_0-\alpha)| \ge |z-\alpha| - |z_0-\alpha| = \rho - R \;\forall\; z \in C_1.$$

Therefore by ML-formula

$$|R_n| = \frac{1}{2\pi i}\left|\oint_{C_1} \frac{f(z)(z_0-\alpha)^n}{(z-\alpha)^n(z-z_0)}\,dz\right| \le \frac{1}{2\pi}\,\frac{MR^n}{\rho^n(\rho-R)}\cdot 2\pi\rho$$

$$= \frac{M\rho}{\rho-R}\left(\frac{R}{\rho}\right)^n \to 0 \text{ as } n \to \infty.$$

Hence from (5.4) it follows that

$$f(z_0) = \sum_{n=0}^{\infty} \frac{f^{(n)}(\alpha)}{n!}(z_0-\alpha)^n.$$

Since z_0 is an arbitrary point within C, we have for every z within C

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(\alpha)}{n!}(z-\alpha)^n$$

$$= \sum_{n=0}^{\infty} a_n (z-\alpha)^n, \text{ where } a_n = \frac{f^{(n)}(\alpha)}{n!}.$$

This proves the theorem.

Note 5.1. The power series representing f is called the Taylor's series of f about the point $z = \alpha$. The Taylor's series for f shows that if f is analytic in a neighbourhood of α then f can be represented in that neighbourhood by a power series in $z - \alpha$ with a positive radius of convergence.

Note 5.2. Let f be analytic at α. Then there exist a circle $C : |z-\alpha| = r$ such that f is analytic within C. Then for each point z within C

$$f(z) = \sum_{n=0}^{\infty} a_n (z-\alpha)^n.$$

The radius of the greatest circle within which the power series $\sum_{n=0}^{\infty} a_n(z-\alpha)^n$ converges to $f(z)$ is the distance of the point α from the singular point of f which is nearest to α.

Laurent's Theorem

Theorem 5.23

If $f(z)$ is analytic inside and on the boundary of the ring-shaped region R bounded by two concentric circles C_1 and C_2 with centre at α and respective radii r_1 and $r_2 (r_1 > r_2)$, then for all z in R,

$$f(z) = \sum_{n=0}^{\infty} a_n(z-\alpha)^n + \sum_{n=1}^{\infty} b_n(z-\alpha)^{-n}$$

where $a_n = \dfrac{1}{2\pi i} \oint_{C_1} \dfrac{f(z)}{(z-\alpha)^{n+1}} dz, \quad n = 0, 1, 2, \cdots$

and $b_n = \dfrac{1}{2\pi i} \oint_{C_2} \dfrac{f(z)}{(z-\alpha)^{-n+1}} dz, \quad n = 1, 2, 3, \cdots$

Proof. Let z_0 be an arbitrary point of R. Then by Cauchy's integral formula we have

$$f(z_0) = \frac{1}{2\pi i} \oint_{C_1} \frac{f(z)}{z-z_0} dz - \frac{1}{2\pi i} \oint_{C_2} \frac{f(z)}{z-z_0} dz. \qquad \ldots(5.5)$$

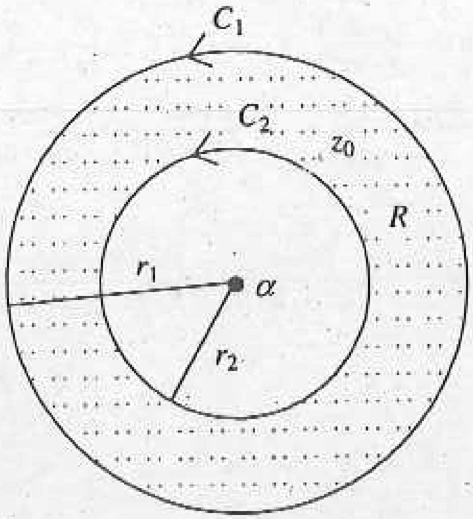


Fig. 5.2

We consider the first integral in (5.5). We have

$$\frac{1}{z-z_0} = \frac{1}{z-\alpha}\left(1-\frac{z_0-\alpha}{z-\alpha}\right)^{-1}$$

$$= \frac{1}{z-\alpha}+\frac{z_0-\alpha}{(z-\alpha)^2}+\cdots+\frac{(z_0-\alpha)^{n-1}}{(z-\alpha)^n}+\left(\frac{z_0-\alpha}{z-\alpha}\right)^n\cdot\frac{1}{z-z_0} \qquad ...(5.6)$$

So that

$$\frac{1}{2\pi i}\oint_{C_1}\frac{f(z)}{z-z_0}dz = \frac{1}{2\pi i}\oint_{C_1}\frac{f(z)}{z-\alpha}dz + \frac{z_0-\alpha}{2\pi i}\oint_{C_1}\frac{f(z)}{(z-\alpha)^2}$$

$$+\cdots+\frac{(z_0-\alpha)^{n-1}}{2\pi i}\oint_{C_1}\frac{f(z)}{(z-\alpha)^n}dz+U_n$$

$$= a_0 + a_1(z_0-\alpha)+\cdots+a_{n-1}(z_0-\alpha)^{n-1}+U_n \qquad ...(5.7)$$

where $a_0 = \frac{1}{2\pi i}\oint_{C_1}\frac{f(z)}{z-\alpha}dz$, $\quad a_1 = \frac{1}{2\pi i}\oint_{C_1}\frac{f(z)}{(z-\alpha)^2}dz, \cdots,$

$$a_{n-1} = \frac{1}{2\pi i}\oint_{C_1}\frac{f(z)}{(z-\alpha)^n}dz \text{ and } U_n = \frac{1}{2\pi i}\oint_{C_1}\left(\frac{z_0-\alpha}{z-\alpha}\right)^n\frac{f(z)}{z-z_0}dz.$$

Now we consider the second integral in (5.5). We have on interchanging z and z_0 in (5.6),

$$-\frac{1}{z-z_0}=\frac{1}{z_0-\alpha}\left(1-\frac{z-\alpha}{z_0-\alpha}\right)^{-1}$$

$$=\frac{1}{z_0-\alpha}+\frac{z-\alpha}{(z_0-\alpha)^2}+\cdots+\frac{(z-\alpha)^{n-1}}{(z_0-\alpha)^n}+\left(\frac{z-\alpha}{z_0-\alpha}\right)^n\frac{1}{z_0-z}$$

So that

$$-\frac{1}{2\pi i}\oint_{C_2}\frac{f(z)}{z-z_0}dz=\frac{1}{2\pi i}\oint_{C_2}\frac{f(z)}{z_0-\alpha}dz+\frac{1}{2\pi i}\oint_{C_2}\frac{(z-\alpha)f(z)}{(z_0-\alpha)^2}dz$$

$$+\cdots+\frac{1}{2\pi i}\oint_{C_2}\frac{(z-\alpha)^{n-1}}{(z_0-\alpha)^n}f(z)dz+V_n$$

$$=\frac{b_1}{z_0-\alpha}+\frac{b_2}{(z_0-\alpha)^2}+\cdots+\frac{b_n}{(z_0-\alpha)^n}+V_n \qquad \ldots(5.8)$$

where $b_1=\frac{1}{2\pi i}\oint_{C_2}f(z)dz$, $b_2=\frac{1}{2\pi i}\oint_{C_2}(z-\alpha)\,f(z)dz$, $\cdots$,

$$b_n=\frac{1}{2\pi i}\oint_{C_2}(z-\alpha)^{n-1}f(z)dz \text{ and } V_n=\frac{1}{2\pi i}\oint_{C_2}\left(\frac{z-\alpha}{z_0-\alpha}\right)^n\frac{f(z)dz}{z_0-z}.$$

From (5.5), (5.7) and (5.8) we have

$$f(z_0) = \{a_0+a_1(z_0-\alpha)+\cdots+a_{n-1}(z_0-\alpha)^{n-1}\}$$

$$+\left\{\frac{b_1}{z_0-\alpha}+\frac{b_2}{(z_0-\alpha)^2}+\cdots+\frac{b_n}{(z_0-\alpha)^n}\right\}+U_n+V_n.$$

The required result follows if we can show that $\lim_{n\to\infty}U_n=0$ and $\lim_{n\to\infty}V_n=0$. Since z is on C_1, we have

$$\left|\frac{z_0-\alpha}{z-\alpha}\right|=r<1$$

where r is a constant. Also we have $|f(z)|<M$ where M is a constant, and

$$|z-z_0| = |(z-\alpha)-(z_0-\alpha)| \geq r_1-|z_0-\alpha|$$

where r_1 is the radius of C_1. Hence using ML-formula we get

$$|U_n| = \frac{1}{2\pi}\left|\oint_{C_1}\left(\frac{z_0-\alpha}{z-\alpha}\right)^n\frac{f(z)}{z-z_0}dz\right| \leq \frac{1}{2\pi}\frac{r^nM}{r_1-|z_0-\alpha|}\cdot 2\pi r_1$$

$$\to 0 \text{ as } n\to\infty \quad [\because r<1].$$

Thus $\lim_{n\to\infty}U_n=0$.

To prove $\lim_{n\to\infty} V_n = 0$, since z is an C_2, we have

$$\left|\frac{z-\alpha}{z_0-\alpha}\right| = k < 1$$

where k is a constant. Also we have $|f(z)| < N$ where N is a constant and

$$|z_0 - z| = |(z_0-\alpha)-(z-\alpha)| \geq |z_0-\alpha| - r_2$$

where r_2 is the radius of C_2. Hence using ML-formula we get

$$|V_n| = \frac{1}{2\pi}\left|\oint_{C_2}\left(\frac{z-\alpha}{z_0-\alpha}\right)^n \frac{f(z)dz}{z_0-z}\right| \leq \frac{1}{2\pi}\frac{k^n N}{|z_0-\alpha|-r_2}\cdot 2\pi r_2$$

$$\to 0 \text{ as } n\to\infty \quad [\because k<1].$$

Thus $\lim_{n\to\infty} V_n = 0$. This completes the proof of the theorem.

Note 5.3. Choosing n to $-n$ in a_n we find that

$$a_{-n} = \frac{1}{2\pi i}\oint_{C_1}\frac{f(z)}{(z-\alpha)^{-n+1}}dz = b_n.$$

Hence, the Laurent's series expansion of f can be written as

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z-\alpha)^n,\ r_2 < |z-\alpha| < r_1$$

where $a_n = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-\alpha)^{n+1}}dz$, $n = 0, \pm 1, \pm 2, \cdots$ and C is any circle with centre α and lying in R.

Note 5.4. If $f(z)$ is analytic in $|z-\alpha| = r_1$, then the function

$$\frac{f(z)}{(z-\alpha)^{-n+1}} = f(z)\,(z-\alpha)^{n-1}$$

is analytic within and on C for $n = 1, 2, 3, \cdots$. So by Cauchy's fundamental theorem

$b_n = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-\alpha)^{-n+1}}dz = 0$ for $n = 1, 2, 3, \cdots$ and the Laurent's series reduces to Taylor's series.

5.7 Uniquencess of analytic function in a region :

Given a function $f(z)$, analytic in a domain D, it is known from Cauchy's integral formula that the values of the function $f(z)$ can be determined at any point of the domain D in terms of the specified values of the function on the boundary Γ of the domain D. In this article it will be observed that if a function took a constant value in the neighbourhod of a point lying in its region of analyticity then the function coincides with that constant value throughout the region.

Theorem 5.24 (Uniqueness Theorem)

Let the functions $f(z)$ and $g(z)$ be analytic in a domain D. If there is a sequence $z_1, z_2, \ldots$ of distinct points in D, converging to ζ belonging to D such that $f(z_k) = g(z_k)$ for all $k = 1, 2, \cdots$, then $f(z)$ coincides identically with $g(z)$ throughout in D.

Proof. Let S denotes the sequence of points $\{z_n\}, n = 1, 2, \cdots$. Then the function $\phi(z) = f(z) - g(z)$ is analytic in D and vanishes on the set of points S. We shall have to show that $\phi(z)$ is identically zero in D.

We first assume D to be a circle with its centre at ζ, the limit point of S.

Since $\phi(z)$ is analytic in D, $\phi(z)$ can be expanded in a Taylor series in some neighbourhood of ζ as

$$\phi(z) = a_0 + a_1(z - \zeta) + a_2(z - \zeta)^2 + \cdots$$

Since $\phi(z_k) = 0$ for $k = 1, 2, \cdots$ we have

$$a_0 + a_1(z_k - \zeta) + a_2(z_k - \zeta)^2 + \cdots = 0. \text{ for } k = 1, 2, \cdots$$

Letting $k \longrightarrow \infty$ we obtain $a_0 = 0$. So

$$a_1 + a_2(z_k - \zeta) + a_3(z_k - \zeta)^2 + \cdots = 0. \quad [\because \; z_k \neq \zeta]$$

Applying the limit successively we find that

$$a_1 = a_2 = \cdots = a_k = \cdots = 0.$$

Hence $\phi(z)$ equals to zero identically.

Now we assume that D is any arbitrary domain. Let C be a circle with centre at ζ lying entirely within D and $\phi(z)$ vanishes identically in C. If $\phi(z)$ is not identically zero in D, there exist a point α in D, lying outside C such that $\phi(\alpha) \neq 0$. Let L be any curve lying in D with initial and final points at $z = \zeta$ and $z = \alpha$ respectively. Let d be the minimum distance from the curve L to the boundary of the domain D. We partition the curve L by the points $\zeta = \zeta_0, \zeta_1, \zeta_2, \ldots, \zeta_{k-1}, \zeta_k = \alpha$ so that

$$\max_j |\zeta_j - \zeta_{j-1}| < d.$$

Now we draw circles $C_0, C_1, C_2, \ldots, C_k$ each with radius d taking centres at $\zeta_0, \zeta_1, \zeta_2, \cdots, \zeta_k$ respectively. Inside the circle C_0 with centre at $\zeta_0 = \zeta$, $\phi(z) \equiv 0$ by hypothesis. Again the circle C_0 contains ζ_1, which is also a limit point of the set on which $\phi(z) = 0$ and hence $\phi(z) \equiv 0$ in the circle C_1. Continuing in this way it can be proved that $\phi(z) \equiv 0$ in C_k with centre at $z = \alpha$. Therefore $\phi(z)$ vanishes throughout in D.

Example 5.12. Expand $f(z) = \sin z$ in a Taylor series about $z = \frac{\pi}{4}$ and determine the region of convergence of this series.

Solution : $f(z) = \sin z$.

$$\therefore\ f'(z) = \cos z,\quad f''(z) = -\sin z,\quad f'''(z) = -\cos z, \cdots$$

$$f\left(\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}},\quad f'\left(\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}},\quad f''\left(\frac{\pi}{4}\right)=-\frac{1}{\sqrt{2}},\quad f'''\left(\frac{\pi}{4}\right)=-\frac{1}{\sqrt{2}} \cdots$$

Hence $f(z) = \sum_{n=0}^{\infty} a_n\left(z-\frac{\pi}{4}\right)^n$, where $a_n = \dfrac{f^{(n)}\left(\frac{\pi}{4}\right)}{n!}$

$$= a_0 + a_1\left(z-\frac{\pi}{4}\right) + a_2\left(z-\frac{\pi}{4}\right)^2 + a_3\left(z-\frac{\pi}{4}\right)^3 + \cdots$$

$$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\left(z-\frac{\pi}{4}\right) - \frac{1}{2\sqrt{2}}\left(z-\frac{\pi}{4}\right)^2 - \frac{1}{6\sqrt{2}}\left(z-\frac{\pi}{4}\right)^3 + \cdots$$

This is the Taylor series expansion of $\sin z$.

Since the singularity of $\sin z$ nearest to $\frac{\pi}{4}$ is at infinity, the above series converges for all finite values of z, i.e. $|z| < \infty$.

Example 5.13. Find Laurent series about the indicated singularity for each of the following functions. Determine the region of convergence of each series.

(i) $\dfrac{e^{2z}}{(z-1)^3}$ at $z = 1$, (ii) $(z-3)\sin\dfrac{1}{z+2}$ at $z = -2$, (iii) $\dfrac{1}{z^2(z-3)^2}$ at $z = 3$.

Solution :

(i) Let $z - 1 = u$. Then

$$\frac{e^{2z}}{(z-1)^3} = \frac{e^2}{u^3}\cdot e^{2u}$$

$$= \frac{e^2}{u^3}\left(1 + 2u + \frac{(2u)^2}{2!} + \frac{(2u)^3}{3!} + \frac{(2u)^4}{4!} + \cdots\right)$$

$$= \frac{e^2}{(z-1)^3} + \frac{2e^2}{(z-1)^2} + \frac{2e^2}{z-1} + \frac{4e^2}{3} + \frac{2e^2}{3}(z-1) + \cdots$$

The series converges for all values of $z \neq 1$.

(ii) Let $z + 2 = u$. Then

$$(z-3)\sin\frac{1}{z+2} = (u-5)\sin\frac{1}{u}$$

$$= (u-5)\left(\frac{1}{u} - \frac{1}{3!u^3} + \frac{1}{5!u^5} - \cdots\right)$$

$$= 1 - \frac{5}{u} - \frac{1}{3!u^2} + \frac{5}{3!u^3} + \frac{1}{5!u^4} - \cdots$$

$$= 1 - \frac{5}{z+2} - \frac{1}{6(z+2)^2} + \frac{5}{6(z+2)^3} + \frac{1}{120(z+2)^4} - \cdots$$

The series converges for all values of $z \neq -2$.

(iii) Let $z - 3 = u$. Then

$$\frac{1}{z^2(z-3)^2} = \frac{1}{u^2(u+3)^2} = \frac{1}{9u^2}\left(1+\frac{u}{3}\right)^{-2}$$

$$= \frac{1}{9u^2}\left[1 - \frac{2u}{3} + \frac{u^2}{3} - \frac{4u^3}{27} + \frac{5u^4}{81} - \cdots\right]$$

$$= \frac{1}{9(z-3)^2} - \frac{2}{27(z-3)} + \frac{1}{27} - \frac{4(z-3)}{243} + \frac{5(z-3)^2}{729} - \cdots$$

The series converges for all values of z such that $0 < |z-3| < 3$.

Example 5.14. Expand $f(z) = e^{z/z-2}$ in a Laurent series about $z = 2$. Determine the region of convergence of the series.

Solution : Let $z - 2 = u$. Then

$$f(z) = e^{\frac{z}{z-2}} = e^{\frac{u+2}{u}}$$

$$= e\left(1 + \frac{2}{u} + \frac{1}{2!}\left(\frac{2}{u}\right)^2 + \frac{1}{3!}\left(\frac{2}{u}\right)^3 + \cdots\right)$$

$$= e\left(1 + \frac{2}{z-2} + \frac{2}{(z-2)^2} + \frac{4}{3(z-2)^3} + \cdots\right)$$

The region of convergence of the series is

$$\left|\frac{2}{u}\right| < \infty \quad \text{i.e. } |u| > 0 \quad \text{i.e. } |z-2| > 0.$$

Example 5.15. Expand $f(z) = \dfrac{1}{(z+1)(z+3)}$ in a Laurent series valid for

(i) $|z| < 1$, (ii) $1 < |z| < 3$. (iii) $|z| > 3$. (iv) $0 < |z+1| < 2$.

Solution : Resolving into partial fraction, we have

$$f(z) = \frac{1}{(z+1)(z+3)} = \frac{1}{2(z+1)} - \frac{1}{2(z+3)}.$$

(i) Let $|z| < 1$. Then

$$f(z) = \frac{1}{2}(1+z)^{-1} - \frac{1}{6}\left(1+\frac{z}{3}\right)^{-1}$$

$$= \frac{1}{2}\left(1 - z + z^2 - z^3 + \cdots\right) - \frac{1}{6}\left(1 - \frac{z}{3} + \frac{z^2}{9} - \frac{z^3}{27} + \cdots\right)$$

$$= \frac{1}{3} - \frac{4}{9}z + \frac{13}{27}z^2 - \frac{40}{81}z^3 + \cdots$$

(ii) Let $1 < |z| < 3$. Then

$$f(z) = \frac{1}{2z}\left(1+\frac{1}{z}\right)^{-1} - \frac{1}{6}\left(1+\frac{z}{3}\right)^{-1}$$

$$= \frac{1}{2z}\left(1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + \cdots\right) - \frac{1}{6}\left(1 - \frac{z}{3} + \frac{z^2}{9} - \frac{z^3}{27} + \cdots\right)$$

$$= -\frac{1}{6}\left(1 - \frac{z}{3} + \frac{z^2}{9} - \frac{z^3}{27} + \cdots\right) + \frac{1}{2}\left(\frac{1}{z} - \frac{1}{z^2} + \frac{1}{z^3} - \cdots\right).$$

(iii) Let $|z| > 3$. Then

$$f(z) = \frac{1}{2z}\left(1+\frac{1}{z}\right)^{-1} - \frac{1}{2z}\left(1+\frac{3}{z}\right)^{-1}$$

$$= \frac{1}{2z}\left(1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + \frac{1}{z^4} - \cdots\right) - \frac{1}{2z}\left(1 - \frac{3}{z} + \frac{9}{z^2} - \frac{27}{z^3} + \frac{81}{z^4} - \cdots\right)$$

$$= \frac{1}{z^2} - \frac{4}{z^3} + \frac{13}{z^4} - \frac{40}{z^5} + \cdots$$

(iv) Let $0 < |z+1| < 2$. Then

$$f(z)=\frac{1}{2(z+1)}-\frac{1}{2(z+1+2)}$$

$$=\frac{1}{2(z+1)}-\frac{1}{4}\left(1+\frac{z+1}{2}\right)^{-1}$$

$$=\frac{1}{2(z+1)}-\frac{1}{4}\left[1-\frac{z+1}{2}+\frac{(z+1)^2}{4}-\frac{(z+1)^3}{8}+\cdots\right].$$

Example 5.16. Expand $f(z)=\dfrac{z+3}{(z^2-z-2)z}$ in powers of z

(i) within the unit circle about the origin

(ii) within the annulus between the concentric circles about the origin having radii 1 and 2 respectively,

(iii) the exterior to the circle with centre as origin and radius 2 i.e., for $|z| > 2$.

Solution : Let $f(z)=\dfrac{z+3}{(z^2-z-2)z}=\dfrac{z+3}{z(z+1)(z-2)}$.

Hence resolving into partial fractions,

$$f(z)=-\frac{3}{2z}+\frac{1}{3(z+1)}+\frac{1}{6(z-2)}.$$

(i) For $0 < |z| < 1$, we have

$$f(z)=-\frac{3}{2z}+\frac{2}{3}(1+z)^{-1}-\frac{5}{12}\left(1-\frac{z}{2}\right)^{-1}$$

$$=-\frac{3}{2z}+\frac{2}{3}[1-z+z^2-z^3+\cdots]-\frac{5}{12}\left[1+\frac{z}{2}+\frac{z^2}{2^2}+\frac{z^3}{2^3}+\cdots\right]$$

$$=-\frac{3}{2z}\sum_{n=0}^{\infty}\left[\frac{2}{3}(-1)^n-\frac{5}{12}\left(\frac{1}{2}\right)^n\right]z^n.$$

(ii) For $1 < |z| < 2$, we have

$$f(z)=-\frac{3}{2z}+\frac{2}{3z}\left(1+\frac{1}{z}\right)^{-1}-\left(1-\frac{z}{2}\right)^{-1}$$

$$=-\frac{3}{2z}+\frac{2}{3z}+\left[1-\frac{1}{z}+\frac{1}{z^2}-\frac{1}{z^3}+\cdots\right]-\frac{5}{12}\left[1+\frac{z}{2}+\frac{z^2}{2^2}+\frac{z^3}{2^3}+\cdots\right]$$

$$=-\frac{3}{2z}+\frac{2}{3z}\sum_{n=0}^{\infty}(-1)^n\frac{1}{z^n}-\frac{5}{12}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n.$$

(iii) For $|z| > 2$, we get

$$f(z) = -\frac{3}{2z} + \frac{2}{3z}\left(1+\frac{1}{z}\right)^{-1} + \frac{5}{6z}\left(1-\frac{2}{z}\right)^{-1}$$

$$= -\frac{3}{2z} + \frac{2}{3z}\left[1-\frac{1}{z}+\frac{1}{z^2}-\frac{1}{z^3}+\cdots\right] + \frac{5}{6z}\left[1+\frac{2}{z}+\frac{2^2}{z^2}+\frac{2^3}{z^3}+\cdots\right]$$

$$= -\frac{3}{2z} + \frac{2}{3z}\sum_{n=0}^{\infty}(-1)^n\frac{1}{z^n} + \frac{5}{6z}\sum_{n=0}^{\infty}\left(\frac{2}{z}\right)^n.$$

Example 5.17. For the function $f(z) = \dfrac{2z^3+1}{z^2+z}$, find (i) a Taylor series valid in the neighbourhood of the point $z = i$,

(ii) a Laurent's series valid within the annulus of which centre is the origin.

Solution :

(i) We have $f(z) = 2(z-1) + \dfrac{1}{z} + \dfrac{1}{z+1}$.

We write $f_1(z) = 2(z-1)$, $f_2(z) = \dfrac{1}{z}$, $f_3(z) = \dfrac{1}{z+1}$.

Taylor's expansion for $f_1(z)$ about $z = i$ is given by

$$f_1(z) = 2(z-1) = f_1(i) + \sum_{n=0}^{\infty}\frac{f_1^{(n)}(i)}{n!}(z-i)^n$$

Now $f_1(i) = 2(i-1)$, $f_1'(i) = 2$ and $f_1^{(n)}(i) = 0$ for $n \geq 2$

Hence $f_1(z) = 2(i-1) + 2(z-i)$.

Again Taylor's expansion for $f_2(z) = \dfrac{1}{z}$ about $z = i$ is given by

$$f_2(z) = \frac{1}{z} = f_2(i) + \sum_{n=1}^{\infty}\frac{f_2^{(n)}(i)}{n!}(z-i)^n$$

Now $f_2(i) = \dfrac{1}{i}$ and $f_2^{(n)}(z) = (-1)^n\dfrac{n!}{z^{n+1}}$

so that $f_2^{(n)}(i) = \dfrac{(-1)^n n!}{i^{n+1}}$

Hence $f_2(z) = \dfrac{1}{i} + \displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{i^{n+1}}(z-i)^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{i^{n+1}}(z-i)^n$.

Similarly, we can show that

$$f_3(z) = \sum_{n=0}^{\infty} (-1)^n \frac{(z-i)^n}{(1+i)^{n+1}}.$$

Thus the Taylor's expansion for $f(z)$ is given by

$$f(z) = 2(i-1) + 2(z-i) + \sum_{n=0}^{\infty} \left[\frac{(-1)^n}{i^{n+1}} + \frac{(-1)^n}{(1+i)^{n+1}}\right](z-i)^n.$$

(ii) For $|z| < 1$, Laurent's series for $f(z)$ is given by

$$f(z) = 2(z-1) + \frac{2}{z} + (1+z)^{-1}$$

$$= 2(z-1) + \frac{1}{z} + (1 - z + z^2 - z^3 + \cdots).$$

Exercise - 5

1. Prove that $\sum_{n=1}^{\infty} \frac{z^n}{n(n+1)}$ converges (absolutely) for $|z| \le 1$.

2. Find the radii of convergence of the following power series.

   (i) $\sum_{n=1}^{\infty} \frac{n!}{n^n} z^n$ (ii) $\sum_{n=1}^{\infty} \frac{z^n}{2^n+1}$ (iii) $\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^{n^2} z^n$

3. Find the region of convergence of the following series :

   (i) $\sum_{n=0}^{\infty} n^2 \left(\frac{z^2+1}{1+i}\right)^n$ (ii) $\sum_{n=0}^{\infty} \frac{(z+i)^n}{(n+1)(n+2)}$ (iii) $\sum_{n=1}^{\infty} n! z^n$

4. For what values of z does the series

   $\sum_{n=0}^{\infty} (-1)^n (z^n + z^{n+1})$ converge and find its sum.

5. Discuss the uniform convergence of the following series :

   (i) $\sum_{n=1}^{\infty} \frac{(z+2)^{n-1}}{(n+1)^3 4^n}$ (ii) $\sum_{n=1}^{\infty} \frac{z^n}{n\sqrt{n+1}}$, $|z| \le 1$.

6. For the following power series, find a number R such that the series converges for $|z| < R$ and diverges for $|z| > R$.

   $$\sum_{n=0}^{\infty} \frac{2^n}{n^2+n} z^{2n}.$$

7. Investigate the (a) absolute and (b) uniform convergence of the series

$$\frac{z}{3}+\frac{z(3-z)}{3^2}+\frac{z(3-z)^2}{3^3}+\frac{z(3-z)^3}{3^4}+\cdots$$

8. Investigate the region of absolute convergence of

$$\sum_{n=1}^{\infty}\frac{(-1)^n n(z-i)^n}{4^n(n^2+1)^{5/2}}.$$

9. Expand each of the following functions in a Taylor series about the indicated points and determine the region of convergence.

(i) $\cos z;\ z=\frac{\pi}{2}$ (ii) $\frac{1}{1+z};\ z=1$ (iii) $\frac{z+3}{(z-1)(z-4)};\ z=2$

(iv) $\frac{z}{e^z+1};\ z=0$ (v) $\frac{e^z}{z(z-1)};\ z=4i$ (vi) $\frac{\sin z}{z^2+4};\ z=0$

10. Expand $f(z)=\frac{z}{(z-1)(2-z)}$ in a Laurent series valid for :

(i) $|z|<1$, (ii) $1<|z|<2$, (iii) $|z|>2$, (iv) $|z-1|>1$, (v) $0<|z-2|<1$.

11. Expand each of the following function in a Laurent series about $z=0$.

(i) $\frac{e^{z^2}}{z^3}$ (ii) $\frac{1-\cos z}{z}$ (iii) $z^2e^{-z^4}$

12. Find an expansion of $f(z)=\frac{z}{z^2+1}$ valid for $|z-3|>2$.

13. Show that if $\tan z$ is expanded into a Laurent series about $z=\frac{\pi}{2}$, (a) the principal part is $-\frac{1}{z-\frac{\pi}{2}}$, (b) the series converges for $0<\left|z-\frac{\pi}{2}\right|<\frac{\pi}{2}$, (c) $z=\frac{\pi}{2}$ is a simple pole.

14. Show that the function $e^{1/z}$ actually takes every value except zero an infinite number of times in the neighbourhood of $z=0$.

Unit-6 ❑ Classification of Singularities

6.1 Introduction :

Consider the functions $\frac{1}{x^2}$, $x\sin\frac{1}{x}$, $\frac{1}{x^2(x^2+2)}$. Then we see that the point $x = 0$ is a singular point for each of these functions in the sense that the function is defined in a deleted neighbourhood of zero. The problem of classifying singularities is not satisfactory for functions defined only on R. On the otherhand the situation is quite different for functions defined on domains in C.

6.2 Definitions

Definition 6.1

A point α is called a regular point or an ordinary point of a function f if f is analytic at α, otherwise α *is called a singular point of f.*

Definition 6.2

Let f be an analytic function with a singular point at $z = \alpha$. If there exist a neighbourhood of α which contains no other singular point of f except α, then α is called an isolated singular point or an isolated singularity of the function f.

Definition 6.3

Suppose that α is an isolated singularity of the function f. Then f can be expanded in a Laurent's series of the form

$$f(z) = \sum_{n=0}^{\infty} a_n (z-\alpha)^n + \sum_{n=1}^{\infty} b_n (z-\alpha)^{-n}$$

in a domain $0 < |z - \alpha| < r$, where r is the distance of α from the nearest singularity of f other than α itself.

The portion of the series involving negative powers of $(z - \alpha)$ i.e. $\sum_{n=1}^{\infty} b_n (z-\alpha)^{-n}$ is called the principal part of f at α, while the series of non-negative powers of $(z - \alpha)$ i.e. $\sum_{n=0}^{\infty} a_n (z-\alpha)^n$ is called the regular part of f at α.

Now we discuss the following three cases separately.

Case 1. Suppose that all the coefficients b_n in the principal part are zero. We then call $z = \alpha$ is a removable singularity of f because we can make f regular in $|z - \alpha| < r$ by suitably defining its value at α.

Case 2. Suppose that the principal part of f at α contains a finite number of terms only. Then f is said to have a pole at $z = \alpha$. If b_m $(m \geq 1)$ is the last non-vanishing coefficient in the principal part then

$$f(z)=\sum_{n=0}^{\infty} a_n(z-\alpha)^n+\frac{b_1}{z-\alpha}+\frac{b_2}{(z-\alpha)^2}+\ldots+\frac{b_m}{(z-\alpha)^m},\quad 0<|z-\alpha|<r,$$

and the pole is said to be of order m. If $m = 1$, then we call the pole is a simple pole and if $m = 2$, then we call the poles as double pole.

Case 3. Suppose that the principal part of f at $z = \alpha$ contains infinitely many non-zero terms. The point $z = \alpha$ is then called an essential singularity of f. In this case

$f(z)=\sum_{n=0}^{\infty} a_n(z-\alpha)^n+\sum_{n=1}^{\infty} b_n(z-\alpha)^{-n}, 0<|z-\alpha|<r$, where the last series does not terminate but is convergent.

An alternate definition of Removable singularity, Pole and Essential Singularity.

Removable Singularity

The singular point z_0 is called a removable singularity of $f(z)$ if $\lim\limits_{z\to z_0} f(z)$ exists.

Pole

The singular point z_0 is called a pole of $f(z)$ of order or multiplicity n if $\lim\limits_{z\to z_0}(z-z_0)^n f(z)=A\neq 0$. If $n = 1$, z_0 is called a simple pole.

Essential Singularity

The singular point z_0 is called an essential singularity if there exists no finite value of n such that $\lim\limits_{z\to z_0}(z-z_0)^n f(z)=A\neq 0$

Example 6.1. Let $f(z)=\dfrac{\sin z}{z}$ if $z \neq 0$

$= 0$ if $z = 0$.

The function is analytic everywhere except at $z = 0$. The Laurent's expansion about $z = 0$ has the form

$$f(z) = \frac{\sin z}{z}$$

$$= \frac{1}{z}\left[z-\frac{z^3}{3!}+\frac{z^5}{5!}-\ldots\right]$$

$$= 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \ldots$$

Since no negative power of z appears, the point $z = 0$ is a removable singularity of f. If we define

$$f(z) = 1, \text{ at } z = 0,$$

the modified function becomes analytic at $z = 0$.

Example 6.2. The function $f(z) = \dfrac{z^2 - 2z + 5}{z - 2}$

$$= 2 + (z-2) + \frac{5}{z-2}, (z \neq 2)$$

has a simple pole at $z = 2$

Example 6.3. The function $f(z) = e^{\frac{1}{z}}$

$$= 1 + \frac{1}{z} + \frac{1}{2!}\frac{1}{z^2} + \cdots$$

has an essential singularity at $z = 0$.

Note 6.1. By definition a pole is an isolated singularity. If a singularity is non-isolated then also we call it an essential singularity.

Theorem 6.1. The function f has a pole of order m at α if and only if f can be expressed in the form

$$f(z) = \frac{\Psi(z)}{(z - \alpha)^m}$$

in some neighbourhood of α, where ψ is analytic at α and $\psi(\alpha) \neq 0$.

Proof. Let α be a pole of f of order m. Then in some neighbourhood of α, f has Laurent's expansion of the form

$$f(z) = \sum_{n=0}^{\infty} a_n (z-\alpha)^n + \sum_{n=1}^{m} b_n (z-\alpha)^{-n}.$$

Putting $\Phi(z) = \sum_{n=0}^{\infty} a_n (z-\alpha)^n$ we get

$$f(z) = \Phi(z) + \frac{b_1}{z-\alpha} + \frac{b_2}{(z-\alpha)^2} + \ldots + \frac{b_m}{(z-\alpha)^m}$$

$$= \frac{(z-\alpha)^m \Phi(z) + b_1(z-\alpha)^{m-1} + \ldots + b_m}{(z-\alpha)^m} = \frac{\Psi(z)}{(z-\alpha)^m},$$

where $\Psi(z) = (z-\alpha)^m \Phi(z) + b_1(z-\alpha)^{m-1} + \ldots + b_m$ is analytic $z = \alpha$ and $\Psi(\alpha) = b_m \neq 0$.

Next we suppose that in some neighbourhood of α,

$$f(z) = \frac{\Psi(z)}{(z-\alpha)^m}$$

where ψ is analytic at $z = \alpha$ and $\psi(\alpha) \neq 0$. Then we can expand ψ in the Taylor's series around α to get

$$\Psi(z) = \sum_{n=0}^{\infty} a_n (z-\alpha)^n$$

$$= a_0 + a_1(z-\alpha) + a_2(z-\alpha)^2 + \ldots + a_{m-1}(z-\alpha)^{m-1} + \sum_{n=m}^{\infty} a_n (z-\alpha)^n,$$

where $a_0 = \Psi(\alpha) \neq 0$. Therefore

$$f(z) = \frac{\Psi(z)}{(z-\alpha)^m} = \frac{a_0}{(z-\alpha)^m} + \frac{a_1}{(z-\alpha)^{m-1}} + \ldots + \frac{a_{m-1}}{z-\alpha} + \sum_{n=m}^{\infty} a_n (z-\alpha)^{n-m}$$

which is the Laurent's expansion of f about α. Since $a_0 \neq 0$, clearly α is a pole of f of order m. This completes the proof of the theorem.

Theorem 6.2. Let z_0 be a pole of $f(z)$. Then there exists a neighbourhood of z_0 which contains no other pole of $f(z)$, that is, poles are isolated.

Proof. Recall that if $f(z)$ has a pole of order m at z_0, then there exists a deleted neighbourhood $0 < |z - z_0| < \rho$ of z_0 in which $f(z)$ is analytic and has Laurent's expansion of the form

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} b_n (z-z_0)^{-n}.$$

Thus in the neighbourhood of $0 < |z - z_0| < \rho$, $f(z)$ contains no other pole. Hence poles are isolated.

Definition 6.4. If α is a regular point of an analytic function f and if $f(\alpha) = 0$, then α is called a zero of f.

Definition 6.5. The point $z = \alpha$ is called a zero of f of order m if in some neighbour of α, f can be expanded in a Taylor series of the form

$$f(z)=\sum_{n=m}^{\infty} a_n(z-\alpha)^n, \text{ where } a_m \neq 0.$$

Theorem 6.3. The point α is a zero of f of order m, if and only if f can be expressed in the form $f(z)=(z-\alpha)^m\Psi(z)$, where ψ is analytic at α and $\psi(\alpha) \neq 0$ and this representation is valid in some neighbourhood of α.

Proof. Let α is a zero of f of order m. Then in some neighbourhood of α we can expand f as

$$f(z)=\sum_{n=m}^{\infty} a_n(z-\alpha)^n, \text{ where } a_m \neq 0.$$

Then $f(z)=(z-\alpha)^m \sum_{n=m}^{\infty} a_n(z-\alpha)^{n-m}$

$$= (z-\alpha)^m \sum_{p=0}^{\infty} A_p(z-\alpha)^p, \text{ where } m+n = \text{p and } A_p = a_{p+m}$$

$$= (z-\alpha)^m \Psi(z),$$

where $\Psi(z)=\sum_{p=0}^{\infty} A_p(z-\alpha)^p$ is analytic at α and $\Psi(\alpha)=A_0=a_m \neq 0$.

Next we suppose that in some neighbourhood of α, $f(z)=(z-\alpha)^m\Psi(z)$, where ψ is analytic at α and $\psi(\alpha) \neq 0$. Then we expand ψ in a Taylor series around α to obtain

$$\Psi(z)=\sum_{n=0}^{\infty} a_n(z-\alpha)^n,$$

where $a_0=\Psi(\alpha)\neq 0$. So in some neighbourhood of α, we get

$$f(z)=(z-\alpha)^m\sum_{n=0}^{\infty} a_n(z-\alpha)^n = \sum_{n=0}^{\infty} a_n(z-\alpha)^{n+m}$$

$$= \sum_{p=m}^{\infty} B_p(z-\alpha)^p, \text{ where } n+m=p \text{ and } B_p = a_{p-m}. \text{ Since } B_m = a_0 \neq 0,$$

$z = \alpha$ is a zero of f of order m. This proves the theorem.

Theorem 6.4. The zeros of an analytic function are isolated points.

Proof. Let α be a zero of f of order m. Then $f(z) = (z - \alpha)^m \phi(z)$ in some neighbourhood of α, ϕ is analytic at α and $\phi(\alpha) \neq 0$. Let $\varepsilon = \frac{1}{2}|\phi(\alpha)| > 0$. Since ϕ is continuous at α there exist a positive number δ such that

$$|\phi(z) - \phi(\alpha)| < \varepsilon = \frac{1}{2}|\phi(\alpha)| \text{ for } |z - \alpha| < \delta.$$

Therefore, $|\phi(\alpha)| - |\phi(z)| \leq |\phi(z) - \phi(\alpha)| < \frac{1}{2}|\phi(\alpha)|$ for $|z - \alpha| < \delta$.

i.e., $\frac{1}{2}|\phi(\alpha)| < |\phi(z)|$ for $|z - \alpha| < \delta$.

So, $\phi(z) \neq 0$ in $|z - \alpha| < \delta$. Hence f cannot vanish in $0 < |z - \alpha| < \delta$. Thus α is an isolated zero of f. Therefore the zeros of an analytic function are isolated points.

Relation between zeros and poles

Theorem 6.5. A point α is a pole of order m of a function f if and only if it is a zero of order m of the function $\frac{1}{f}$.

Proof. Let f has a pole of order m at $z = \alpha$. Then in some neighbourhood of α.

$$f(z) = \frac{\Psi(z)}{(z-\alpha)^m}$$

where Ψ is analytic at α and $\Psi(\alpha) \neq 0$.

Therefore, $\frac{1}{f(z)} = (z-\alpha)^m \phi(z)$ where $\phi(z) = \frac{1}{\Psi(z)}$ is analytic at $z = \alpha$ and $\phi(\alpha) \neq 0$. So, α is a zero of $\frac{1}{f}$ of order m.

Conversely let $z = \alpha$ is a zero of $\frac{1}{f}$ of order m. Then in some neighbourhood of α

$$\frac{1}{f(z)} = (z-\alpha)^m g(z), \text{ where } g(z) \text{ is analytic at } z = \alpha \text{ and } g(\alpha) \neq 0.$$

Then $f(z) = \frac{h(z)}{(z-\alpha)^m}$, where $h(z) = \frac{1}{g(z)}$ is analytic at α and $h(\alpha) \neq 0$. Hence α is a pole of f of order m.

Theorem 6.6. If a function f has an essential singularity at α, then $\frac{1}{f}$ has also an essential singularity at α.

Proof. If possible, let α be a regular point of $\frac{1}{f}$ and α be not a zero of $\frac{1}{f}$. Then α is a regular point of f which contradicts the nature of α.

Let α be a regular point of $\frac{1}{f}$ and a zero of $\frac{1}{f}$ of order m. Then α is a pole of order m of f which contradicts the nature of α.

If possible, let α be a pole of order m of $\frac{1}{f}$. Then α is a zero of f of order m contradicting the nature of α. Hence the only possibility that remains for α is to be an essential singularity of $\frac{1}{f}$.

Behaviour of a function near a pole

Theorem 6.7. If α is a pole of the function f, then $|f(z)| \to \infty$ as $z \to \alpha$.

Proof. Let α be a pole of f of order m. *Then* in some neighbourhood of α

$$f(z) = \frac{\phi(z)}{(z-\alpha)^m}$$

where ϕ is analytic at α and $\phi(\alpha) \neq 0$. Since ϕ is continuous at α, for $\varepsilon = \frac{1}{2}|\phi(\alpha)| > 0$, we can find a $\delta > 0$ such that

$$|\phi(z) - \phi(\alpha)| < \varepsilon = \frac{1}{2}|\phi(\alpha)| \text{ for } |z - \alpha| < \delta.$$

Therefore,

$$|\phi(z)| = |\phi(z) - \phi(\alpha) + \phi(\alpha)| \geq |\phi(\alpha)| - |\phi(z) - \phi(\alpha)|$$

$$> |\phi(\alpha)| - \frac{1}{2}|\phi(\alpha)| = \frac{1}{2}|\phi(\alpha)| \text{ for } |z - \alpha| < \delta.$$

Hence for $|z - \alpha| < \delta$, $|f(z)| > \dfrac{\frac{1}{2}|\phi(\alpha)|}{|z-\alpha|^m}$.

Let G be a positive number however large, Then $|f(z)| > G$

$$\text{if } \frac{\frac{1}{2}|\phi(\alpha)|}{|z-\alpha|^m} > G \text{ and } |z - \alpha| < \delta,$$

$$\text{i.e., if } |z-\alpha| < \left(\frac{|\phi(\alpha)|}{2G}\right)^{\frac{1}{m}} \text{ and } |z-\alpha| < \delta,$$

$$\text{i.e. if } |z-\alpha| < \delta_1 \text{ where } \delta_1 = \min\left\{\left(\frac{|\phi(\alpha)|}{2G}\right)^{1/m}, \delta\right\}.$$

This shows that $|f(z)| \to \infty$ as $z \to \alpha$ and the theorem is proved.

Limit points of zeros and poles

Theorem 6.8. The limit point of the zeros of an analytic function is an essential singularity of the function, unless the function is identically zero.

Proof. Let α be a limit point of the zeros of a function f. Then an infinity of zeros of f lies in every deleted neighbourhood of α. If possible, let α be a regular point of f. Then f is continuous at α. So for given $\varepsilon > 0$, $\exists$ a $\delta > 0$ such that $|f(z) - f(\alpha)| < \varepsilon$ for $|z - \alpha| < \delta$.

Since there is an infinity of zeros of f in $0 < |z - \alpha| < \delta$, for all these zeros we have $|f(\alpha)| < \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we get $f(\alpha) = 0$.

Hence α is a zero of f which is impossible unless f is identically zero. Hence α is not a regular point of f and so α must be a singularity of f.

Let α be a pole of f. Then given any positive number G we can find a number $\eta > 0$ such that $|f(z)| > G \quad \forall z$ in $0 < |z - \alpha| < \eta$.

This is impossible as the deleted neighbourhood $0 < |z - \alpha| < \eta$ contains an infinity of zeros of f. Hence α can not be a pole of f. Therefore α is an essential singularity of f unless f is identically zero. This proves the theorem.

Theorem 6.9. The limit point of poles of an analytic function f is a non-isolated essential singularity of f.

Proof. Let α be a limit point of the poles of f. Since every neighbourhood of α contains an infinity of poles of f, α can not be a regular point of f. Hence α is a singularity of f which is non-isolated. Since a pole is an isolated singularity, α can not be a pole of f. Therefore α is a non-isolated essential singularity of f. This proves the theorem.

Riemann's Theorem

Theorem 6.10. If a function f is bounded and analytic throughout a domain $0<|z-\alpha|<\delta$ then either f is analytic at α or else α is a removable singularity of f.

Proof. Under the given hypothesis, f can be represented in the Laurent series in the given domain about α in the form

$$f(z)=\sum_{n=0}^{\infty}a_n(z-\alpha)^n+\sum_{n=1}^{\infty}b_n(z-\alpha)^{-n}$$

Let C denotes the circle $|z-\alpha|=r$ $(<\delta)$. Then putting $z-\alpha=re^{i\theta}$, $0\le\theta\le 2\pi$, we obtain

$$b_n=\frac{1}{2\pi i}\int_C\frac{f(z)}{(z-\alpha)^{-n+1}}dz=\frac{r^n}{2\pi}\int_0^{2\pi}f(\alpha+re^{i\theta})e^{in\theta}d\theta,\quad n=1,2,3,\ldots$$

Since f is bounded there exist a positive number M such that $|f(z)|\le M$ $\forall z$ in the given domain. Hence

$$|b_n|=\frac{r^n}{2\pi}\left|\int_0^{2\pi}f(\alpha+re^{i\theta})\,e^{in\theta}d\theta\right|\le\frac{r^n}{2\pi}.2\pi M=M.r^n \text{ for } n=1,2,3\ldots.$$

Since r can be chosen arbitraryly small,

$b_n=0$ for $n=1,2,3,\ldots.$ Thus Laurent series for f reduces to

$$f(z)=\sum_{n=0}^{\infty}a_n(z-\alpha)^n \text{ in } 0<|z-\alpha|<\delta.$$

This shows that either f is analytic at α or else α is a removable singularity of f. This completes the proof of the theorem.

Theorem of Weierstrass and Casorati

Theorem 6.11. If α is an isolated essential singularity of an analytic function f, then given any positive number r and ε and any finite complex number c, there is a point z in $0<|z-\alpha|<r$ at which $|f(z)-c|<\varepsilon$.

If $c=\infty$, then $\lim_{n\to\infty}f(z_n)=\infty$ for a sequence $\{z_n\}$ tending to α.

Proof. Let $c=\infty$. We note that there is not a single neighbourhood of α in which f is bounded. For, otherwise, the point α, by Riemann Theorem would be a removable singularity. This means that for each positive integer n there is a point z_n in the annulus $0<|z-\alpha|<\frac{1}{n}$ Such that $|f(z_n)|>n$. That is $z_n \to \alpha$ and $f(z_n)\to\infty$ as $n\to\infty$.

Let $c\neq\infty$. If possible let us suppose that the theorem is not true. Then there is a positive number ε_0 and a positive number r_0 such that

$$|f(z)-c|\geq\varepsilon_0 \ \ \forall z \text{ in } 0<|z-\alpha|<r_0 . \qquad \text{..... (6.1)}$$

$$\text{Let } g(z)=\frac{1}{f(z)-c}. \qquad \text{..... (6.2)}$$

From (6.1) and (6.2) we get

$$|g(z)|=\frac{1}{|f(z)-c|}\leq\frac{1}{\varepsilon_0} \text{ in } 0<|z-\alpha|<r_o .$$

Since α is an isolated singular point of *f*, it is an isolated singular point of *g* also. Again *g* is bounded in $0<|z-\alpha|<r_0$ and so by Riemann theorem, $z=\alpha$ is a removable singularity of *g*. Clearly α cannot be an essential singularity of $\frac{1}{g}$ because otherwise α becomes an essential singularity of *g* which is not the case.

From (6.2) we have

$$f(z)=c+\frac{1}{g(z)}$$

and this shows that α is not an essential singularity of *f*, a contradiction. This proves the theorem.

Rational Function

A function *f*(*z*) is called a rational function of *z* if it is of the form

$$f(z)=\frac{\Phi(z)}{\psi(z)}$$

where $\Phi(z)$ and $\Psi(z)$ are polynomials.

Entire function

A function which is analytic everywhere in the finite complex plane (i.e. everywhere except at ∞) is called an entire function or integral function. The functions sin*z*, cos*z*, e^z are entire functions.

An entire function can be represented by a Taylor series which has an infinite radius of convergence. Conversely if a power series has an infinite radius of convergence, it represents an entire function.

Meromorphic Function

A function which is analytic everywhere in the finite complex plane except at a finite number of poles is called a meromorphic function.

Example 6.4. The function $f(z)=\dfrac{z}{(z-2)(z+3)^2}$ which is analytic everywhere in the finite complex plane except at the poles $z = 2$ (simple pole) and $z = -3$ (pole of order 2) is a meromorphic function.

Singularities at infinity

The behaviour of a function $f(z)$ at $z = \infty$ is considered by making the substitution $z=\frac{1}{w}$ and examine the nature of $f\left(\frac{1}{w}\right)$ at $w = 0$.

Clearly $f(z)$ is regular or has a pole or has an essential singularity at $z = \infty$ if $f\left(\frac{1}{w}\right)$ has the same property at $w = 0$.

Example 6.5. The function $f(z) = z^3$ has a pole of order 3 at $z = \infty$. The function $f(z) = e^z$ has an essential singularity at $z = \infty$, since $f\left(\frac{1}{w}\right)=e^{\frac{1}{w}}$ has an essential singularity at $w = 0$.

Theorem 6.12. A function which is analytic everywhere including the point at infinity is constant.

Proof. Since f is analytic for all finite z, by Taylor's theorem

$$f(z)=\sum_{n=0}^{\infty} a_n z^n$$

where the series converges for all finite z. Then

$$f\left(\frac{1}{w}\right)=a_0+\frac{a_1}{w}+\frac{a_2}{w^2}+\ldots$$

Since $f\left(\frac{1}{w}\right)$ is regular at $w = 0$, $a_n = 0$ for $n = 1, 2, \ldots$. Hence $f(z) = a_0 =$ constant. This proves the theorem.

Branch points and Branch Lines

A point $z = z_0$ is called a branch point of the multiple-valued function $f(z)$ if the branches of $f(z)$ are interchanged when z describes a closed path about z_0.

Suppose that we are given the function $w = z^{1/2}$. Suppose further that we allow z to make a complete circuit (counterclockwise) around the origin starting from point A. We have $z = re^{i\theta}$, $w = \sqrt{r}e^{i\theta/2}$, so that at A, $\theta = \theta_1$ and $w = \sqrt{r}e^{i\theta_1/2}$.

After a complete circuit back to A, $\theta = \theta_1 + 2\pi$ and $w = \sqrt{r}e^{\frac{i(\theta_1+2\pi)}{2}} = -\sqrt{r}e^{\frac{i\theta_1}{2}}$. Thus we have not achieved the same value of w with which we started. By making a second complete circuit back to A, i.e., $\theta = \theta_1 + 4\pi$, $w = \sqrt{r}e^{\frac{i(\theta_1+4\pi)}{2}} = \sqrt{r}e^{\frac{i\theta_1}{2}}$ and we then obtain the same value of w with which we started.

We can describe the above by stating that if $0 \le \theta < 2\pi$ we are on one branch of the multiple-valued function $z^{\frac{1}{2}}$, while if $2\pi \le \theta < 4\pi$ we are on the other branch of the function.

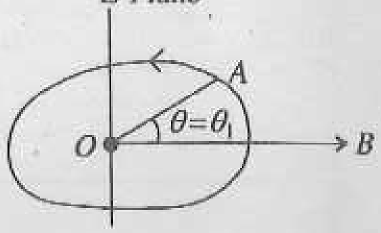


Fig. 6.1

It is clear that each branch of the function is single-valued. In order to keep the function single-valued, we set up an artificial barrier such as OB where B is at infinity (although any other line from O can be used). This barrier is called a branch line or branch cut, and point O is called a branch point. Here $z = 0$ is the only finite branch point since a circuit around any point other that $z = 0$ does not lead to different values.

Example 6.6. The function $f(z) = (z-3)^{1/2}$ has a branch point at $z = 3$.

Example 6.7. The function $f(z) = \log(z^2 + z - 2)$ has branch points where $z^2 + z - 2 = 0$; i.e. at $z = 1$ and $z = -2$.

Example 6.8. Examine the singularity of the function

$$f(z) = \sin\frac{1}{z-1} + \sin\frac{1}{z+1}.$$

Solution. Zeros of $\sin\frac{1}{z-1}$ are given by $z - 1 = \frac{1}{n\pi}$, $n = 0, \pm1, \pm2, \ldots$

The limit point of the zeros is the point $z = 1$. So $z = 1$ is an essential singularity of $\sin\frac{1}{z-1}$.

Again zeros of $\sin\frac{1}{z+1}$ are given by $z + 1 = \frac{1}{n\pi}$, $n = 0, \pm1, \pm2, \ldots$

The limit points of the zeros is the point $z = -1$. So $z = -1$ is an essential singularity of $\sin\frac{1}{z+1}$. Hence $z = 1, -1$ are two essential singularities of $f(z)$.

Example 6.9. Locate and name all the singularities of $f(z)=\frac{z^8+z^4+2}{(z-1)^3(3z+2)^2}$.

Solution. The singularities in the finite z-plane are located at $z = 1$ and $z=-\frac{2}{3}$; $z = 1$ is a pole of order 3 and $z=-\frac{2}{3}$ is pole of order 2.

To determine whether there is a singularity at $z=\infty$, let $z=\frac{1}{w}$. Then

$$f\left(\frac{1}{w}\right)=\frac{1+w^4+2w^8}{w^3(1-w)^3(3+2w)^2}$$

Since $w = 0$ is a pole of order 3 for the function $f\left(\frac{1}{w}\right)$, it follows that $z = \infty$ is a pole of order 3 for the function $f(z)$. Thus the given function has three singularities : a pole of order 3 at $z = 1$, a pole of order 2 at $z=-\frac{2}{3}$ and a pole of order 3 at $z = \infty$.

Example 6.10. Prove that the function $f(z)=e^{-1/z^2}$ has no singularities.

Solution. $f(z)=e^{-1/z^2}=\frac{1}{e^{1/z^2}}$

Poles of $f(z)$ are given by $e^{1/z^2} = 0$. But this is not possible for any value of z, real or complex and hence $f(z)$ has no poles.

Again zeros of $f(z)$ are given by $e^{-1/z^2} = 0$, i.e. $z = 0$ (twice). Thus $z = 0$ is a zero of order two and as such there is no limit of zeros and hence no singularity. Therefore $f(z)$ has no singularity.

Example 6.11. Locate and name all the singularities of $f(z)=\sec\frac{1}{z}$ in the finite z-plane and determine whether they are isolated singularities or not.

Solution. Since $\sec\frac{1}{z} = \dfrac{1}{\cos\frac{1}{z}}$, the singularities occur where $\cos\frac{1}{z} = 0$, i.e.,

$\dfrac{1}{z} = (2n+1)\dfrac{\pi}{2}$ or $z = \dfrac{2}{(2n+1)\pi}$, where $n = 0, \pm1, \pm2, \pm3, \ldots\ldots$ Also, since $f(z)$ is not defined at $z = 0$, it follows that $z = 0$ is also a singularity.

Now by L. Hospital's rule

$$\lim_{z \to \frac{2}{(2n+1)\pi}} \left\{z - \frac{2}{(2n+1)\pi}\right\} f(z) = \lim_{z \to \frac{2}{(2n+1)\pi}} \frac{z - \frac{2}{(2n+1)\pi}}{\cos\frac{1}{z}}$$

$$= \lim_{z \to \frac{2}{(2n+1)\pi}} \frac{1}{\sin\frac{1}{z}\left\{\frac{1}{z^2}\right\}}$$

$$= \frac{\left\{\frac{2}{(2n+1)\pi}\right\}^2}{\sin(2n+1)\pi/2} = \frac{4(-1)^n}{(2n+1)^2\pi^2} \neq 0$$

Thus the singularities $z = \dfrac{2}{(2n+1)\pi}$, $n = 0, \pm1, \pm2, \ldots$ are simple poles. Note that these poles are located on the real axis at $z = \pm\dfrac{2}{\pi}, \pm\dfrac{2}{3\pi}, \pm\dfrac{2}{5\pi}, \ldots$ and that there are infinitely many in a finite interval which includes 0.

Since we can surround each of these by a circle of radius δ which contains no other singularity, it follows that they are isolated singularities. It should be noted that the δ required is smaller the closer the singularity is to the origin.

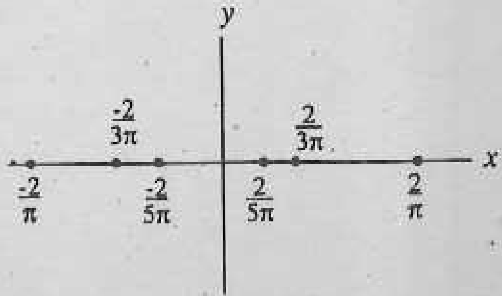


Fig. 6.2

Since we cannot find any positive integer n such that $\lim_{z \to 0}(z-0)^n f(z) = A \neq 0$, it follows that $z = 0$ is an essential singularity. Also since every circle of radius δ with centre at $z = 0$ contains singular points other than $z = 0$, no matter how small we take δ, we see that $z = 0$ is a non-isolated singularity.

Exercises-6

1. For each of the following functions locate and name the singularities in the finite z-plane.

   (i) $\frac{z^2-3z}{z^2+2z+2}$, (ii) $\frac{\sin z}{z(z-1)(z-2)^2}$ (iii) $\frac{\cos z}{(z+i)^3}$ (iv) $\frac{z-2}{z^2}\sin\frac{1}{z-1}$

   (v) $\sin^{-1}\left(\frac{1}{z}\right)$ (vi) $\tan\frac{1}{z}$.

2. Show that $f(z)=\frac{(z-3i)^5}{(z^2-2z+5)^2}$ has double poles at $z = 1 \pm 2i$ and a simple pole at infinity.

3. Show that $f(z)=e^{z^2}$ has an essential singularity at infinity.

4. Locate and name all the singularities of each of the following functions.

   (i) $\frac{z+3}{z^2-1}$ (ii) $\cosh\left(\frac{1}{z}\right)$ (iii) $\frac{1-e^z}{1+e^z}$.

5. Let $f(z)=(z^2+1)^{1/2}$. Show that $z = \pm i$ are branch points of $f(z)$.

6. Show that the point at infinity is a simple zero of

$$f(z)=\frac{z^2-2}{z^3+3z+4}.$$

Unit-7 ❑ Calculus of Residues and Contour Integration

7.1. Introduction :

If f is analytic at a point $z = \alpha$, then there exists a neighbourhood N of α inside which f is analytic. Let C be a positively oriented closed contour contained in N. Then Cauchy theorem tells us that $\int_C f(z)dz = 0$. If, however, f fails to be analytic at finitely many isolated singularities inside C, then the above argument fails; which means each of these singularities contribute a specified value to the value of the integral. This motivates us to generalize the Cauchy theorem to functions which have isolated singularities. This generalization results in the Residue theorem. This result is one of the most important and often used, tools that applied scientists need, from the theory of complex functions.

7.2. Residue at a Finite Point

Let a function f has an isolated singularity at $z = \alpha \neq \infty$, with Laurent's expansion

$$f(z) = \sum_{n=0}^{\infty} a_n (z-\alpha)^n + \frac{b_1}{z-\alpha} + \frac{b_2}{(z-\alpha)^2} + \dots$$

valid in some neighbourhood $0 < |z-\alpha| < r$, then the coefficient of $\frac{1}{z-\alpha}$ i.e. b_1 in the Laurent's expansion is called the residue of f at the singularity α and is usually denoted by Res (f; α).

Thus $b_1 = \frac{1}{2\pi i} \oint_C f(z)dz$

where C is any circle given by $C: |z-\alpha| = \rho < r$. The residue at the isolated singularity α may also be defined as

$$b_1 = \frac{1}{2\pi i} \oint_C f(z)dz$$

where C is any simple closed contour in the domain $0 < |z-\alpha| < r$ which enclose α and no other singularity of f.

In the next theorem we see that how can residue be calculated for a function f having a pole of order m at $z = \alpha$.

Theorem 7.1. If f has a pole of order m at $z = \alpha$, then

$$\text{Res}(f;\alpha) = \frac{1}{\lfloor m-1}\lim_{z\to\alpha}\frac{d^{m-1}}{dz^{m-1}}[(z-\alpha)^m f(z)].$$

Proof. Since α is a pole of f of order m, in some deleted neighbourhood of α we can write

$$f(z) = \phi(z) + \frac{b_1}{z-\alpha} + \frac{b_2}{(z-\alpha)^2} + \ldots + \frac{b_m}{(z-\alpha)^m},$$

where $\phi(z)$ is analytic at $z = \alpha$ and $b_m \neq 0$. So,

$$(z-\alpha)^m f(z) = (z-\alpha)^m \phi(z) + b_1(z-\alpha)^{m-1} + b_2(z-\alpha)^{m-2} + \ldots + b_m.$$

Thus, $\dfrac{d^{m-1}}{dz^{m-1}}[(z-\alpha)^m f(z)] = \dfrac{d^{m-1}}{dz^{m-1}}[(z-\alpha)^m \phi(z)] + b_1 \lfloor m-1$

Since $\displaystyle\lim_{z\to\alpha}\frac{d^{m-1}}{dz^{m-1}}[(z-\alpha)^m \phi(z)] = 0$, we obtain

$$\text{Res}(f;\alpha) = b_1 = \frac{1}{\lfloor m-1}\lim_{z\to\alpha}\frac{d^{m-1}}{dz^{m-1}}[(z-\alpha)^m f(z)].$$

This proves the theorem.

Note : If α is a simple pole of f, then

$$\text{Res}(f;\alpha) = \lim_{z\to\alpha}(z-\alpha)f(z).$$

Theorem 7.2. Suppose ϕ is analytic at α with $\phi(\alpha) \neq 0$ and Ψ has a simple zero at α. Then $\text{Res}\left(\dfrac{\phi(z)}{\Psi(z)};\alpha\right) = \dfrac{\phi(\alpha)}{\Psi'(\alpha)}$.

Proof. Let $f(z) = \dfrac{\phi(z)}{\Psi(z)}$. Since α is a simple pole of f,

$$\text{Res}(f;\alpha) = \lim_{z\to\alpha}(z-\alpha)f(z) = \lim_{z\to\alpha}(z-\alpha)\frac{\phi(z)}{\Psi(z)}$$

$$= \lim_{z\to\alpha}\frac{\phi(z)}{\dfrac{\Psi(z)-\Psi(\alpha)}{z-\alpha}} = \frac{\phi(\alpha)}{\Psi'(\alpha)}.$$

This proves the result.

Example 7.1. Find the residues of $f(z)=\dfrac{z^3+5}{z(z-1)^3}$ at its singularities.

Solution. f has a simple pole at $z = 0$ and a pole of order 3 at $z = 1$. Now.

$$\text{Res}(f;0)=\lim_{z\to 0} z\cdot\frac{z^3+5}{z(z-1)^3} = \lim_{z\to 0}\frac{z^3+5}{(z-1)^3}=-5$$

$$\text{Res}(f;1)=\frac{1}{\underline{|2}}\lim_{z\to 1}\frac{d^2}{dz^2}\left[(z-1)^3\frac{z^3+5}{z(z-1)^3}\right]$$

$$=\frac{1}{2}\lim_{z\to 1}\frac{d^2}{dz^2}\left(z^2+\frac{5}{z}\right)=\frac{1}{2}\lim_{z\to 1}\left(2+\frac{10}{z^3}\right)=6.$$

Example 7.2. Find the residues of $f(z)=\dfrac{1}{(z^3-1)(z+1)^2}$ at its singularities.

Solution. Clearly f has a double pole at $z = -1$ and simple poles at $z = 1$, w, w^2, where w is a cube root of unity.

Now, $$\text{Res}\,(f;-1) = \frac{1}{\underline{|2-1}}\lim_{z\to -1}\frac{d}{dz}[(z+1)^2 f(z)]$$

$$=\lim_{z\to -1}\frac{d}{dz}\left(\frac{1}{z^3-1}\right)=-\frac{3}{4}.$$

$$\text{Res}\,(f;1) = \lim_{z\to 1}(z-1)f(z)$$

$$=\lim_{z\to 1}\frac{1}{(z+1)^2(1+z+z^2)}=\frac{1}{12}.$$

$$\text{Res}(f;w)=\lim_{z\to w}(z-w)f(z)$$

$$=\lim_{z\to w}\frac{1}{(z+1)^2(z-1)(z-w^2)}=\frac{1}{3}.$$

$$\text{Res}\,(f;w^2)=\lim_{z\to w^2}\frac{1}{(z+1)^2(z-1)(z-w)}=\frac{1}{3}.$$

Example 7.3. Find the residue of $f(z) = \cot z$ at $z = 0$.

Solution. $f(z)=\cot z=\dfrac{\cos z}{\sin z}=\dfrac{\phi(z)}{\psi(z)}$, say.

Clearly $\phi(z)$ is analytic at $z = 0$ and $\phi(0) \neq 0$. Again $\Psi(z)$ is analytic at $z = 0$ and $\Psi(0) = 0$, $\Psi'(0) \neq 0$.

Hence $\text{Res}(\cot z; 0) = \left[\dfrac{\phi(z)}{\psi'(z)}\right]_{z=0} = 1.$

7.3. Residue at an essential singularity

In this case one usually has to resort to the Laurent's series expansion if it can be found. For example $z = 0$ is an essential singularity of $f(z)=e^{-1/z}$ and the Laurent's expansion is equal to

$$1-\frac{1}{z}+\frac{1}{2!}\frac{1}{z^2}-\frac{1}{3!}\frac{1}{z^3}+\ldots$$

So, $\text{Res}(e^{-1/z}; 0) = -1.$

Residue at the point at infinity

Let the point $z = \infty$ be an isolated singularity of the analytic function f. Then the residue of f at $z = \infty$ is defined as follows :

$$\text{Res}(f;\infty)=\frac{1}{2\pi i}\oint_C f(z)dz$$

$$=-\frac{1}{2\pi i}\oint_C f(z)dz,$$

where the contour C is an arbitrary closed contour outside of which the function f is analytic and does not have any singularity other than the point at infinity.

Example 7.4. Determine the singularities of $\sec\frac{1}{z}$ and find residues at its pole.

Solution. Let $f(z)=\sec\frac{1}{z}=\dfrac{1}{\cos\frac{1}{z}}$

The poles of $f(z)$ occurs at the zeros of $\cos\frac{1}{z}$.

Now $\cos\frac{1}{z}=0 \Rightarrow z=\dfrac{1}{(2n+1)\pi/2}$, $n = 0, \pm1, \pm2, \ldots$

$z=0$ is a non-isolated essential singularity of $f(z)$ because $z=0$ is a pole of $f(z)$.

Now, $\left[\frac{d}{dz}\left(\cos\frac{1}{z}\right)\right]_{z=\frac{1}{(2n+1)\pi/2}} \neq 0$. Hence $z=\frac{1}{(2n+1)\pi/2}$ is a simple pole of $f(z)$.

$$\therefore \operatorname{Res}\left(f;\frac{1}{(2n+1)\pi/2}\right)=\lim_{z\to\frac{1}{(2n+1)\pi/2}}\left(z-\frac{1}{(2n+1)\pi/2}\right)\frac{1}{\cos\frac{1}{z}}$$

$$=\lim_{z\to\frac{1}{(2n+1)\pi/2}}\frac{1}{\left(-\sin\frac{1}{z}\right)\left(-\frac{1}{z^2}\right)}$$

$$=\lim_{z\to\frac{1}{(2n+1)\pi/2}}\frac{z^2}{\sin\frac{1}{z}}=(-1)^n\frac{4}{(2n+1)^2\pi^2}.$$

Example 7.5. Find the residues of the function $f(z)=\frac{e^{iz}}{1+z^2}$ at its singular points.

Solution. $f(z)$ has simple poles at $z=\pm i$.

Now, $\operatorname{Res}(f;i)=\lim_{z\to i}(z-i)f(z)=\lim_{z\to i}\frac{e^{iz}}{z+i}=\frac{e^{-1}}{2i}$.

$$\operatorname{Res}(f;-i)=\lim_{z\to -i}(z+i)f(z)=\lim_{z\to -i}\frac{e^{iz}}{z-i}=-\frac{e}{2i}.$$

7.5. Residue Theorem

The effectiveness of the residue theorem depends, of course, on how effectively we can evaluate residues at various singularities. However, caution must be exercised to avoid a hasty conclusion based on appearances. Having identified the type of singularities, we have to choose a proper contour. Most often the following theorem will be applied in the next chapter to evaluate different types of integrals.

Cauchy's Residue Theorem

Theorem 7.3. Let f be a single valued analytic function within and on a simple closed contour C, except at a finite number of isolated singular points $z_1, z_2, ..., z_n$ with C. Then

$$\oint_C f(z)dz = 2\pi i \sum_{k=1}^{n} \text{Res}(f; z_k).$$

Proof. Let $C_1, C_2, ..., C_n$ be n circles with centres at $z_1, z_2, ..., z_n$ and radii so small that they lie entirely within C and do not overlap. Then f is analytic in the region bounded by C and the circles $C_1, C_2, ..., C_n$. So by Cauchy's fundamental theorem

$$\oint_C f(z)dz = \sum_{k=1}^{n} \oint_{C_k} f(z)\,dz \quad \text{................. (7.1)}$$

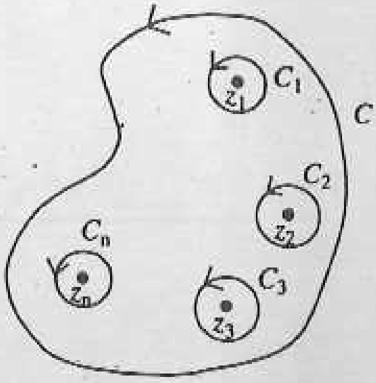


Fig. 7.1

Since $\text{Res}(f; z_k) = \frac{1}{2\pi i} \oint_{C_k} f(z)\,dz$, $k = 1, 2, ..., n$, it follows from (7.1) that

$$\oint_C f(z)\,dz = 2\pi i \sum_{k=1}^{n} \text{Res}(f : z_k).$$

This proves the theorem.

Theorem 7.4. Let the function f be analytic in the extended complex plane with the exception of a finite number of isolated singular points $z_1, z_2, ..., z_{N-1}, z_N$ including the point at infinity (say, $z_N = \infty$). Then

$$\sum_{k=1}^{N} \text{Res}(f; z_k) = 0.$$

Proof. Consider the closed contour C containing all $N-1$ singularities $z_1, z_2, ..., z_{N-1}$ located at a finite distance from the point $z = 0$. So by Cauchy's Residue Theorem

$$\frac{1}{2\pi i} \oint_C f(z)\,dz = \sum_{K=1}^{N-1} \text{Res}(f; z_k). \quad \text{....... (7.2)}$$

Also, $\text{Res}(f; z_N) = \text{Res}(f; \infty) = -\frac{1}{2\pi i} \oint_C f(z)\,dz.$ .... (7.3)

From (7.2) and (7.3), it follows that

$$\sum_{K=1}^{N-1} \mathrm{Res}(f; z_k) = -\mathrm{Res}(f; z_N).$$

Thus, $\sum_{K=1}^{N} \mathrm{Res}(f; z_k) = 0$. This proves the theorem.

Example 7.6. Evaluate $\oint_{|z|=2} \frac{e^z}{z(z-1)^2} dz$.

Solution. Let $f(z) = \frac{e^z}{z(z-1)^2}$.

The curve of integration is a circle with centre at origin and radius 2. Clearly, $f(z)$ has a simple pole at $z = 0$ and a double pole at $z = 1$ within the curve of integration. Hence by Cauchy's Residue Theorem

$$\oint_{|z|=2} \frac{e^z}{z(z-1)^2} dz = 2\pi i[\mathrm{Res}(f;0) + \mathrm{Res}(f;1)].$$

Now, $\mathrm{Res}(f;0) = \lim_{z \to 0} z . \frac{e^z}{z(z-1)^2} = 1$.

$$\mathrm{Res}(f;1) = \frac{1}{(2-1)!} \lim_{z \to 1} \frac{d}{dz}\left[(z-1)^2 \frac{e^z}{z(z-1)^2}\right] = \lim_{z \to 1} \frac{ze^z - e^z}{z^2} = 0.$$

So, $\oint_{|z|=2} \frac{e^z}{z(z-1)^2} dz = 2\pi i$.

Example 7.7. Evaluate $\oint_{|z|=2} \frac{z dz}{z^4 - 1}$.

Solution. Let $f(z) = \frac{z}{z^4 - 1}$.

The curve of integration is a circle with centre at origin and redius 2. Clearly, $f(z)$ has simple poles at $z = \pm 1$ and $z = \pm i$, all lies within the curve of integration $|z| = 2$. Hence by Cauchy's Residue Theorem

$$\oint_{|z|=2} \frac{z}{z^4 - 1} dz = 2\pi i[\mathrm{Res}(f;1) + \mathrm{Res}(f;-1) + \mathrm{Res}(f;i) + \mathrm{Res}(f;-i)].$$

Now, $\text{Res}(f;1) = \lim_{z\to 1}(z-1)\dfrac{z}{z^4-1} = \dfrac{1}{4}$.

$$\text{Res}(f;-1) = \lim_{z\to -1}(z+1)\frac{z}{z^4-1} = \frac{1}{4}.$$

$$\text{Res}(f;i) = \lim_{z\to i}(z-i)\frac{z}{z^4-1} = -\frac{1}{4}.$$

$$\text{Res}(f;-i) = \lim_{z\to -i}(z+i)\frac{z}{z^4-1} = -\frac{1}{4}.$$

So, $\oint_{|z|=2} \dfrac{z}{z^4-1}\,dz = 0$.

Example 7.8 Evaluate the residues of $\dfrac{z^3}{(z-1)(z-2)(z-3)}$ at $z = 1, 2, 3$ and infinity and show that their sum is zero.

Solution. $f(z) = \dfrac{z^3}{(z-1)(z-2)(z-3)}$ has simple poles at $z = 1$, 2 and 3.

The residue at $z = 1$ is

$$\lim_{z\to 1}(z-1)f(z) = \lim_{z\to 1}(z-1)\frac{z^3}{(z-1)(z-2)(z-3)} = \lim_{z\to 1}\frac{z^3}{(z-2)(z-3)} = \frac{1}{2}.$$

Similarly the residues of $f(z)$ at $z = 2$ and $z = 3$ are -8 and $\dfrac{27}{2}$ respectively.

To find the residue of $f(z)$ at the point at infinity, we expand $f(z)$ in the neighbourhood of $z = \infty$ as follows :

$$f(z) = \left(1-\frac{1}{z}\right)^{-1}\left(1-\frac{2}{z}\right)^{-1}\left(1-\frac{3}{z}\right)^{-1} = 1+\frac{6}{z} + \text{higher powers of } \frac{1}{z}.$$

Hence the residue of $f(z)$ at the point at infinity is -6. Therefore the sum of residues of $f(z)$ at $z = 1, 2, 3$ and at the point at infinity is $\dfrac{1}{2} - 8 + \dfrac{27}{2} - 6 = 0$.

Deduction. Obtain Cauchy's Integral Formula from Cauchy's Residue Theorem.

Solution. Let the function $f(z)$ be analytic in a domain D with boundary Γ.

Let $f(z)$ be continuous on Γ and α be any point within Γ. Let $g(z)=\dfrac{f(z)}{z-\alpha}$.

Then $g(z)$ has a simple pole at $z=\alpha$ in D and

$$\text{Res}(g;\alpha)=\lim_{z\to\alpha}(z-\alpha)g(z)=f(\alpha).$$

Hence applying Cauchy's Residue Theorem on $g(z)$ we get

$$\oint_{\Gamma} g(z)\,dz=2\pi i.f(\alpha).$$

Thus, $f(\alpha)=\dfrac{1}{2\pi i}\displaystyle\oint_{\Gamma}\dfrac{f(z)}{z-\alpha}dz$, which is the Cauchy's Integral Formula.

The Argument Theorem.

Here we shall establish that the residue theorem can be applied to find the difference of number of zeros and poles of a meromorphic function.

Theorem 7.5. Let $f(z)$ be analytic inside and on a simple closed contour C except for a finite number of poles within C and let $f(z)\neq 0$ anywhere on C. Then

$$\frac{1}{2\pi i}\oint_{C}\frac{f'(z)}{f(z)}dz=N-P$$

where N and P are respectively the number of zeros and poles of $f(z)$ within C, a zero or pole of order m being counted m times.

Proof. Let $\alpha_1, \alpha_2, \ldots, \alpha_j$ and $\beta_1, \beta_2, \ldots \beta_k$ be the respective poles and zeros of $f(z)$ within C and suppose their multiplicities are $p_1, p_2, \ldots p_j$ and $n_1, n_2, \ldots, n_k$. We enclose each pole and zeros by non-overlapping circles $C_1, C_2, \text{---}\ C_j$ and $\Gamma_1, \Gamma_2, \ldots, \Gamma_k$. This can always be done since the poles and zeros are isolated.

Since $f(z)$ has a pole of order p_l at $z=\alpha_l$, we have

$$f(z)=\frac{\phi(z)}{(z-\alpha_l)^{p_l}},\ l=1,2,\ldots,j, \quad \ldots\ldots\ldots\ldots\ldots (7.4)$$

where $\phi(z)$ is analytic and different from zero inside and on C_l. Then taking logarithms in (7.4) and differentiating, we find

$$\frac{f'(z)}{f(z)}=\frac{\phi'(z)}{\phi(z)}-\frac{p_l}{z-\alpha_l},\quad l=1,2,\ldots,j.$$

Thus $\frac{1}{2\pi i}\oint_{C_l}\frac{f'(z)}{f(z)}dz=\frac{1}{2\pi i}\oint_{C_l}\frac{\phi'(z)}{\phi(z)}dz-\frac{p_l}{2\pi i}\oint_{C_l}\frac{dz}{z-\alpha_l}=-p_l$, $l=1, 2, \ldots, j$. .....(7.5)

Again, since $f(z)$ has a zero of order n_r at $z=\beta_r$, we have

$$f(z)=(z-\beta)^{n_r}\Psi(z),\ r=1, 2, \ldots, K,$$

where $\Psi(z)$ is analytic and different from zero inside and on Γ_r. Then by logarithmic differentiation, we get

$$\frac{f'(z)}{f(z)}=\frac{n_r}{z-\beta_r}+\frac{\Psi'(z)}{\Psi(z)},\ r=1, 2, \ldots K.$$

Then $\frac{1}{2\pi i}\oint_{\Gamma_r}\frac{f'(z)}{f(z)}dz=\frac{1}{2\pi i}\oint_{\Gamma_r}\frac{\Psi'(z)}{\Psi(z)}dz+\frac{n_r}{2\pi i}\oint_{\Gamma_r}\frac{dz}{z-\beta_r}=n_r$, $r=1, 2, \ldots, K$ ... (7.6)

Then using (7.5) and (7.6) we obtain

$$\frac{1}{2\pi i}\oint_C\frac{f'(z)}{f(z)}dz=\sum_{r=1}^{K}\frac{1}{2\pi i}\oint_{\Gamma_r}\frac{f'(z)}{f(z)}dz+\sum_{l=1}^{j}\frac{1}{2\pi i}\oint_{C_l}\frac{f'(z)}{f(z)}dz$$

$$=\sum_{r=1}^{K}n_r-\sum_{l=1}^{j}p_l=N-P.$$

This proves the theorem.

Remark 7.1. We note that

$$\frac{1}{2\pi i}\int_C\frac{f'(z)}{f(z)}dz=\frac{1}{2\pi i}[\log f(z)]_C$$

$$=\frac{1}{2\pi i}[\log|f(z)|+i\arg f(z)]_C$$

$$=\frac{1}{2\pi}[\arg f(z)]_C$$

since $\log|f(z)|$ returns to its original value as we go once around C.

Remark 7.2. If $f(z)$ is analytic within and on C and $f(z)\neq 0$ on C, then the number of zeros of $f(z)$ within C is given by

$$N=\frac{1}{2\pi}[\log f(z)]_C.$$

Rouche's Theorem

Theorem 7.6. If f and g are analytic within and on a simple closed contour C and if $|g(z)| < |f(z)|$ for any z on C, then $f(z)$ and $f(z) + g(z)$ have the same number of zeros within C.

Proof. We first show that none of f and $f + g$ can vanish on C.

Suppose that $f(\alpha) = 0$ for some α on C. Then $|g(\alpha)| < |f(\alpha)| = 0$,

a contradiction. We now suppose that $f(\alpha) + g(\alpha) = 0$ for some α on C. Then

$$f(\alpha) = -g(\alpha) \quad \text{i.e. } |f(\alpha)| = |g(\alpha)|,$$

a contradiction.

Let N_1 and N_2 be the number of zeros of f and $f + g$ respectively within C. Then by Argument Principle

$$N_1 = \frac{1}{2\pi}[\arg f(z)]_C, \quad N_2 = \frac{1}{2\pi}[\arg(f(z)+g(z))]_C.$$

$$\text{Now, } N_2 = \frac{1}{2\pi}\left[\arg\left\{f(z)\left(1+\frac{g(z)}{f(z)}\right)\right\}\right]_C$$

$$= \frac{1}{2\pi}[\arg f(z)]_C + \frac{1}{2\pi}\left[\arg\left(1+\frac{g(z)}{f(z)}\right)\right]_C$$

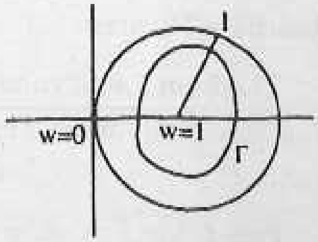
$$= N_1 + \frac{1}{2\pi}\left[\arg\left(1+\frac{g(z)}{f(z)}\right)\right]_C. \qquad \text{............ (7.7)}$$

Let $w(z) = 1+\dfrac{g(z)}{f(z)}$ $\forall$ z on C. Then $|w(z)-1| = \left|\dfrac{g(z)}{f(z)}\right| < 1$ $\forall$ z on C.

This shows that as z describes the closed curve C, the corresponding point w describes the curve Γ lying entirely within the circle $|w(z) - 1| = 1$. Hence the origin ($w = 0$) lies outside the curve Γ. So,

$$\arg w(z) = \arg\left(1 + \frac{g(z)}{f(z)}\right)$$

returns to its original value as z describes the curve C.



. Fig. 7.2

Thus $\left[\arg\left(1 + \frac{g(z)}{f(z)}\right)\right]_C = 0$.

So from (7.7) we have $N_1 = N_2$. This proves the theorem.

Example 7.9. Show that if $|a| > e$, the equation $az^n - e^z = 0$ has n roots inside the circle $|z| = 1$.

Solution. Let $f(z) = az^n$ & $g(z) = -e^z$.

On $|z| = 1$, $|f(z)| = |az^n| = |a||z|^n = |a|$

and $|g(z)| = |-e^z| = |e^z| \le e^{|z|} = e$.

So, on $|z| = 1$, $|g(z)| \le e < |a| = |f(z)|$.

So by Rouche's theorem $f(z)$ and $f(z) + g(z)$ have the same number of zeros within $|z| = 1$. Since $f(z)$ has n zeros within $|z| = 1$, $f(z) + g(z) = az^n - e^z = 0$ has n roots within $|z| = 1$.

Example 7.10. How many zeros of $F(z) = 3z^9 - 8z^6 + 2z^5 + z^3 + 1$ lie in the annulus $1 < |z| < 2$?

Solution. Let N_1 & N_2 be the number of zeros of $F(z)$ inside $|z| = 1$ and $|z| = 2$ respectively. To find N_1, we take

$$f(z) = -8z^6 \text{ and } g(z) = 3z^9 + 2z^5 + z^3 + 1.$$

On $|z| = 1$, $|f(z)| = 8$ and $|g(z)| \le 7$.

Thus on $|z| = 1$, $|g(z)| < |f(z)|$. So by Rouche's theorem we have $f(z)$ and $f(z) + g(z)$ has the same number of zeros within $|z| = 1$. Hence $N_1 = 6$.

To find N_2, we consider

$$f(z) = 3z^9 \text{ and } g(z) = -8z^6 + 2z^5 + z^3 + 1.$$

On $|z| = 2$, $|f(z)| = 1536$ and $|g(z)| \le 585$.

So on $|z| = 2$, $|g(z)| \le |f(z)|$. So by Rouche's theorem $f(z)$ and $f(z) + g(z)$ have the same number of zeros within $|z| = 2$. Hence $N_2 = 9$.

Therefore $F(z) = 0$ has 3 roots in the annulus $1 < |z| < 2$.

Example 7.11. If $K > 1$, then show that the equation $z^n e^{K-z} = 1$ has n roots within $|z| = 1$.

Solution. $z^n e^{K-z} = 0$

i.e. $z^n e^K - e^z$:

Let $f(z) = z^n e^K$ and $g(z) = -e^z$.

On $|z| = 1$, $|f(z)| = |z^n e^K| = |e^K| > e$ $(\because K > 1)$

and $|g(z)| = |-e^z| \le e^{|z|} = e$.

Hence on $|z| = 1$, $|g(z)| \le e < |f(z)|$.

So, by Rouche's theorem $f(z)$ and $f(z) + g(z)$ have the same number of zeros within $|z| = 1$.

Since $f(z) = z^n e^K$ has n zeros within $|z| = 1$, so $z^n e^K - e^z = 0$ has n zeros within $|z| = 1$. Thus the equation $z^n e^{K-z} = 1$ has n roots within $|z| = 1$.

Example 7.12. Prove that all the roots of $z^7 - 5z^3 + 12 = 0$ lie between the circles $|z| = 1$ and $|z| = 2$.

Solution. Let C_1, C_2 denote respectively the circles $|z| = 1$ and $|z| = 2$. We write $f(z) = 12$ and $g(z) = z^7 - 5z^3$. Then on C_1, $f(z)$ and $g(z)$ are analytic and as $|z| = 1$ on C_1 we have

$$\left|\frac{g(z)}{f(z)}\right| = \left|\frac{z^7 - 5z^3}{12}\right| \le \frac{|z|^7 + |-5z^3|}{12} = \frac{|z|^7 + 5|z|^3}{12} = \frac{1+5}{12} = \frac{1}{2}.$$

Thus $\left|\frac{g(z)}{f(z)}\right| < 1$ i.e. $|g(z)| < |f(z)|$ on C_1.

Hence by Rouche's theorem, $f(z) + g(z) = z^7 - 5z^3 + 12$ has the same number of zeros inside C_1 as $f(z) = 12$. But $f(z) = 12$ has no zeros inside C_2. It follows that $z^7 - 5z^3 + 12$ has no zeros inside C_1.

We next consider the circle C_2. We write $F(z) = z^7$ and $\phi(z) = 12 - 5z^3$. Evidently $F(z)$ and $\phi(z)$ are analytic within and on C_2. Further we have on C_2,

$$\left|\frac{\phi(z)}{f(z)}\right| = \frac{|12 - 5z^3|}{|z|^7} \le \frac{12 + 5|z|^3}{|z|^7} = \frac{12 + 5.2^3}{2^7} = \frac{52}{128} < 1, \text{ as } |z| = 2 \text{ on } C_2.$$

Thus $|\phi(z)|<|F(z)|$ on C_2. Hence by Rouche's Theorem $F(z) + \phi(z) = z^7 - 5z^3 + 12$ has the same number of zeros as $F(z)=z^7$ inside C_2. But all the seven zeros of z^7 lie inside $|z| = 2$ since they are all located at the origin. It follows that all seven zeros of z^7-5z^3+12 lie inside $|z|=2$.

Thus we have shown that the equation $z^7 - 5z^3 + 12 = 0$ has no roots inside $|z| = 1$ but has all the seven roots inside $|z| = 2$. It follows that all the roots of this equation lie between the circles $|z| = 1$ and $|z| = 2$ as required.

Example 7.13. Use Rouche's theorem to show that the equation $z^5 + 15z + 1 = 0$ has one root in the disc $|z|<\frac{3}{2}$ and four roots in the annulus $\frac{3}{2}<|z|<2$.

Solution. Let $f(z)=z^5$ and $g(z)=15z+1$. On the circle $|z|=2$, we have

$|f(z)|=|z|^5=2^5=32$ and $|g(z)|=|15z+1|\le 15|z|+1=15\times 2+1=31$.

Thus $|g(z)|<|f(z)|$ on the circle $|z|=2$.

If follows by Rouche's theorem that the function $f(z)+g(z)=z^5+15z+1$ has as many zeros in $|z|<2$ as the function $f(z)=z^5$. Since the latter function has a zero of order 5 at $z = 0$, it follows that all the five roots of $z^5 + 15z + 1 = 0$ must lie in the disc $|z| < 2$.

On the other hand, since for $|z|=\frac{3}{2}$,

$$|z^5+1|\le |z|^5+1=\frac{243}{32}+1<\frac{45}{2}=|15z|,$$

the function $z^5 + 15z + 1$ has as many zeros in $|z|<\frac{3}{2}$ as the function $15z$, that is, it has exactly one zero there. Consequently four of the zeros of $z^5 + 15z + 1$ must lie in the ring $\frac{3}{2}<|z|<2$ as required.

Fundamental Theorem of Classical Algebra

Theorem 7.7. Every polynomial of degree n has n zeros in the complex plane.

Proof. Let $\phi(z)=a_0+a_1z+\ldots+a_nz^n$, $(a_n\ne 0)$ be a polynomial of degree n.

Let $f(z)=a_nz^n$ and $g(z)=a_0+a_1z+\ldots+a_{n-1}z^{n-1}$. Then $\phi(z)=f(z)+g(z)$

Let C denote the circle $|z| = R$, $R > 1$. Then we see that f has n zeros within C, all the zeros being at the origin.

Now on C,

$$|f(z)| = |a_n||z|^n = |a_n| R^n$$

and $$|g(z)| = |a_0 + a_1 z + ... + a_{n-1} z^{n-1}|$$

$$\le |a_0| + |a_1| R + ... + |a_{n-1}| R^{n-1}$$

$$< R^{n-1}(|a_0| + |a_1| + ... + |a_{n-1}|), \quad (\because R > 1).$$

So, on C, $|g(z)| < |f(z)|$ if $R^{n-1}(|a_0| + |a_1| + ... + |a_{n-1}|) < |a_n| R^n$

i.e. if $$R > \frac{|a_0| + |a_1| + ... + |a_{n-1}|}{|a_n|}. \qquad ...... (7.8)$$

Hence by Rouche's theorem $f(z)$ and $f(z) + g(z) = \phi(z)$ have the same number of zeros within a circle having centre at origin and radius R satisfying (7.8). Thus $\phi(z)$ has n zeros within such a circle and as such $\phi(z)$ has also n zeros in the entire complex plane.

This proves the theorem.

Maximum Modulus Theorem

Theorem 7.8. If $f(z)$ is analytic inside and on a simple closed curve C and is not identically equal to a constant, then the maximum value of $|f(z)|$ occurs on C.

Proof. Since $f(z)$ is analytic and hence continuous inside and on C, it follows that $|f(z)|$ does have a maximum value M for at least one value of z inside or on C. Suppose that this maximum value is attained at an interior point a, i.e. $|f(a)| = M$. Let C_1 be a circle inside C with centre at a. If we exclude $f(z)$ from being a constant inside C_1, then there must be a point inside C_1, say b, such that $|f(b)| < M$, i.e. $|f(b)| = M - \varepsilon, \varepsilon > 0$.

Now by the continuity of $|f(z)|$ at b, we see that for any $\varepsilon > 0, \exists$ a $\delta > 0$ such that

$$||f(z)| - |f(b)|| < \tfrac{1}{2}\varepsilon$$

whenever $|z - b| < \delta$ ..... (7.9)

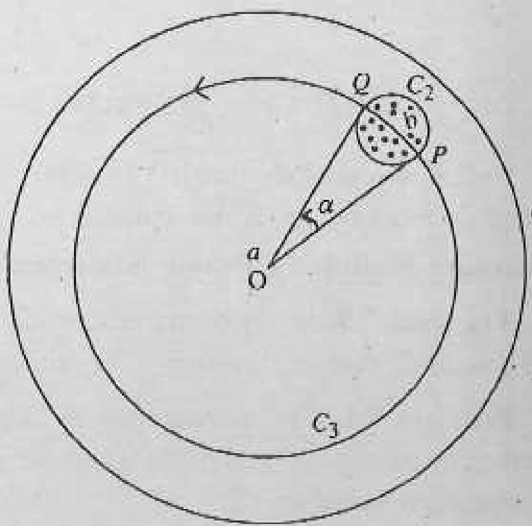


Fig. 7.3

i.e. $|f(z)| < |f(b)| + \frac{1}{2}\varepsilon = M - \frac{1}{2}\varepsilon,$ ..... (7.10)

for all points interior to a circle C_2 with centre at b and radius δ, as shown shaded in the figure.

We construct a circle C_3 with centre at a passing through b. On part of this circle [namely that part PQ included in C_2] we have from (7.10), $|f(z)| < M - \frac{1}{2}\varepsilon$. On the remaining part of the circle we have $|f(z)| \le M$.

We measure θ counterclockwise from OP and let $\angle POQ = \alpha$ and $r = |b - a|$.

Then $f(a) = \frac{1}{2\pi}\int_0^{\alpha} f(a + re^{i\theta})d\theta + \frac{1}{2\pi}\int_{\alpha}^{2\pi} f(a + re^{i\theta})d\theta$

i.e. $|f(a)| \le \frac{1}{2\pi}\int_0^{\alpha} |f(a + re^{i\theta})|\,d\theta + \frac{1}{2\pi}\int_{\alpha}^{2\pi} |f(a + re^{i\theta})|\,d\theta$

$$\le \frac{1}{2\pi}\int_0^{\alpha} (M - \frac{1}{2}\varepsilon)d\theta + \frac{1}{2\pi}\int_{\alpha}^{2\pi} M d\theta$$

$$\le \frac{\alpha}{2\pi}\left(M - \frac{1}{2}\varepsilon\right) + \frac{M}{2\pi}(2\pi - \alpha)$$

$$= M - \frac{\alpha\varepsilon}{4\pi}$$

i.e. $|f(a)| = M \le M - \frac{\alpha\varepsilon}{4\pi}$, a contradiction.

Hence we conclude that $|f(z)|$ cannot attain its maximum at any interior point of C and so must attain its maximum on C.

Minimum Modulus Theorem (statement only)

Theorem 7.9. If $f(z)$ is analytic inside and on a simple closed curve C and $f(z) \neq 0$ inside C, then $|f(z)|$ assumes its minimum value on C.

Problem 7.1. Give an example to show that if $f(z)$ is analytic inside and on a simple closed curve C and $f(z) = 0$ at some point inside C, then $|f(z)|$ need not assume its minimum value on C.

Solution. Let $f(z) = z$ for $|z| \le 1$, so that C is a circle with centre at the origin and radius one. We have $f(z) = 0$ at $z = 0$. If $z = re^{i\theta}$, then $|f(z)| = r$ and it is clear that the minimum value of $|f(z)|$ does not occur on C but occurs inside C where $r = 0$, i.e. at $z = 0$.

7.7. Contour Integration

A variety of real definite integrals can be evaluated with the help of Cauchy's residue theorem. Here we illustrate the methods together with a suitable function f and a suitable closed contour C; the choice, nevertheless, depends on the problem.

7.8. Integration around the unit circle

An integral of the form

$$\int_0^{2\pi} f(\sin\theta, \cos\theta)d\theta \qquad \text{....... (7.11)}$$

where the integrand is a rational function of $\sin\theta$ and $\cos\theta$, can be evaluated by writing $z = e^{i\theta}$.

Since $\sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}$ and $\cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2}$, the integral (7.11) takes the form

$$\int_C f(z)dz$$

where f is a rational function of z and C is the unit circle $|z| = 1$.

Example 7.14. Show that $\displaystyle\int_0^{2\pi} \frac{\cos 2\theta}{1 - 2a\cos\theta + a^2} d\theta = \frac{2\pi a^2}{1 - a^2}, (a^2 < 1)$.

Solution. Let $I = \displaystyle\int_0^{2\pi} \frac{\cos 2\theta}{1 - 2a\cos\theta + a^2} d\theta$

Let $z = e^{i\theta}$. Then

$$\sin\theta = \frac{1}{2i}\left(z - \frac{1}{z}\right), \cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right) \text{ and } dz = ie^{i\theta}d\theta.$$

Hence $\cos 2\theta = \dfrac{e^{i2\theta} + e^{-i2\theta}}{2} = \dfrac{1}{2}\left(z^2 + \dfrac{1}{z^2}\right)$.

Therefore, $I = \int\limits_{|z|=1} \dfrac{\frac{1}{2}\left(z^2+\frac{1}{z^2}\right)}{1-a\left(z+\frac{1}{z}\right)+a^2}.\dfrac{dz}{iz}$

$$= \frac{1}{2i}\int\limits_{|z|=1} \frac{z^4+1}{z^2(z-a)(1-az)}dz$$

$$= \frac{1}{2i}\int\limits_{|z|=1} f(z)dz\text{, where } f(z)=\frac{z^4+1}{z^2(z-a)(1-az)}.$$

Now, f has simple poles at $z=a,\frac{1}{a}$ and a double pole at $z=0$, of which the poles at $z=0$ and $z=a$ lie within the circle $|z|=1$.

$$\text{Res}(f;a)=\lim_{z\to a}(z-a)f(z)=\lim_{z\to a}\frac{z^4+1}{z^2(1-az)}=\frac{a^4+1}{a^2(1-a^2)}.$$

$$\text{Res}(f;0)=\lim_{z\to 0}\frac{d}{dz}\left[z^2 f(z)\right]=\lim_{z\to 0}\frac{d}{dz}\left[\frac{z^4+1}{(z-a)(1-az)}\right]=-\frac{1+a^2}{a^2}.$$

Therefore by Cauchy's Residue Theorem we obtain

$$I=2\pi i.\frac{1}{2i}\left[\frac{a^4+1}{a^2(1-a^2)}-\frac{1+a^2}{a^2}\right]=\frac{2\pi a^2}{1-a^2}.$$

7.9. Integral around a semicircle

To evaluate $\int\limits_{-\infty}^{\infty} f(x)dx$, we consider $\int\limits_{C} f(z)dz$ where C is the contour consisting of the semicircle $C_R : |z| = R(\text{Im}(z) \geq 0)$ together with the diameter that encloses it. Supposing that f has no singular points on the real axis, we have by the Residue Theorem

$$\int\limits_{C_R} f(z)dz+\int\limits_{-R}^{R} f(x)dx=2\pi i\sum_{a}\text{Res}(f;a).$$

Finally making $R\to\infty$, we find the value of $\int\limits_{-\infty}^{\infty} f(x)dx$, provided $\int\limits_{C_R} f(z)dz\to 0$ as $R\to\infty$.

Example 7.15. Evaluate $\int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)(x^2+4)} dx$

Solution. We consider

$$\int_C \frac{z^2}{(z^2+1)(z^2+4)} dz = \int_C f(z)dz, \text{ say,} \qquad \text{..... (7.12)}$$

where C is the contour consisting of the semicircle C_R of radius R together with the part of the real axis from $-R$ to R.

The function $f(z)$ has simple poles at $z = \pm i$, $z = \pm 2i$ of which $z = i$ and $z = 2i$ lie inside C. Now,

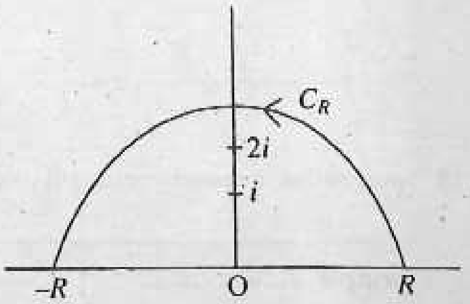


Fig. 7.4

$$\text{Res}(f;i) = \lim_{z\to i}(z-i)f(z)$$

$$= \lim_{z\to i} \frac{z^2}{(z+i)(z^2+4)} = -\frac{1}{6i}.$$

$$\text{Res}(f;2i) = \lim_{z\to 2i}(z-2i)f(z)$$

$$= \lim_{z\to 2i} \frac{z^2}{(z^2+1)(z+2i)} = \frac{1}{3i}.$$

Thus $2\pi i[\text{Res}(f;i) + \text{Res}(f;2i)] = \frac{\pi}{3}$. .... (7.13)

From (7.12) and (7.13), by Cauchy's Residue Theorem

$$\int_{C_R} f(z)dz + \int_{-R}^{R} f(x)dx = \frac{\pi}{3}. \qquad \text{.......(7.14)}$$

On C_R, $|f(z)| = \frac{|z|^2}{|1+z^2||4+z^2|} \le \frac{|z|^2}{(|z|^2-1)(|z|^2-4)}$

$$= \frac{R^2}{(R^2-1)(R^2-4)} = \frac{1}{R^2} \cdot \frac{1}{\left(1-\frac{1}{R^2}\right)\left(1-\frac{4}{R^2}\right)}$$

So, by ML formula,

$$\left|\int_{C_R} f(z)dz\right| \le \frac{1}{R^2}\frac{1}{\left(1-\frac{1}{R^2}\right)\left(1-\frac{4}{R^2}\right)}.\pi R \to 0 \text{ as } R \to \infty.$$

i.e. $\lim_{R\to\infty} \int_{C_R} f(z)dz = 0$.

So, form (7.14) we get making $R \to \infty$

$$\int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)(x^2+4)}dx = \frac{\pi}{3}.$$

7.10. Integration around rectangular contour

Example 7.16. Evaluate $\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x}dx, 0<a<1.$

Solution. Consider $\int_C \frac{e^{az}}{1+e^z}dz = \int_C f(z)dz,$

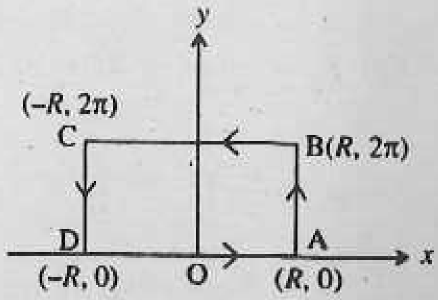


Fig. 7.5

where C is the rectangle ABCD with vertices at $(R,0)$, $(R,2\pi)$, $(-R,2\pi)$, $(-R,0)$, R being positive.

$f(z)$ has simple poles given by

$$e^z + 1 = 0 \text{ i.e. } z = (2n+1)\pi i,\ n = 0, \pm 1, \pm 2, \ldots$$

The only pole inside the rectangle ABCD is $z = \pi i$.

Then by Cauchy's residue theorem

$$\int_C f(z)dz = 2\pi i \operatorname{Res}(f;\pi i). \qquad \ldots\ldots (7.15)$$

Now, $\operatorname{Res}(f;\pi i) = \lim_{z\to\pi i} \frac{e^{az}}{\frac{d}{dz}(1+e^z)} = \frac{e^{a\pi i}}{e^{\pi i}} = -e^{a\pi i}$

So from (7.15) we have

$$\int_C f(z)dz = -2\pi i e^{a\pi i}$$

i.e. $$\int_{AB} f(z)dz + \int_{BC} f(z)dz + \int_{CD} f(z)dz + \int_{DA} f(z)dz = -2\pi i\, e^{a\pi i} \qquad \text{.... (7.16)}$$

On AB, $z = R + iy, 0 \le y \le 2\pi$. So

$$|f(z)| = \left|\frac{e^{az}}{1+e^z}\right| = \left|\frac{e^{a(R+iy)}}{1+e^{R+iy}}\right| \le \frac{e^{aR}}{e^R - 1}.$$

So, by ML formula,

$$\left|\int_{AB} f(z)dz\right| \le \frac{e^{aR}}{e^R - 1} \cdot 2\pi \to 0 \text{ as } R \to \infty \text{ and } 0 < a < 1.$$

On CD, $z = -R + iy$, $2\pi \ge y \ge 0$. So,

$$|f(z)| = \left|\frac{e^{az}}{1+e^z}\right| = \left|\frac{e^{a(-R+iy)}}{1+e^{-R+iy}}\right| \le \frac{e^{-aR}}{-e^{-R}+1}.$$

So by ML-formula,

$$\left|\int_{CD} f(z)dz\right| \le \frac{e^{-aR}}{1-e^{-R}} \cdot 2\pi \to 0 \text{ as } R \to \infty \text{ and } 0 < a < 1.$$

Letting $R \to \infty$ we get from (7.16)

$$\int_{\infty}^{-\infty} \frac{e^{a(x+i2\pi)}}{1+e^{x+i2\pi}}dx + \int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x}dx = -2\pi i\, e^{a\pi i}$$

i.e. $$\left(1 - e^{2\pi a i}\right) \int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x}dx = -2\pi i\, e^{a\pi i}$$

i.e. $$\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x}dx = -\frac{2\pi i\, e^{a\pi i}}{1-e^{2\pi a i}} = -\frac{2\pi i}{e^{-a\pi i} - e^{a\pi i}} = \frac{\pi}{\sin \pi a}.$$

Jordan Inequality

For $0 \le \theta \le \frac{\pi}{2}$, we have $\frac{2\theta}{\pi} \le \sin\theta \le \theta$.

The above inequality is known as Jordan Inequality.

7.11. Indenting the contour having poles on the real axis

Example 7.17. Evaluate $\int_0^\infty \frac{\sin mx}{x} dx$.

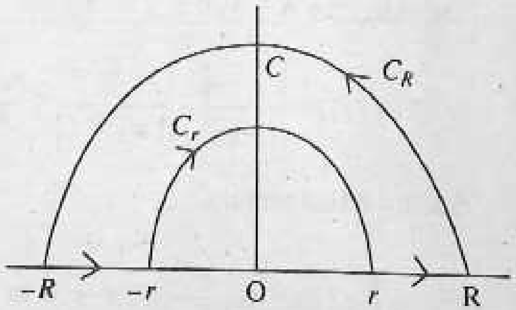


Fig. 7.6

Solution. We consider the integral

$$\int_C \frac{e^{imz}}{z} dz = \int_C f(z)dz$$

where C is a contour consisting of the following :

i) the real axis from r to R,

ii) the upper half of the circle $C_R : |z| = R$,

iii) the real axis from $-R$ to $-r$,

iv) the upper half of the circle $C_r : |z| = r$.

Since f has no singularity inside C (its only singularity being a simple pole at $z = 0$, which is outside the contour C), we get by Cauchy fundamental theorem

$$\int_C f(z)dz = 0.$$

i.e., $$\int_r^R f(x)dx + \int_{C_R} f(z)dz + \int_{-R}^{-r} f(x)dx + \int_{C_r} f(z)dz = 0 \qquad \text{..... (7.17)}$$

Now, $\int_{C_R} f(z)dz = \int_0^\pi \frac{e^{imRe^{i\theta}}}{Re^{i\theta}} . iRe^{i\theta} d\theta$, putting $z = Re^{i\theta}, 0 \le \theta \le \pi$

$$= i\int_0^\pi e^{imR(\cos\theta + i\sin\theta)} d\theta.$$

Since $\left|e^{imR(\cos\theta + i\sin\theta)}\right| = \left|e^{imR\cos\theta} . e^{-mR\sin\theta}\right| = e^{-mR\sin\theta}$, we have

$$\left|\int_{C_R} f(z)dz\right| = \left|i\int_0^\pi e^{imR(\cos\theta+i\sin\theta)}d\theta\right| \le \int_0^\pi \left|e^{imR(\cos\theta+i\sin\theta)}\right|d\theta$$

$$= \int_0^\pi e^{-mR\sin\theta}d\theta = 2\int_0^{\pi/2} e^{-mR\sin\theta}d\theta$$

$$\le 2\int_0^{\pi/2} e^{-\frac{2Rm}{\pi}\theta}d\theta, \text{ by Jordan's inequality.}$$

$$= 2.\frac{\pi}{2Rm}\left[e^{-\frac{2Rm}{\pi}\theta}\right]_{\pi/2}^{0}$$

$$= \frac{\pi}{Rm}\left(1-e^{-Rm}\right) \to 0 \text{ as } R \to \infty.$$

Also $z = 0$ is a simple pole of $f(z)$. Hence near $z = 0$, $f(z)$ has a Laurent expansion of the form

$$f(z) = \frac{a}{z} + \phi(z),$$

where $\phi(z)$ is analytic at $z = 0$ and

$$a = \text{Res}(f;0) = \lim_{z\to 0} zf(z) = \lim_{z\to 0} e^{imz} = 1.$$

Hence $\int_{C_r} f(z)dz = \int_{C_r} \frac{1}{z}dz + \int_{C_r} \phi(z)dz$.

On C_r, $z = re^{i\theta}$, $\pi \ge \theta \ge 0$. Then

$$\int_{C_r} f(z)dz = \int_\pi^0 i d\theta + \int_{C_r} \phi(z)dz = -i\pi + \int_{C_r} \phi(z)dz.$$

Since ϕ is analytic at the origin, $\exists$ a positive number M such that in some neighbourhood of the origin $|\phi(z)| \le M$. We take r so small that C_r lies entirely in this neighbourhood. Hence $|\phi(z)| \le M$ on C_r. Therefore by ML formula

$$\left|\int_{C_r} \phi(z)dz\right| \leq M.\pi r \to 0 \quad \text{as } r \to 0.$$

So, $\lim\limits_{r\to 0} \int_{C_r} f(z)dz = -i\pi$.

Hence proceeding to the limit as $r \to 0$ and $R \to \infty$, we get form (7.17)

$$\int_{-\infty}^{\infty} f(x)dx = i\pi$$

i.e. $$\int_{-\infty}^{\infty} \frac{e^{imx}}{x} dx = i\pi$$

i.e. $$\int_{-\infty}^{\infty} \frac{\cos mx + i\sin mx}{x} dx = i\pi.$$

Equating the imaginary part in both sides we get

$$\int_{-\infty}^{\infty} \frac{\sin mx}{x} dx = \pi$$

i.e. $$\int_{0}^{\infty} \frac{\sin mx}{x} dx = \frac{\pi}{2}.$$

7.12 Other types

Example 7.18. Show by the method of contour integration

$$\int_0^{\infty} \sin x^2 dx = \int_0^{\infty} \cos x^2 dx = \frac{\sqrt{2\pi}}{4}.$$

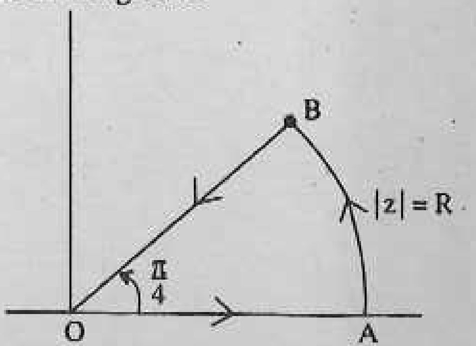


Fig. 7.7

Solution. Let $f(z) = e^{-z^2}$.

We integrate f round a closed contour C consisting of

(i) the line segment $L_1 : z = x,\ 0 \leq x \leq R$;

(ii) the circular arc $C_R : z = Re^{i\theta}, 0 \leq \theta \leq \frac{\pi}{4}$;

(iii) the line segment $L_2 : z = xe^{i\frac{\pi}{4}}, R \ge x \ge 0$.

Since f is analytic within and on the closed contour C, by Cauchy's fundamental theorem we get

$$0 = \int_C f(z)\,dz = \int_{L_1} e^{-z^2} dz + \int_{C_R} e^{-z^2} dz + \int_{L_2} e^{-z^2} dz$$

$$= \int_0^R e^{-x^2} dx + \int_{C_R} e^{-z^2} dz + \int_R^0 e^{-x^2 e^{i\pi/2}} . e^{i\pi/4} dx$$

$$= \int_0^R e^{-x^2} dx + \int_{C_R} e^{-z^2} dz - e^{i\pi/4} \int_0^R e^{-ix^2} dx \qquad \text{.... (7.18)}$$

Now, on C_R, $z = Re^{i\theta}, 0 \le \theta \le \frac{\pi}{4}$.

So, $\left| \int_{C_R} e^{-z^2} dz \right| = \left| \int_0^{\pi/4} e^{-R^2 e^{i2\theta}} .iRe^{i\theta} d\theta \right| \le R \int_0^{\pi/4} \left| e^{-R^2(\cos 2\theta + i \sin 2\theta)} \right| d\theta = R \int_0^{\pi/4} e^{-R^2 \cos 2\theta} d\theta$.

Putting $\phi = \frac{\pi}{2} - 2\theta$ we get

$$\left| \int_{C_R} e^{-z^2} dz \right| \le \frac{1}{2} R \int_0^{\pi/2} e^{-R^2 \sin\phi} d\phi.$$

For $0 \le \phi \le \frac{\pi}{2}$, we get by Jordan's inequality

$$\frac{2\phi}{\pi} \le \sin\phi$$

i.e. $e^{-R^2 \sin\phi} \le e^{-R^2 . \frac{2\phi}{\pi}}$

So, $\left|\int_{C_R} e^{-z^2} dz\right| \leq \frac{1}{2} R \int_0^{\pi/2} e^{-R^2 \frac{2\phi}{\pi}} d\phi = \frac{\pi}{4R}\left(1 - \frac{1}{e^{R^2}}\right) \to 0$ as $R \to \infty$.

Now we consider the integral $\int_0^\infty e^{-x^2} dx$.

Putting $x^2 = t$ we get

$$\int_0^\infty e^{-x^2} dx = \frac{1}{2}\int_0^\infty e^{-t} t^{\frac{1}{2}-1} dt = \frac{\Gamma\left(\frac{1}{2}\right)}{2} = \frac{\sqrt{\pi}}{2}.$$

Proceeding to the limit as $R \to \infty$ we get from (7.18)

$$e^{\frac{i\pi}{4}} \int_0^\infty e^{-ix^2} dx = \frac{\sqrt{\pi}}{2}$$

i.e. $\left(\cos\frac{\pi}{4} + i\sin\pi/4\right)\int_0^\infty (\cos x^2 - i\sin x^2)dx = \frac{\sqrt{\pi}}{2}$

i.e. $\int_0^\infty (1+i)(\cos x^2 - i\sin x^2)dx = \sqrt{\frac{\pi}{2}}$.

Equating real and imaginary part we obtain

$$\int_0^\infty (\cos x^2 + \sin x^2)dx = \sqrt{\frac{\pi}{2}} \quad \text{and} \quad \int_0^\infty (\cos x^2 - \sin x^2)dx = 0.$$

Adding and subtracting above two results we obtain the solution of the problem.

Exercises-7

1. Find the residues of $f(z)=\dfrac{z^2-2z}{(z+1)^2(z^2+4)}$ at all its poles in the finite plane.

2. Find the residue of the function $f(z)=z^3\sin\frac{1}{z-1}$ at its singular points.

3. Evaluate the following integral.

$$\oint_{|z|=2}\frac{\sin z}{(z-1)^2(z^2+9)}dz$$

4. Show that $\displaystyle\int_0^{2\pi}\frac{\cos 3\theta}{5-4\cos\theta}d\theta=\frac{\pi}{12}$.

5. Prove that the roots of the equation $z^5+az+1=0$ lie within $|z|=r$ if $|a|<r^4-\dfrac{1}{r}$.

6. Show that the equation $z^3+iz+1=0$ has a root in each of the first, second and fourth quadrants.

7. How many zeros have the complex polynomial z^4+3z^2+z+1 in the right half plane?

8. Show that $\displaystyle\int_0^{2\pi}\frac{\cos 2\theta}{1-2a\cos\theta+a^2}d\theta=\frac{2\pi}{a^2(a^2-1)}$, $(a^2>1)$.

9. Show that $\displaystyle\int_0^{2\pi}\frac{d\theta}{5+3\cos\theta}=\frac{\pi}{2}$.

10. Show that $\displaystyle\int_0^{\infty}\frac{dx}{x^6+1}=\frac{\pi}{3}$.

11. Show that $\int_{-\infty}^{\infty} \frac{x \sin x}{a^2 + x^2} dx = \pi e^{-a}, a > 0.$

12. Show that $\int_{0}^{\infty} \frac{dx}{x^4 + a^4} = \frac{\pi\sqrt{2}}{4a^3}$, $a > 0.$

13. Prove that $\int_{0}^{\infty} \frac{\log(x^2 + 1)}{x^2 + 1} dx = \pi \log 2.$

14. Prove that $\int_{0}^{\pi/2} \log \sin x \, dx = \int_{0}^{\pi/2} \log \cos x \, dx = -\frac{\pi}{2} \log 2.$

Unit-8 ❑ Bilinear Transformation

8.1 Definition

If a, b, c, d are complex constants then the transformation

$$w = w(z) = \frac{az+b}{cz+d} \qquad \text{..... (8.1)}$$

where $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \neq 0$, is called a Bilinear Transformation or a linear fractional transformation or a Mobius Transformation. The expression $ab - bc$ is called the determinant of the transformation.

Remark 8.1. When $c = 0$, (8.1) represents simply a linear transformation. When $c \neq 0$, (8.1) can be written as

$$w = \frac{az+b}{cz+d} = \frac{\frac{a}{c}(cz+d) - \frac{ad}{c} + b}{cz+d} = \frac{a}{c} - \frac{ad-bc}{c} \cdot \frac{1}{cz+d}.$$

If $ad - bc = 0$, then $w = \frac{a}{c}$ = constant.

Thus the condition $ad - bc \neq 0$ imply that the function w is non-constant.

8.2. Properties of B.T

i) The inverse of a B.T is also a B.T.

Proof. Let (8.1) is a B.T. Solving for z we get

$$z = \frac{-dw+b}{cw-a}, \qquad \text{.... (8.2)}$$

where $\begin{vmatrix} -d & b \\ c & -a \end{vmatrix} = ad - bc \neq 0.$

The transformation (2) is the inverse of the B.T (8.1) which is also a B.T.

Conformal Mapping

A mapping or transformation which preserves angles in magnitude and sense between every pair of curves through a point is said to be conformal at the point.

Theorem 8.1. At each point where a function f is analytic and $f'(z) \neq 0$, the mapping $w = f(z)$ is conformal.

ii) A bilinear transformation is a conformal mapping of the extended z-plane onto the extended w-plane.

Proof. Let $w = w(z) = \dfrac{az+b}{cz+d}$, $ab - bc \neq 0$ be a bilinear transformation. Then

$$w'(z) = \frac{ad-bc}{(cz+d)^2} \neq 0$$

and so $w = w(z)$ is a conformal mapping.

iii) The composition of two B.T is again a B.T.

Proof. Let $\zeta = \dfrac{a_1 z + b_1}{c_1 z + d_1}$, $a_1 d_1 - b_1 c_1 \neq 0$ ..... (8.3)

and $w = \dfrac{a_2 \zeta + b_2}{c_2 \zeta + d_2}$, $a_2 d_2 - b_2 c_2 \neq 0$ ..... (8.4)

be two bilinear transformations. Substituting we get

$$w = \frac{a_2 \cdot \frac{a_1 z + b_1}{c_1 z + d_1} + b_2}{c_2 \cdot \frac{a_1 z + b_1}{c_1 z + d_1} + d_2} = \frac{(a_2 a_1 + b_2 c_1) z + (a_2 b_1 + b_2 d_1)}{(a_1 c_2 + c_1 d_2) z + (b_1 c_2 + d_1 d_2)}$$

$$= \frac{az+b}{cz+d} \qquad \text{...... (8.5)}$$

where $a = a_1 a_2 + b_2 c_1$, $b = a_2 b_1 + b_2 d_1$, $c = a_1 c_2 + c_1 d_2$, $d = b_1 c_2 + d_1 d_2$.

Again $ad - bc = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a_1 & b_1 \\ c_1 & d_1 \end{vmatrix} \begin{vmatrix} a_2 & b_2 \\ c_2 & d_2 \end{vmatrix} \neq 0.$

Thus the composition of two B.T is again a B.T

iv) The identity mapping $w = z$ is a B.T.

Proof. $w = z = \dfrac{1.z+0}{0.z+1}$, which is obviously a B.T.

v) The associative law for composition of B.T holds.

Proof. Let $T_1 : \zeta = \dfrac{a_1 z + b_1}{c_1 z + d_1}$, $a_1 d_1 - b_1 c_1 \neq 0$,

$$T_2 : w = \frac{a_2 \zeta + b_2}{c_2 \zeta + d_2}, \; a_2 d_2 - b_2 c_2 \neq 0,$$

$$\text{and } T_3 : \lambda = \frac{a_3 w + b_3}{c_3 w + d_3}, \; a_3 d_3 - b_3 c_3 \neq 0,$$

be three bilinear transformations. Then it is easily verified that

$$T_3(T_2 T_1) = (T_3 T_2) T_1$$

and so the associative property holds for the composition of B.T.

In view of the above properties the next theorem follows :

Theorem 8.2. The set of all B.T forms a group with respect to the composition.

8.3. Fixed Points or Invariant Points of a Transformation.

The points which coincide with their transformations are called fixed or invariant points of the transformation.

It z be a fixed point of the transformation T, then $T(z) = z$. As for example, the fixed or invariant points of the transformation $w = z^2$ are solutions of $z^2 = z$., i.e. $z = 0, 1$.

8.4. Cross Ratio

If z_1, z_2, z_3, z_4 are distinct points taken in the order in which they are written then the cross ratio of these points is defined as

$$(z_1, z_2, z_3, z_4) = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_2 - z_3)(z_4 - z_1)}.$$

Note 8.1. The cross ratio will change if the order of the factors is changed.

Since four letter z_1, z_2, z_3, z_4 can be arranged in $4! = 24$ ways, there will be 24 cross ratios but as matter of fact there will be only six distinct cross ratios. The six distinct cross ratios are

(z_1, z_2, z_3, z_4), (z_1, z_2, z_4, z_3), (z_1, z_3, z_2, z_4), (z_1, z_3, z_4, z_2), (z_1, z_4, z_2, z_3), (z_1, z_4, z_3, z_2).

Theorem 8.3. A bilinear transformation leaves a cross ratio invariant.

Proof. Let $w = \dfrac{az + b}{cz + d}$, $ad - bc \neq 0$ .... (8.6) be a bilinear transformation. Let w_1, w_2, w_3, w_4 be the images of the four points z_1, z_2, z_3, z_4 under the B.T (8.6). We have to show that

$$(w_1, w_2, w_3, w_4) = (z_1, z_2, z_3, z_4).$$

Now $(w_1, w_2, w_3, w_4) = \dfrac{(w_1 - w_2)(w_3 - w_4)}{(w_2 - w_3)(w_4 - w_1)}$.

$$w_1 - w_2 = \frac{(ad - bc)(z_1 - z_2)}{(cz_1 + d)(cz_2 + d)},$$

$$w_2 - w_3 = \frac{(ad - bc)(z_2 - z_3)}{(cz_2 + d)(cz_3 + d)},$$

$$w_3 - w_4 = \frac{(ad - bc)(z_3 - z_4)}{(cz_3 + d)(cz_4 + d)},$$

and $w_4 - w_1 = \dfrac{(ad - bc)(z_4 - z_1)}{(cz_4 + d)(cz_1 + d)}$.

Hence $(w_1, w_2, w_3, w_4) = \dfrac{(z_1 - z_2)(z_3 - z_4)}{(z_2 - z_3)(z_4 - z_1)} = (z_1, z_2, z_3, z_4)$.

Theorem 8.4. Prove that in general there are two values of z for which $w = z$ but there is only one if $(a - d)^2 + 4bc = 0$.

Show that if there are distinct invariant points p and q the transformation may be put in the form

$$\frac{w - p}{w - q} = K\left(\frac{z - p}{z - q}\right)$$

and that if there is only one invariant point p, the transformation may be put in the form

$$\frac{1}{w - p} = \frac{1}{z - p} + K.$$

Proof. Let $w = \dfrac{az + b}{cz + d}$, $ad - bc \neq 0$ .... (8.7)

be a bilinear transformation. For fixed points we have $w = z$, i.e. $cz^2 - (a-d)z - b = 0$. .... (8.8)

In general (8.8) gives two values of z and hence two fixed points. In case of $(a-d)^2 + 4bc = 0$, there will be only one fixed point.

Let p and q be two fixed points. Then from (8.8) we have

$$\left.\begin{array}{r} cp^2 - ap = b - pd \\ \text{and } cq^2 - aq = b - qd \end{array}\right\}. \qquad \text{...... (8.9)}$$

Now $w - p = \dfrac{(a-cp)z + (b-dp)}{cz+d} = \dfrac{(a-cp)(z-p)}{cz+d}$, by (8.9).

Similarly, $w - q = \dfrac{(a-cq)(z-q)}{cz+d}$.

$$\therefore \frac{w-p}{w-q} = \frac{(a-cp)(z-p)}{(a-cq)(z-q)} = K\left(\frac{z-p}{z-q}\right), \text{ where } K = \frac{a-cp}{a-cq}.$$

If there is only one fixed point then

$$p = q = \frac{p+q}{2} = \frac{a-d}{2c}, \text{ by (8.9).} \qquad \text{.... (8.10)}$$

$$\therefore \frac{1}{w-p} = \frac{cz+d}{(a-cp)(z-p)} = \frac{cz+a-2cp}{(a-cp)(z-p)}, \text{ by (8.10)}$$

$$= \frac{c(z-p)+(a-cp)}{(a-cp)(z-p)} = K + \frac{1}{z-p}, \text{ where } K = \frac{c}{a-cp}.$$

This proves the theorem.

Note 8.2.

i) A bilinear transformation $w = \dfrac{az+b}{cz+d}$ having only one fixed point is called a parabolic transformation and as shown above the transformation is of the form $\dfrac{1}{w-p} = \dfrac{1}{z-p} + K$, where p is the fixed point.

ii) A bilinear transformation $w = \dfrac{az+b}{cz+d}$, $ad - bc \neq 0$ having two fixed points p and q can be put in the form

$$\frac{w-p}{w-q} = K\left(\frac{z-p}{z-q}\right).$$

If $|K| = 1$, then it is called elliptic and if K is real then it is called hyperbolic.

iii) A bilinear transformation $w = \frac{az+b}{cz+d}, ad - bc \neq 0$ which is neither hyperbolic nor elliptic nor parabolic is called loxodromic.

8.5. Inverse Point

Let C be a circle of radius R with centre at z_0. Two points P and Q are said to be inverse points with respect to the circle C if they are collinear with the centre and lie on the same side of it and the product of their distances from the centre is equal to R^2.

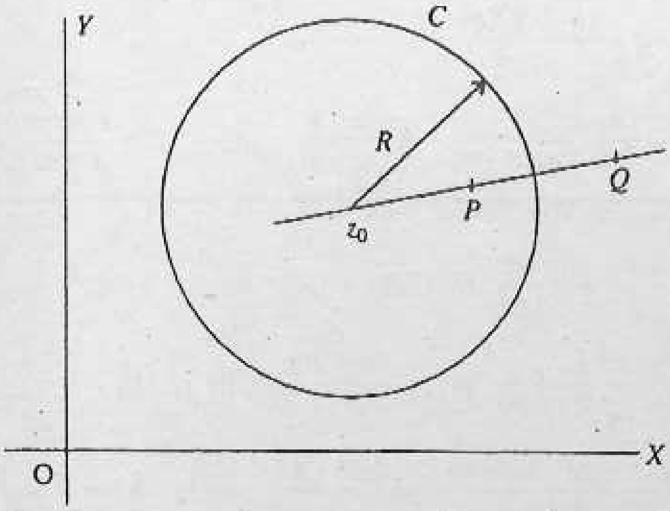


Fig. 8.1

Clearly Q is exterior to C if and only if P is interior to C. If Q is on C, then Q coincides with P.

Note 8.3. $z = 0$ and $z = \infty$ are consider as a pair of inverse points.

Thus if $p = z_0 + \rho e^{i\lambda}$, then $q = z_0 + \frac{R^2}{\rho} e^{i\lambda}$.

If z is any point on C, then $z = z_0 + Re^{i\theta}, 0 \leq \theta \leq 2\pi$.

$$\text{Therefore, } \left|\frac{z-p}{z-q}\right| = \left|\frac{Re^{i\theta} - \rho e^{i\lambda}}{Re^{i\theta} - \frac{R^2}{\rho} e^{i\lambda}}\right| = \frac{\rho}{R}.$$

This is a new form of the equation of a circle.

Conservely, it can be shown that

$$\left|\frac{z-p}{z-q}\right| = K(\neq 1)$$

represents a circle in the z-plane with respect to which p and q are inverse points. In the particular case when $K = 1$, z is equidistant from the points p and q, and hence lies on the perpendicular bisectors of the line joining them.

Theorem 8.5. A B.T transforms a circle into a circle and inverse points into inverse points. In the particular case in which the circle becomes a straight line, inverse points becomes points symmetric about the line.

Proof. Let $\left|\frac{z-p}{z-q}\right| = K$ be a circle (or a straight line for $K = 1$) with p and q as inverse points (or symmetric points).

Let $w = \frac{az+b}{cz+d}$, $ad - bc \neq 0$ ..... (8.11)

be a B.T. Solving for z we get

$$z = \frac{-dw+b}{cw-a}.$$

Then the circle is transformed into

$$\left|\frac{\frac{-dw+b}{cw-a} - p}{\frac{-dw+b}{cw-a} - q}\right| = K$$

i.e. $$\left|\frac{-dw+b-cpw+ap}{-dw+b-cqw+aq}\right| = K$$

i.e. $$\left|\frac{w - \frac{ap+b}{cp+d}}{w - \frac{aq+b}{cq+d}}\right| = K\left|\frac{cq+d}{cp+d}\right|$$

i.e. $$\left|\frac{w-\alpha}{w-\beta}\right| = K',$$

where $\alpha = \frac{ap+b}{cp+d}$, $\beta = \frac{aq+b}{cq+d}$ and $K' = K\left|\frac{cq+d}{cp+d}\right|$.

Thus the map of the circle $\left|\frac{z-p}{z-q}\right| = K$ under the B.T. (8.11) is a circle or a straight line $\left|\frac{w-\alpha}{w-\beta}\right| = K'$ with respect to which α and β are inverse points or symmetric points which are respectively the images of p and q. This proves the theorem.

Example 8.1. Show that the B.T. which carries the points $z = i, 0, -i$ into $w = 0, -1, \infty$ respectively maps

i) the real axis Im(z) = 0 on $|w| = 1$,

ii) the upper half plane Im(z) > 0 on $|w| < 1$,

iii) the lower half plane Im(z) < 0 on $|w| > 1$.

Solution. Let $w = \frac{az+b}{cz+d}$, $ad - bc \neq 0$ ..... (8.12)

be the required B.T.

Now, $i \to 0 \Rightarrow \frac{ai+b}{ci+d} = 0$

i.e. $ai + b = 0$. ...... (8.13)

$0 \to -1 \Rightarrow \frac{b}{d} = -1$

i.e. $b + d = 0$. .... (8.14)

$-i \to \infty \Rightarrow \frac{-ai+b}{-ci+d} = \infty$

i.e. $-ci + d = 0$. .... (8.15)

From (8.12), (8.13), (8.14) and (8.15) we get

$w = \frac{z-i}{z+1}$, which is the required B.T.

Now we consider following three cases separately.

Case 1. Any point on the real axis can be taken as $z = x$.

Then its image is $w = \frac{x-i}{x+i}$. Thus $|w| = \left|\frac{x-i}{x+i}\right| = 1$.

Case 2. Any point on the upper half plane can be taken as

$$z = x + iy, \ \ y > 0.$$

Then its image is

$$w = \frac{x + i(y-1)}{x + i(y+1)}.$$

$$\text{Thus } |w| = \left|\frac{x+i(y-1)}{x+i(y+1)}\right| = \sqrt{\frac{x^2+y^2-2y+1}{x^2+y^2+2y+1}} < 1. \ (\because y > 0)$$

So the image of the upper half plane Im(z) > 0 is the region $|w| < 1$.

Case 3. In this case $z = x + iy$, $y < 0$.

Then its image is

$$w = \frac{x + i(y-1)}{x + i(y+1)}$$

$$\text{and} \quad |w| = \sqrt{\frac{x^2+y^2-2y+1}{x^2+y^2+2y+1}} > 1. \quad (\because y < 0)$$

Thus the image of the lower half plane Im(z) < 0 is the region $|w| > 1$.

Example 8.2. Show that the transformation $w = \dfrac{2z+3}{z-4}$ maps the circle $x^2 + y^2 - 4x = 0$ onto the line $4u + 3 = 0$.

Solution. Given transformation is clearly a B.T. The inverse transformation is given by

$$z = \frac{4w+3}{w-2}. \qquad \text{....... (8.16)}$$

The equation of the circle can be written as

$$x^2 + y^2 - 4x = 0.$$

$$\text{i.e.} \quad |z|^2 - 4Rl(z) = 0$$

$$\text{i.e.} \quad z\bar{z} - 2(z + \bar{z}) = 0. \qquad \text{....... (8.17)}$$

Substituting for z and $\bar{z}$ from (8.16) in (8.17),

$$\frac{4w+3}{w-2} \cdot \frac{4\bar{w}+3}{\bar{w}-2} - 2\left(\frac{4w+3}{w-2} + \frac{4\bar{w}+3}{\bar{w}-2}\right) = 0$$

i.e. $(4w+3)(4\bar{w}+3)-2\{(4w+3)(\bar{w}-2)+(4\bar{w}+3)(w-2)\}=0$

i.e. $2(w+\bar{w})+3=0$

i.e. $4u + 3 = 0$, $[\because w = u + iv]$

which is the required line.

Example 8.3. Let $f(z)$ be a bilinear transformation such that $f(\infty) = 1$, $f(i) = i$ and $f(-i) = -i$. Find the image of the unit disc $\{Z \in c : |z| < 1\}$ under $f(z)$.

Solution. Let $w=\dfrac{az+b}{cz+d}$, $ad - bc \neq 0$ ..... (8.18)

be the required bilinear transformation.

Now $\infty \to 1 \Rightarrow \dfrac{a}{c}=1$

$$\Rightarrow a-c=0. \qquad \text{...... (8,.19)}$$

$$i \to i \Rightarrow \frac{ai+b}{ci+d}=i$$

$$\Rightarrow (a-d)i+(b+c)=0. \qquad \text{....... (8.20)}$$

$$-i \to -i \Rightarrow \frac{-ai+b}{-ci+d}=-i$$

$$\Rightarrow (a-d)i-(b+c)=0. \qquad \text{...... (8.21)}$$

From (8.19), (8.20) and (8.21) we obtain

$a = c = d = -b.$

Thus $w=f(z)=\dfrac{z-1}{z+1}$. Let $z=re^{i\theta}$. Then

$$w=\frac{re^{i\theta}-1}{re^{i\theta}+1}=\frac{(r\cos\theta-1)+ir\sin\theta}{(r\cos\theta+1)+ir\sin\theta}=\frac{r^2-1}{r^2+2r\cos\theta+1}+i\frac{2r}{r^2+2r\cos\theta+1}.$$

The map of a point lying on $|z| = 1$ lies on the imaginary axis in the w-plane because for $r = 1$, $\text{Re}(w) = 0$. If $r < 1$, then $\text{Re}(w) < 0$, i.e. the image of a point lying inside the unit circle $|z| = 1$ lies in the left half plane of w-plane. (See Fig. 8.2 and 8.3).

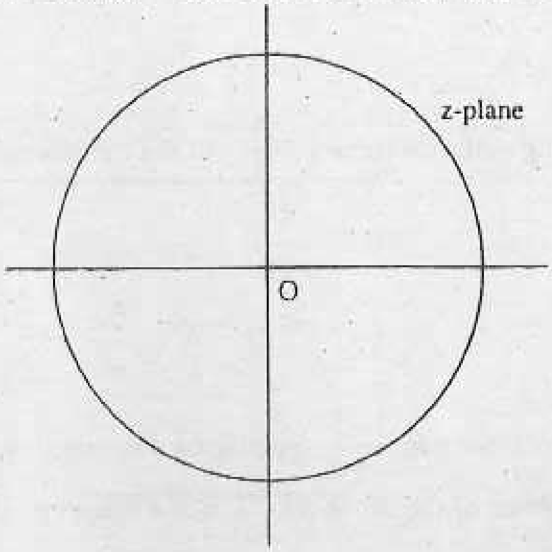


Fig. 8.2

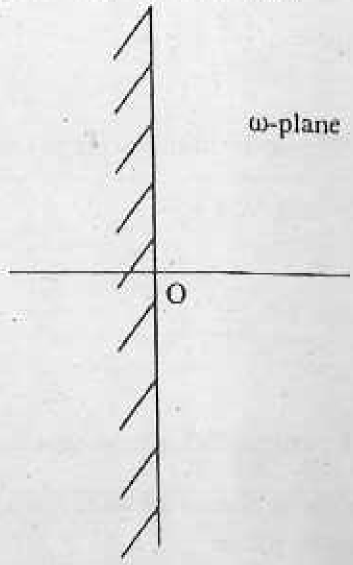


Fig. 8.3

Example 8.4. Show that the transformation $w = \dfrac{iz+2}{4z+i}$ maps the real axis in the z-plane into a circle in the w-plane. Find the centre and the radius of the circle and the point in the z-plane which is mapped on the centre of the circle.

Solution. The inverse transformation is $z = \dfrac{2-iw}{4w-i}$.

Now the equation of the real axis in the z-plane is $z - \bar{z} = 0$.

Hence the transformation curve is given by

$$\frac{2-iw}{4w-i} - \frac{2+i\bar{w}}{4\bar{w}+i} = 0$$

i.e. $8\bar{w} + 2i - 4iw\bar{w} + w - 8w + 2i - 4iw\bar{w} - \bar{w} = 0$

i.e. $8iw\bar{w} + 7(w - \bar{w}) - 4i = 0$

i.e. $8i(u^2 + v^2) + 14iv - 4i = 0$

i.e. $u^2 + v^2 + \dfrac{7}{4}v - \dfrac{1}{2} = 0$

which is a circle in the w-plane whose centre is $\left(0, -\frac{7}{8}\right)$ i.e., $w = -\frac{7}{8}i$ and radius $= \sqrt{\frac{49}{64} + \frac{1}{2}} = \frac{9}{8}$.

The point in the z-plane corresponding to the centre $w = -\frac{7}{8}i$ of the transformed circle is given by

$$z = \frac{2 - i\left(-\frac{7}{8}i\right)}{4\left(-\frac{7}{8}i\right) - i} = \frac{9}{-36i} = \frac{1}{4}i.$$

Example 8.5. Prove that the transformation $w = i\frac{1-z}{1+z}$ transforms the circle $|z|=1$ onto the real axis of w-plane and the interior of the circle $|z| < 1$ into the upper half of the w-plane.

Solution. The inverse transformation is

$$z = \frac{i - w}{i + w}.$$

Under the transformation, the circular disc $|z| \le 1$ is transformed into the region

$$\left|\frac{i - w}{i + w}\right| \le 1$$

i.e. $\left(\frac{i - w}{i + w}\right)\left(\frac{-i - \bar{w}}{-i + \bar{w}}\right) \le 1$

i.e. $1 + (w - \bar{w})i + w\bar{w} \le 1 - (w - \bar{w})i + w\bar{w}$

i.e. $2i(w - \bar{w}) \le 0$

i.e. $2i2iv \le 0$

i.e. $-4v \le 0$

i.e. $v \ge 0$.

Thus the boundary of the circle $|z| = 1$ is transformed into the real axis $v = 0$ and the interior $|z| < 1$ is transformed into $v > 0$ i.e., into the upper half plane as required.

Also it is evident that the exterior $|z| > 1$ is transformed into the lower half plane defined by $v < 0$.

Exercises-8

1. Find the bilinear transformation which maps $z_1 = i$, $z_2 = 1$, $z_3 = \infty$ into $w_1 = -i$, $w_2 = \infty$, $w_3 = -1$.

   Ans. $\dfrac{-z+(2i+1)}{z-1}$.

2. Find a bilinear transformation which maps points $z = 0, -i, -1$ into $w = i, 1, 0$.

   Ans. $w = -i\left(\dfrac{z+1}{z-1}\right)$.

3. Find a bilinear transformation which maps the points $1, i, -1$ in the z-plane into the points $0, 1, \infty$ in a w-plane. Show that by means of this transformation the area of the circle $|z| = 1$ is represented in the w-plane by the half plane above the real axis.

4. Find the fixed or invariant points of the transformation $w = \dfrac{2z-5}{z+4}$.

5. Show that the transformation $w = \dfrac{1-iz}{z-i}$ maps $|z| = r$, where $r < 1$, into a circle in the w-plane, whose centre is on the imaginary axis.

6. Show that the transformation $w = \dfrac{5-4z}{4z-2}$ transforms the circle $|z| = 1$ into a circle of unit radius in the w-plane and find the centre of the circle.

   Ans. $(-½, 0)$

7. If $a \neq b$ are the two fixed points of the bilinear transformation, show that it can be written in the form

$$\frac{w-a}{w-b} = K\left(\frac{z-a}{z-b}\right)$$

   where K is a constant.

8. If $a = b$ in problem 7, show that the transformation can be written in the form

$$\frac{1}{w-a} = \frac{1}{z-a} + K$$

   where K is a constant.

9. Prove that the most general bilinear transformation which maps $|z| = 1$ onto $|w| = 1$ is

$$w = e^{i\theta}\left(\frac{z-p}{\bar{p}z-1}\right)$$

where p is a complex number.

10. Show that the transformation of Problem 9 maps $|z| < 1$ onto (a) $|w| < 1$ if $|p| < 1$ and (b) $|w| > 1$ if $|p| > 1$.

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